Int. J. Nonlinear Anal. Appl. 15 (2024) 7, 93–100
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.26754.3407

Collocation method for solving system of non-linear Abel integral equations

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(Communicated by Asadollah Aghajani)

Abstract

In this paper, a special system of non-linear Abel integral equations (SNAIEs) is studied, which arises in astrophysics. Here, the well-known collocation method is extended to obtain an approximate solution of the SNAIEs. The existence and uniqueness conditions of the solution are investigated. Finally, some examples are solved to illustrate the accuracy and efficiency of the proposed method.

Keywords: Abel integral equations, System, Collocation Method, Existence and uniqueness
2020 MSC: 65R20, 97N40

1 Introduction

Integral equations arise in many applied sciences such as physics, astrophysics and engineering [19], [20]. One of the most important integral equations is Abel integral equation. This equation, which has a weakly singular kernel, has many important applications in physics and mechanics. Many problems in heat transfer, non-linear diffusion, the propagation of non-linear waves, astrophysics, solid, plasma physics, scattering theory and elasticity theory can be formulated as Abel integral equation [6], [7], [8], [10], [13]. The existence and uniqueness of the solution for different kinds of Abel integral equations were investigated in [4] and [8]. Also, differential properties of the solution of these equations were examined in [17], [22]. Up to now, many different methods have been used to obtain approximate solution of the Abel integral equation. For example, two numerical schemes such as linear scheme and quadratic scheme were proposed in [11]. Also, Barycentric rational interpolation method in [2], operational method based on Jacobi polynomials in [16] and Euler-Maclaurin formula in [21] were investigated.

Mandal et al. in [12] used the fractional operators and their application to obtain a closed form solution of the system of generalized Abel integral equations. A numerical method based on the Laguerre polynomials was proposed in [18]. Sing et al. suggested an approach for non-linear system of generalized integral equations by using Legendre scaling functions with convergence analysis in [19]. In [14], a numerical method based on the Bernstein polynomials wavelet was proposed to solve the system of linear Abel integral equations.

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Received: January 2022 *Accepted:* April 2022

As mentioned above, in this paper, we study the system of generalized Abel integral equations as following

$$\begin{cases} a_{11}(x)\int_0^x \frac{F_1(u_1(t),u_2(t))}{(x-t)^b}dt + a_{12}(x)\int_x^1 \frac{F_2(u_1(t),u_2(t))}{(t-x)^b}dt = f_1(x) \\ a_{21}(x)\int_0^x \frac{F_3(u_1(t),u_2(t))}{(t-x)^b}dt + a_{22}(x)\int_x^1 \frac{F_4(u_1(t),u_2(t))}{(x-t)^b}dt = f_2(x) \end{cases} \quad x\in[0,1], \tag{1.1}$$

where $F_i,\ i=1,...,4$ are known continuous functions. Also $a_{i,j}\ i=j=1,2$ are known continuous functions such that the determinant of coefficients of the above system, is not vanishing. The rest of this paper is organized as follows. In section 2, an approach for solving the system (1.1) by the collocation method is constructed. In section 3, the existence and uniqueness conditions of the solution are investigated. In Section 4, some numerical examples are given to illustrate the accuracy of the method. Results obtained by the proposed method and results obtained by the Legendre method in [19] are also compared. Finally, in Section 5, the conclusion of this paper is given.

2 Description of the method

In this section, we describe the method of this paper based on the collocation method. To this end, consider a uniform mesh

$$I_h := \{x_i :\ 0\le x_0 < ... < x_N \le 1\}$$

on the interval $[0,1]$ with $x_i := ih,\ i=0,...,N$ and $h:=\frac{1}{N}$ where $N\in\mathbb{N}$. We want to approximate the solution of the system (1.1) in the piecewise polynomial space

$$S_{m-1}^{(-1)}(I_h) := \{v : v|_{\sigma_i}\in\Pi_{m-1},\ i=0,...,N-1\},$$

where $\sigma_0 := [x_0,x_1],\ \sigma_i := (x_i,x_{i+1}]$ and Π_{m-1} denotes the space of polynomials of degree not exceeding $m-1$. Also, consider the set of collocation points as

