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Some Lie theory on Shearlet group

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Abstract

In this work, using some tools of Lie theory, we compute the Lie algebra of the Shearlet group regarding as a 3-fold semidirect product Lie group. As we will see, it is a 3-fold semidirect sum of Lie algebras.

Keywords: Shearlet group; Lie group; Lie algebra; Exponential map; Semidirect product
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1 Introduction

Lie algebras are favorite objects because they are one kind of "linearization" of Lie groups so that reduce geometry to linear algebra. In recent years, Shearlet group which is the locally compact group related to Shearlet transform has investigated from various points of view such as group theory, harmonic analysis and image processing (see [1], [2], [5], [6], [7],[8], and [9]). As it is shown that in [13] page 17, the Shearlet group is related to semidirect product of groups. The authors proved in [3] and [12] that Shearlet group can be made of a 3-fold semidirect product group. In this work, we use this fact to extract some results about Lie algebra of Shearlet group.

2 Preliminaries

We first recall some prerequisites from [11]:

Let G be a Lie group, $p \in G$ and (ϕ, U) a chart of G with $p \in U$. Let $\gamma : I \to G$ be a smooth curve, where $I \subseteq \mathbb{R}$ is an interval containing 0 and $\gamma(0) = p$. We call two such curves $\gamma_i : I_i \to G, i = 1, 2$ *equivalent* and denoted by $\gamma_1 \sim \gamma_2$, if $(\phi o \gamma_1)'(0) = (\phi o \gamma_2)'(0)$. Clearly, this defines an equivalence relation. The equivalence classes are called *tangent vectors* at p. We denote $T_p(G)$ for the set of all tangent vectors at p and $[\gamma] \in T_p(G)$ for the equivalence class of the curve γ.

For a Lie group G, we call $T_1(G)$ *the Lie algebra* of G and it denotes by $\mathbf{L}(G)$. We refer the willing reader to [11] for seeing structure of $T_1(G)$.

For a given smooth map of Lie groups such as $\phi : G_1 \to G_2$ with $p \in G_1$, we obtain a linear map

$$T_p(\phi) : T_p(G_1) \to T_{\phi(p)}(G_2), \quad [\gamma] \mapsto [\phi o \gamma],$$

that is called *the differential* of ϕ at p.

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Proposition 2.1. If $\phi : G_1 \to G_2$ is a homomorphism of Lie groups, then its differential at $\mathbf{1}$, i.e.,

$$\mathbf{L}(\phi) := T_1(\phi) : \mathbf{L}(G_1) \to \mathbf{L}(G_2),$$

is a homomorphism of Lie algebras. Moreover, $(exp_{G_2})o\mathbf{L}(\phi) = \phi o(exp_{G_1})$.

Proof . See Propositions 9.1.8 and 9.2.10 of [11]. □

Let V be a finite-dimensional vector space. We write $GL(V)$ for the group of all linear automorphisms on V. It is well-known in the literature that $GL(V)$ carries a natural Lie group structure.

For a given Lie group G, a homomorphism $\phi : G \to GL(V)$ is called a *representation* of G on V. Similarly, if $\mathfrak{g}$ is a Lie algebra, then a homomorphism of Lie algebras; $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ is called a representation of $\mathfrak{g}$ on V, where by $\mathfrak{gl}(V)$ we mean $\mathbf{L}(GL(V))$ which is equal to $End(V)$ the group of all linear endomorphisms on V.

Proposition 2.2. If $\phi : G \to GL(V)$ is a representation of G, then $\mathbf{L}(\phi) : \mathbf{L}(G) \to \mathfrak{gl}(V)$ is a representation of its Lie algebra.