$$X_h := \{x_i + c_a h :\ 0\le c_1 < ... < c_m \le 1,\ i=0,...,N-1\},$$

in which $c_1,...,c_m$ are collocation parameters and assume that u_h and $\overline{u}_h$ are collocation approximations of solutions u_1 and u_2 of the system (1.1), respectively. By using the Lagrange polynomial interpolation, we have [3]

$$u_h(x_i+\nu h) := \sum_{\alpha=1}^{m} L_\alpha(\nu)U_{i\alpha},\ \nu\in(0,1], \tag{2.1}$$

$$\overline{u}_h(x_i+\nu h) := \sum_{\alpha=1}^{m} L_\alpha(\nu)\overline{U}_{i\alpha},\ \nu\in(0,1], \tag{2.2}$$

where

$$L_\alpha(\nu) = \prod_{\substack{k=1\\k\ne\alpha}}^{m}\frac{\nu-c_k}{c_\alpha-c_k},\ \alpha=1,...,m,$$

and $U_{i\alpha}$ and $\overline{U}_{i\alpha}$ are unknown coefficients. The collocation solution for (1.1) is defined by the collocation equations [3]

$$\begin{cases} a_{11}(x)\int_0^x \frac{F_1(u_1(t),u_2(t))}{(x-t)^b}dt + a_{12}(x)\int_x^1 \frac{F_2(u_1(t),u_2(t))}{(t-x)^b}dt = f_1(x), \\ a_{21}(x)\int_0^x \frac{F_3(u_1(t),u_2(t))}{(t-x)^b}dt + a_{22}(x)\int_x^1 \frac{F_4(u_1(t),u_2(t))}{(x-t)^b}dt = f_2(x), \end{cases} \quad x\in X_h, \tag{2.3}$$

and setting $x = x_{ia} := x_i + c_a h$ in (2.3) implies

$$\begin{cases} a_{11}(x_{ia})\int_0^{x_{ia}} \frac{F_1(u_h(t),\overline{u}_h(t))}{(x_{ia}-t)^b}dt + a_{12}(x_{ia})\int_{x_{ia}}^1 \frac{F_2(u_h(t),\overline{u}_h(t))}{(t-x_{ia})^b}dt = f_1(x_{ia}), \\ a_{21}(x_{ia})\int_0^{x_{ia}} \frac{F_3(u_h(t),\overline{u}_h(t))}{(t-x_{ia})^b}dt + a_{22}(x_{ia})\int_{x_{ia}}^1 \frac{F_4(u_h(t),\overline{u}_h(t))}{(x_{ia}-t)^b}dt = f_2(x_{ia}), \end{cases} \tag{2.4}$$

for $i = 0, ..., N-1$ and $a = 1, ..., m$. The integrals in (2.4) can be written as

$$\begin{cases} a_{11}(x_{ia}) \sum_{l=0}^{i-1} \int_{x_l}^{x_{l+1}} \frac{F_1(u_h(t),\overline{u}_h(t))}{(x_{ia}-t)^b} dt + a_{11}(x_{ia}) \int_{x_i}^{x_{ia}} \frac{F_1(u_h(t),\overline{u}_h(t))}{(x_{ia}-t)^b} dt \\ +a_{12}(x_{ia}) \int_{x_{ia}}^{x_{i+1}} \frac{F_2(u_h(t),\overline{u}_h(t))}{(t-x_{ia})^b} dt + a_{12}(x_{ia}) \sum_{p=i+1}^{N-1} \int_{x_p}^{x_{p+1}} \frac{F_2(u_h(t),\overline{u}_h(t))}{(t-x_{ia})^b} dt \\ \qquad = f_1(x_{ia}), \\ a_{21}(x_{ia}) \sum_{l=0}^{i-1} \int_{x_l}^{x_{l+1}} \frac{F_3(u_h(t),\overline{u}_h(t))}{(t-x_{ia})^b} dt + a_{21}(x_{ia}) \int_{x_i}^{x_{ia}} \frac{F_3(u_h(t),\overline{u}_h(t))}{(t-x_{ia})^b} dt \\ +a_{22}(x_{ia}) \int_{x_{ia}}^{x_{i+1}} \frac{F_4(u_h(t),\overline{u}_h(t))}{(x_{ia}-t)^b} dt + a_{22}(x_{ia}) \sum_{p=i+1}^{N-1} \int_{x_p}^{x_{p+1}} \frac{F_4(u_h(t),\overline{u}_h(t))}{(x_{ia}-t)^b} dt \\ \qquad = f_2(x_{ia}), \end{cases} \tag{2.5}$$

by changing of variables in (2.5) and using (2.1) and (2.2), we have

$$\begin{cases} ha_{11}(x_{ia}) \sum_{l=0}^{i-1} \int_0^1 \frac{F_1(\sum_{\alpha=1}^m L_\alpha(t)U_{l\alpha},\sum_{\alpha=1}^m L_\alpha(t)\overline{U}_{l\alpha})}{(x_{ia}-x_l-th)^b} dt + h^{1-b}a_{11}(x_{ia}) \int_0^{c_a} \frac{F_1(\sum_{\alpha=1}^m L_\alpha(t)U_{i\alpha},\sum_{\alpha=1}^m L_\alpha(t)\overline{U}_{i\alpha})}{(c_a-t)^b} dt \\ +h^{1-b}a_{12}(x_{ia}) \int_{c_a}^1 \frac{F_2(\sum_{\alpha=1}^m L_\alpha(t)U_{i\alpha},\sum_{\alpha=1}^m L_\alpha(t)\overline{U}_{i\alpha})}{(t-c_a)^b} dt + ha_{12}(x_{ia}) \sum_{p=i+1}^{N-1} \int_0^1 \frac{F_2(\sum_{\alpha=1}^m L_\alpha(t)U_{p\alpha},\sum_{\alpha=1}^m L_\alpha(t)\overline{U}_{p\alpha})}{(x_p+th-x_{ia})^b} dt \\ \qquad = f_1(x_{ia}), \\ ha_{21}(x_{ia}) \sum_{l=0}^{i-1} \int_0^1 \frac{F_3(\sum_{\alpha=1}^m L_\alpha(t)U_{l\alpha},\sum_{\alpha=1}^m L_\alpha(t)\overline{U}_{l\alpha})}{(x_l+th-x_{ia})^b} dt + h^{1-b}a_{21}(x_{ia}) \int_0^{c_a} \frac{F_3(\sum_{\alpha=1}^m L_\alpha(t)U_{i\alpha},\sum_{\alpha=1}^m L_\alpha(t)\overline{U}_{i\alpha})}{(t-c_a)^b} dt \\ +h^{1-b}a_{22}(x_{ia}) \int_{c_a}^1 \frac{F_4(\sum_{\alpha=1}^m L_\alpha(t)U_{i\alpha},\sum_{\alpha=1}^m L_\alpha(t)\overline{U}_{i\alpha})}{(c_a-t)^b} dt + ha_{22}(x_{ia}) \sum_{p=i+1}^{N-1} \int_0^1 \frac{F_4(\sum_{\alpha=1}^m L_\alpha(t)U_{p\alpha},\sum_{\alpha=1}^m L_\alpha(t)\overline{U}_{p\alpha})}{(x_{ia}-x_p-th)^b} dt \\ \qquad = f_2(x_{ia}), \end{cases} \tag{2.6}$$

for $i = 0, ..., N-1,\ a = 1, ..., m$, which is a system of $2Nm$ non-linear equations with $2Nm$ unknowns $U_{i\alpha}$ and $\overline{U}_{i\alpha}$. By solving the system (2.6) and substituting values $U_{i\alpha}$ and $\overline{U}_{i\alpha}$ in (2.1) and (2.2), the approximations u_h and $\overline{u}_h$ for $x \in \sigma_i,\ i = 0, ..., N-1$ can be obtained.