Proof . See Proposition 9.2.19 of [11]. □

The representation $\mathbf{L}(\phi)$ obtained in Proposition 2.2, from group representation ϕ is called the *derived representation.* This is motivated by the fact that for each $x \in \mathbf{L}(G)$, we have

$$\mathbf{L}(\phi)(x) = \frac{d}{dt}|_{t=0} e^{t\mathbf{L}(\phi)(x)} = \frac{d}{dt}|_{t=0}\phi(exp_G(tx)).$$

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and define

$$Aut(\mathfrak{g}) := \{g \in GL(\mathfrak{g}) : (\forall x, y \in \mathfrak{g}) g[x.y] = [gx.gy]\}.$$

By Lemma 4.2.2 in [11], we deduce Lie algebra of $Aut(\mathfrak{g})$ as follows:

$$\mathfrak{aut}(\mathfrak{g}) = \mathbf{L}(Aut(\mathfrak{g})) = \{D \in \mathfrak{gl}(\mathfrak{g}) : (\forall x, y \in \mathfrak{g}) D[x.y] = [Dx.y] + [x.Dy]\}.$$

The elements of this Lie algebra are called *derivations* of $\mathfrak{g}$. Also, $\mathfrak{aut}(\mathfrak{g})$ is denoted by $der(\mathfrak{g})$.

3 Main results

In this section, we consider locally compact group $(\mathbb{R}^+ \ltimes_\tau \mathbb{R}) \ltimes_\lambda \mathbb{R}^2$ as a Lie group with the natural Lie structure and then we compute its Lie algebra as a 3-fold semidirect sum of Lie algebras. We begin with a definition.

Definition 3.1. Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras and $\alpha : \mathfrak{g} \to der(\mathfrak{h})$ be a homomorphism. Then the direct sum $\mathfrak{g} \oplus \mathfrak{h}$ of the vector spaces $\mathfrak{g}$ and $\mathfrak{h}$ is a Lie algebra with respect to the bracket:

$$[(x, y), (x', y')] := ([x, x'], [y, y'] + \alpha(x)y' - \alpha(x')y).$$

This Lie algebra is called the *semidirect sum* with respect to α of $\mathfrak{g}$ and $\mathfrak{h}$ and denotes by $\mathfrak{g} \oplus_\alpha \mathfrak{h}$. If $\alpha = 0$, then $\mathfrak{g} \oplus_\alpha \mathfrak{h}$ is called the *direct sum* of $\mathfrak{g}$ and $\mathfrak{h}$, and it is denoted by $\mathfrak{g} \oplus \mathfrak{h}$.

Let G and H be Lie groups and $\tau : G \to Aut(H)$ be a group homomorphism defining a smooth action $(g, h) \mapsto \tau_g(h)$ of G on H. Then the product manifold $G \times H$ is a group with respect to the following product

$$(g, h)(g', h') = (gg', h\tau_g(h'))$$

and its inverse:

$$(g, h)^{-1} = (g^{-1}, \tau_{g^{-1}}(h^{-1})).$$

Since multiplication and its inverse are smooth, this group is a Lie group, also it is called the semidirect product of G and H with respect to τ. It denotes by $G \ltimes_\tau H$.

We establish our results on the next main theorem ([11],Theorem 9.2.25).

Theorem 3.2. . Let G and H be Lie groups. Lie algebra of the semidirect product group $G \ltimes_\tau H$ is defined by

$$\mathbf{L}(G \ltimes_\tau H) \cong \mathbf{L}(G) \oplus_\alpha \mathbf{L}(H),$$

where $\alpha : \mathbf{L}(G) \to der(\mathbf{L}(H))$ is the derived representation of $\mathbf{L}(G)$ on $\mathbf{L}(H)$ corresponding to the representation of G on $\mathbf{L}(H)$ given by $g.x := \mathbf{L}(\tau_g)x$.

Theorem 3.3. Let $Aff(\mathbb{R}^n) := GL_n(\mathbb{R}) \ltimes_\tau \mathbb{R}^n$ be the group of affine transformations of n-dimensional Euclidean space, where τ is defined by $\tau_R : \mathbb{R}^n \to \mathbb{R}^n, \tau_R(x) = Rx$. The Lie algebra of $Aff(\mathbb{R}^n)$ is given by

$$\mathbf{L}(GL_n(\mathbb{R}) \ltimes_\tau \mathbb{R}^n) \cong \mathfrak{gl}_n(\mathbb{R}) \oplus_\beta \mathbb{R}^n,$$

where, $\beta : \mathfrak{gl}_n(\mathbb{R}) \to \mathfrak{gl}_n(\mathbb{R})$ is given by $\beta(M) = M$.

Proof . Suppose that $\mathbf{L}(\mathbb{R}^n) \cong \mathbb{R}^n$ is the Lie algebra of $\mathbb{R}^n$ and $\mathbf{L}(\tau_R)$ is the derived Lie algebra homomorphism given by

$$\mathbf{L}(\tau_R) : \mathbf{L}(\mathbb{R}^n) \cong \mathbb{R}^n \to \mathbf{L}(\mathbb{R}^n) \cong \mathbb{R}^n :$$
$$x \mapsto \tau_R(x) = Rx.$$

Also, suppose π is the representation of $GL_n(\mathbb{R})$ on $\mathbf{L}(\mathbb{R}^n) \cong \mathbb{R}^n$ induced by $\mathbf{L}(\tau_R)$, more precisely,

$$\pi : GL_n(\mathbb{R}) \to GL(\mathbf{L}(\mathbb{R}^n)) \cong GL_n(\mathbb{R}) :$$

$$\pi(R)(x) = \mathbf{L}(\tau_R)(x) = Rx.$$

Finally, assume that β be the derived representation of $\mathbf{L}(GL_n(\mathbb{R}))$ on $\mathbf{L}(\mathbb{R}^n) \cong \mathbb{R}^n$ corresponding to π, i.e.,

$$\beta : \mathfrak{gl}_n(\mathbb{R}) \to der(\mathbf{L}(\mathbb{R}^n)) :$$

$$\beta(M) = \mathbf{L}(\pi)(M) = \frac{d}{dt}|_{t=0}\pi(\exp_{GL_n(\mathbb{R})}(tM)) = M,$$

where $der(\mathbf{L}(\mathbb{R}^n))$ is defined by $\mathbf{L}(Aut(\mathbf{L}(\mathbb{R}^n))) \cong \mathbf{L}(Aut(\mathbb{R}^n))$, since in the Lie algebra $\mathbb{R}^n$ bracket is equal to zero, we have $Aut(\mathbb{R}^n) = GL(\mathbb{R}^n)$, which this implies $der(\mathbf{L}(\mathbb{R}^n)) \cong \mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$.

Now, Theorem 3.2 yields that:

$$\mathbf{L}(GL_n(\mathbb{R}) \ltimes_\tau \mathbb{R}^n) \cong \mathfrak{gl}_n(\mathbb{R}) \oplus_\beta \mathbb{R}^n,$$

where the Lie algebra in the right-hand side endows with this bracket:

$$\begin{aligned}[(M, x).(M', x')] &= ([M.M'], [x.x'] + \beta(M)x' - \beta(M')x) \\ &= (MM' - M'M, Mx' - M'x).\end{aligned}$$

□

Since $O_n(\mathbb{R}) = \{g \in GL_n(\mathbb{R}) : (\forall x \in \mathbb{R}^n), \|gx\| = \|x\|\}$, the group of isometries of $\mathbb{R}^n$ consisting $Iso(\mathbb{R}^n) := O_n(\mathbb{R}) \ltimes_\tau \mathbb{R}^n$ is a subgroup of affine group. The group of *Euclidean motions* of $\mathbb{R}^n$ is the subgroup $Mot(\mathbb{R}^n) := SO_n(\mathbb{R}) \ltimes_\tau \mathbb{R}^n$ of isometries preserving orientation. On the other hand, the identity component of $O_n(\mathbb{R})$ is just $SO_n(\mathbb{R})$. Moreover, since the exponential of a matrix in the Lie algebra is automatically in the identity component of the corresponding Lie group, the Lie algebra of $O_n(\mathbb{R})$ is the same as the Lie algebra of $SO_n(\mathbb{R})$. So, we can deduce:

Corollary 3.4.

$$\mathbf{L}(Iso(\mathbb{R}^n)) = \mathbf{L}(Mot(\mathbb{R}^n)) \cong \mathfrak{o}_n(\mathbb{R}) \oplus_\beta \mathbb{R}^n.$$

As it is shown that in [10], $\mathfrak{o}_n(\mathbb{R})$ is the space of all $n \times n$ real matrices X with $X^{tr} = -X$.