3 Existence and Uniqueness of the Solution

In this section, the existence and uniqueness of the solution of the system (1.1) with constant coefficients are investigated, which has recently been studied numerically in literature such as [19]

$$\begin{cases} \alpha \int_0^x \frac{F_1(u_1(t),u_2(t))}{(x-t)^b} dt + \beta \int_x^1 \frac{F_2(u_1(t),u_2(t))}{(t-x)^b} dt = f_1(x) \\ \gamma \int_0^x \frac{F_1(u_1(t),u_2(t))}{(t-x)^b} dt + \delta \int_x^1 \frac{F_2(u_1(t),u_2(t))}{(x-t)^b} dt = f_2(x) \end{cases} \tag{3.1}$$

where

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0,$$

and

$$F_1 \neq F_2.$$

First consider the Abel integral equation

$$\frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} u(t) dt = g(x), \quad x \in [0,1],\ 0 < \nu < 1, \tag{3.2}$$

and adjoint Abel integral equation

$$\frac{1}{\Gamma(\nu)} \int_x^1 (t-x)^{\nu-1} u(t) dt = g(x), \quad x \in [0,1],\ 0 < \nu < 1. \tag{3.3}$$

In [4], it is investigated that for $g \in C^1[0,1]$, equation (3.2) has the solution $u \in C(0,1]$ as

$$u(x) = \frac{1}{\Gamma(1-\nu)} \Big(g(0) x^{-\nu} + \int_0^x (x-t)^{-\nu} g'(t) dt \Big), \quad 0 < t \leq 1,$$

and if $g(0)=0$ then $u\in C[0,1]$. Also for $g\in C^1[0,1]$, the equation (3.3) possess a unique solution $u\in C[0,1)$ and the solution has the form as [4]

$$u(x)=\frac{1}{\Gamma(1-\nu)}\Big(g(1)(1-x)^{-\nu}-\int_x^1(t-x)^{-\nu}g^{'}(t)dt\Big),\quad 0\leq t<1$$

if $g(1)=0$ then $u\in C[0,1]$. Now, in the following theorem, we present the conditions under which the system (3.1) has a unique solution.

Theorem 3.1. Let in the system (3.1)

(a) $-\gamma f_1(x)+\frac{\alpha}{(-1)^b}f_2(x)=(1-x)^{1-b}g(x),\ x\in[0,1),\ g\in C^m[0,1],\ m\geq 1,$
(b) $\delta f_1(x)-\frac{\beta}{\overline{(-1)^b}}f_2(x)=x^{1-b}h(x),\ x\in(0,1],\ h\in C^n[0,1],\ n\geq 1,$
(c) $F:\mathbf{R}^2\to\mathbf{R}^2$ *is continuous and satisfies*

$$((F(u)-F(y)),(u-y))>0,\quad u,y\in\mathbf{R}^2,\ u\neq y,$$

(d)

$$\lim_{\|u\|_2\to\infty}\frac{(F(u),u)}{\|u\|_2}\to\infty$$

where

$$F:=(F_1(u),F_2(u))^T,\ u:=(u_1,u_2)^T,$$

$(.,.)$ and $\|u\|_2$ denote the inner product and Euclidian norm in R^2, respectively. Then the system (3.1) has a unique solution $u\in C[0,1]$.

Proof . By setting $z_1(t):=F_1(u_1(t),u_2(t))$ and $z_2(t):=F_2(u_1(t),u_2(t))$, the equation (3.1) can be considered as

$$\begin{cases}\alpha\int_0^x\frac{z_1(t)}{(x-t)^b}dt+\beta\int_x^1\frac{z_2(t)}{(t-x)^b}dt=f_1(x)\\ \\ \gamma\int_0^x\frac{z_1(t)}{(t-x)^b}dt+\delta\int_x^1\frac{z_2(t)}{(x-t)^b}dt=f_2(x).\end{cases}\tag{3.4}$$

Without loss of generality, we assume $\alpha\neq 0$ (since for $\alpha=0$ the first equation of (3.4) turns to Abel integral equation and the system can be solved for z_1 and z_2). So the first equation of (3.4) can be written as

$$\int_0^x(x-t)^{-b}z_1(t)dt=-\frac{\beta}{\alpha}\int_x^1(t-x)^{-b}z_2(t)dt+\frac{f_1(x)}{\alpha}.\tag{3.5}$$