Exponential map related to a semidirect product Lie group has been computed in [14] as follows:

Theorem 3.5. Let $G_1 \ltimes_\tau G_2$ be a semidirect product of Lie groups such that$\mathfrak{g}_1 \ltimes_\alpha \mathfrak{g}_2$ is its Lie algebra. If $x \in \mathfrak{g}_1$ and $y \in \mathfrak{g}_2$, then for sufficiently small $t > 0$, we have:

$$exp_{G_1 \ltimes_\tau G_2}(tx, ty) = (exp_{G_1}(tx), \tau(exp_{G_1}(tx))exp_{G_2}(z)),$$

where

$$z = ty - \frac{1}{2}\alpha(tx)ty + \frac{1}{6}\alpha^2(tx)ty + \frac{1}{12}[ty.\alpha(tx)ty] + O(t^3).$$

Let $(\mathbb{R}^+ \ltimes_\tau \mathbb{R}) \ltimes_\lambda \mathbb{R}^2$ be Shearlet group, where for $a \in \mathbb{R}^+$ and for $(a, s) \in \mathbb{R}^+ \ltimes_\tau \mathbb{R}$, $\tau_a : \mathbb{R} \to \mathbb{R}$ and $\lambda_{(a,s)} : \mathbb{R}^2 \to \mathbb{R}^2$ are given by $\tau_a(s) = \sqrt{a}s$ and $\lambda_{(a,s)}(t_1, t_2) = S_s A_a(t_1, t_2)$, where $S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ and $A_a = \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix}$, respectively, are *shearing* and *anisotropic(parabolic) scaling* matrices acting on the plane.
Now, we are able to compute Lie algebra of Shearlet group in the following theorem.

Theorem 3.6. Lie algebra of Shearlet group is given by:

$$\mathbf{L}\big((\mathbb{R}^+ \ltimes_\tau \mathbb{R}) \ltimes_\lambda \mathbb{R}^2\big) \cong (\mathbb{R} \oplus_\alpha \mathbb{R}) \oplus_\beta \mathbb{R}^2,$$

where, $\alpha : \mathbb{R} \to \mathbb{R}$ and $\beta : \mathbb{R} \oplus_\alpha \mathbb{R} \to \mathfrak{gl}_2(\mathbb{R})$ are defined by $\alpha(x) = \frac{1}{2}x$ and $\beta(x, y) = \begin{pmatrix} x & y \\ 0 & \frac{x}{2} \end{pmatrix}$.

Proof . In first step, we consider the semidirect product Lie group $\mathbb{R}^+ \ltimes_\tau \mathbb{R}$. Let $\mathbf{L}(\mathbb{R}) \cong \mathbb{R}$ be the Lie algebra of $\mathbb{R}$ and $\mathbf{L}(\tau_a)$ be the derived Lie algebra homomorphism given by

$$\mathbf{L}(\tau_a) : \mathbf{L}(\mathbb{R}) \cong \mathbb{R} \to \mathbf{L}(\mathbb{R}) \cong \mathbb{R},$$

$$s \mapsto \tau_a(s) = \sqrt{a}s.$$

Also, suppose that π be the representation of $\mathbb{R}^+$ on $\mathbf{L}(\mathbb{R}) \cong \mathbb{R}$ induced by $\mathbf{L}(\tau_a)$, more precisely,

$$\pi : \mathbb{R}^+ \to GL(\mathbf{L}(\mathbb{R})) \cong \mathbb{R}^*,$$

$$\pi(a)(s) = \mathbf{L}(\tau_a)(s) = \sqrt{a}s.$$

Finally, let α be the derived representation of $\mathbf{L}(\mathbb{R}^+) = \mathbb{R}$ on $\mathbf{L}(\mathbb{R}) \cong \mathbb{R}$ corresponding to π, i.e.,

$$\alpha : \mathbb{R} \to der(\mathbf{L}(\mathbb{R})),$$

$$\alpha(x) = \mathbf{L}(\pi)(x) = \frac{d}{dt}|_{t=0}\pi(\exp(tx)) = \frac{1}{2}x,$$

where, $der(\mathbf{L}(\mathbb{R}))$ is defined by $\mathbf{L}(Aut(\mathbf{L}(\mathbb{R}))) \cong \mathbf{L}(Aut(\mathbb{R}))$. Since in the Lie algebra $\mathbb{R}$ bracket is defined equal to zero, we have $Aut(\mathbb{R}) = GL(\mathbb{R}) = \mathbb{R}^*$, which this implies $der(\mathbf{L}(\mathbb{R})) \cong \mathbb{R}$.
Now, Theorem 3.2 yields that

$$\mathbf{L}(\mathbb{R}^+ \ltimes_\tau \mathbb{R}) \cong \mathbb{R} \oplus_\alpha \mathbb{R},$$

where, the Lie algebra in the right-hand side endows with the following bracket:

$$\begin{aligned}[(x, y).(x', y')] &= \big([x.x'], [y.y'] + \alpha(x)y' - \alpha(x')y\big) \\ &= \big(0, \frac{1}{2}(xy' - x'y)\big).\end{aligned}$$

One should note that the quantity $\frac{1}{2}(xy' - x'y)$ is the best value known and it is called *symplectic form* on $\mathbb{R}^2$.

In second step, we compute $exp_{\mathbb{R}^+ \ltimes_\tau \mathbb{R}}$. Let $x \in \mathbf{L}(\mathbb{R}^+) = \mathbb{R}$ and $y \in \mathbf{L}(\mathbb{R}) = \mathbb{R}$. We have $exp_{\mathbb{R}^+}(tx) = e^{tx}$ and by Example 9.2.3 in [11] $exp_{\mathbb{R}}$ is the identity map. Finally, since in the Lie group $\mathbb{R}$ bracket is defined equal to zero, Theorem 3.5 results that for sufficiently small $t > 0$, we get

$$exp_{\mathbb{R}^+ \ltimes_\tau \mathbb{R}}(tx, ty) = (e^{tx}, e^{\frac{1}{2}tx}(yt - \frac{1}{4}xyt^2 + \frac{1}{24}xyt^3)).$$

In third step, we consider the Shearlet group $(\mathbb{R}^+ \ltimes_\tau \mathbb{R}) \ltimes_\lambda \mathbb{R}^2$. Also, Suppose that ρ is the representation of $\mathbb{R}^+ \ltimes_\tau \mathbb{R}$ on $\mathbf{L}(\mathbb{R}^2) \cong \mathbb{R}^2$ induced by $\mathbf{L}(\lambda_{(a,s)})$, more precisely,

$$\rho : \mathbb{R}^+ \ltimes_\tau \mathbb{R} \to GL(\mathbf{L}(\mathbb{R}^2)) \cong GL_2(\mathbb{R}),$$

$$\rho(a,s)(t_1,t_2) = \mathbf{L}(\lambda_{(a,s)})(t_1,t_2) = \lambda_{(a,s)}(t_1,t_2).$$

Finally, assume that β is the derived representation of $\mathbf{L}(\mathbb{R}^+ \ltimes_\tau \mathbb{R}) \cong \mathbb{R} \oplus_\alpha \mathbb{R}$ on $\mathbf{L}(\mathbb{R}^2) \cong \mathbb{R}^2$ corresponding to ρ, i.e.,

$$\beta : \mathbb{R} \oplus_\alpha \mathbb{R} \to der(\mathbf{L}(\mathbb{R}^2)),$$

$$\begin{aligned}
\beta(x,y) &= \mathbf{L}(\rho)(x,y) \\
&= \frac{d}{dt}|_{t=0}\rho(\exp_{\mathbb{R}^+ \ltimes_\tau \mathbb{R}}(t(x,y))) \\
&= \frac{d}{dt}|_{t=0}\rho(e^{tx}, e^{\frac{1}{2}tx}(yt - \frac{1}{4}xyt^2 + \frac{1}{24}xyt^3)) \\
&= \frac{d}{dt}|_{t=0}\lambda_{(e^{tx}, e^{\frac{1}{2}tx}(yt - \frac{1}{4}xyt^2 + \frac{1}{24}xyt^3))} \\
&= \begin{pmatrix} x & y \\ 0 & \frac{x}{2} \end{pmatrix},
\end{aligned}$$

where $der(\mathbf{L}(\mathbb{R}^2))$ is defined by $\mathbf{L}(Aut(\mathbf{L}(\mathbb{R}^2))) \cong \mathbf{L}(Aut(\mathbb{R}^2))$. Since in the Lie algebra $\mathbb{R}^2$ bracket is defined equal to zero, we have $Aut(\mathbb{R}^2) = GL(\mathbb{R}^2) = GL_2(\mathbb{R})$ which this implies that $der(\mathbf{L}(\mathbb{R}^2)) \cong \mathfrak{gl}_2(\mathbb{R})$.
Now, Theorem 3.2 yields that

$$\mathbf{L}\big((\mathbb{R}^+ \ltimes_\tau \mathbb{R}) \ltimes_\lambda \mathbb{R}^2\big) \cong (\mathbb{R} \oplus_\alpha \mathbb{R}) \oplus_\beta \mathbb{R}^2,$$

where the Lie algebra in the righthand side is endowed with the following bracket

$$\begin{aligned}
[((x,y),z).((x',y'),z')] &= ([(x,y).(x',y')], \beta(x,y)z' - \beta(x',y')z) \\
&= ((0, \frac{1}{2}(xy' - x'y)), \begin{pmatrix} x & y \\ 0 & \frac{x}{2} \end{pmatrix} z' - \begin{pmatrix} x' & y' \\ 0 & \frac{x'}{2} \end{pmatrix} z).
\end{aligned}$$

□

We define *algebraic Shearlet group* as $(\mathbb{R} \ltimes_\tau \mathbb{R}) \ltimes_\lambda \mathbb{R}^2$, where τ and λ are given by $\tau_a(s) = e^{\frac{a}{2}}s$ and $\lambda_{(a,s)}(t_1,t_2) = S_s A_a(t_1,t_2)$, and $S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ and $A_a = \begin{pmatrix} e^a & 0 \\ 0 & e^{\frac{a}{2}} \end{pmatrix}$. This is actually a locally compact group with the left Haar measure as $d\mu(a,s,t) = e^{-2a}$.

Similar to the previous theorem, it can be easily proved that the Lie algebra of the algebraic Shearlet group is $(\mathbb{R} \oplus_\alpha \mathbb{R}) \oplus_\beta \mathbb{R}^2$, where α and β described as the above.

We can deal with $(\mathbb{R} \oplus_\alpha \mathbb{R}) \oplus_\beta \mathbb{R}^2$ as a matrix space. More specifically,

$$(\mathbb{R} \oplus_\alpha \mathbb{R}) \oplus_\beta \mathbb{R}^2 = \{\begin{pmatrix} x & y & z_1 \\ 0 & \frac{x}{2} & z_2 \\ 0 & 0 & 0 \end{pmatrix} : x, y, z_1, z_2 \in \mathbb{R}\},$$

along with the standard matrix bracket, i.e., $[X.Y] = XY - YX$.
As it is shown in the first few lines of [4], one can consider Shearlet group as a linear group which a typical element of it is in the form of

$$\begin{pmatrix} a & s\sqrt{a} & t_1 \\ 0 & \sqrt{a} & t_2 \\ 0 & 0 & 1 \end{pmatrix},$$

where $a \in \mathbb{R}^+, s \in \mathbb{R}$ and $(t_1,t_2) \in \mathbb{R}^2$. Comparing matrix form of Shearlet group and its Lie algebra, we see that the latter is simpler than the former. In fact, this is a reason why we search for Lie algebra of Shearlet group.

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