On the other hand

$$\frac{1}{(t-x)^b}=\frac{1}{(-(x-t))^b}=\frac{1}{(i^2(x-t))^b}=\frac{\overline{(-1)^b}}{(x-t)^b},\quad t<x,$$

where i is imaginary unit and $\overline{(-1)^b}$ denotes complex conjugate of $(-1)^b$. So the second equation of (3.4) can be written as

$$\gamma\overline{(-1)^b}\int_0^x(x-t)^{-b}z_1(t)dt+\delta\overline{(-1)^b}\int_x^1(t-x)^{-b}z_2(t)dt=f_2(x).\tag{3.6}$$

Now, substituting from (3.5) into (3.6), yields

$$\overline{(-1)^b}\Big(\delta-\frac{\gamma\beta}{\alpha}\Big)\int_x^1(t-x)^{-b}z_2(t)dt=f_2(x)-\frac{\gamma\overline{(-1)^b}}{\alpha}f_1(x),\tag{3.7}$$

and since $\Big(\delta-\frac{\gamma\beta}{\alpha}\Big)\neq 0$, (3.7) can be simplifed as

$$\int_x^1(t-x)^{-b}z_2(t)dt=\frac{\alpha}{(\delta\alpha-\gamma\beta)\overline{(-1)^b}}f_2(x)-\frac{\gamma}{(\delta\alpha-\gamma\beta)}f_1(x).\tag{3.8}$$

As stated in [1], multiply the equation (3.8) by $(t-x)^{b-1}$ and integrate from x to 1, implies

$$\int_x^1 z_2(t)dt = \frac{1}{(\delta\alpha-\gamma\beta)\Gamma(b)\Gamma(1-b)}\int_x^1 (t-x)^{b-1}(1-t)^{1-b}g(t)dt, \tag{3.9}$$

differentiating of (3.9) with respect to x, reveal that (3.8) has a unique solution $z_2 \in C^m[0,1]$ and it can be written as

$$z_2(x) = \Big(a + (1-x)\widehat{g}(x)\Big),\ \widehat{g} \in C^{m-1}[0,1],$$

where

$$a := \frac{(1-b)}{(\delta\alpha-\gamma\beta)}g(1),$$

$$\widehat{g}(x) := \frac{-1}{(\delta\alpha-\gamma\beta)\Gamma(b)\Gamma(1-b)}\int_0^1 \omega^{b-1}(1-\omega)^{2-b}\Big(g^{'}(x+(1-x)\omega) + \frac{g(1)-g(x+(1-x)\omega)}{(1-x)(1-\omega)}\Big)d\omega.$$

Finally, by substituting from (3.8) into the first equation of the system (3.4), we get

$$\int_0^x (x-t)^{-b}z_1(t)dt = \Big(\frac{\delta}{(\delta\alpha-\gamma\beta)}\Big)f_1(x) - \Big(\frac{\beta}{(\delta\alpha-\gamma\beta)\overline{(-1)^b}}\Big)f_2(x), \tag{3.10}$$

by using the hypotheses and lemma in [1], it can be concluded that the unique solution of (3.10) has the form as

$$z_1(x) = \Big(c + x\widehat{h}(x)\Big),\quad \widehat{h} \in C^{n-1}[0,1].$$

where

$$c := \frac{(1-b)}{(\delta\alpha-\gamma\beta)}h(0),$$

$$\widehat{h}(x) := \frac{1}{(\delta\alpha-\gamma\beta)\Gamma(b)\Gamma(1-b)}\int_0^1 (1-\omega)^{b-1}\omega^{2-b}\Big(h^{'}(x\omega) + \frac{h(x\omega)-h(0)}{x\omega}\Big)d\omega,$$

in the end, the assumptions (c) and (d) guarantee that the system $F(u) = z$ with $z := (z_1, z_2)^T$ has a unique solution $u \in C[0,1]$ (see theorem 6.1.16 in [3]). Moreover, the value of u at $x = 0$ and $x = 1$ is given implicitly by

$$\begin{cases} F_1(u_1(0),u_2(0)) = \frac{(1-b)}{(\delta\alpha-\gamma\beta)}\lim_{x\to 0} h(x) \\ F_2(u_1(0),u_2(0)) = \lim_{x\to 0}\Big(a + (1-x)\widehat{g}(x)\Big) \end{cases}$$

and

$$\begin{cases} F_1(u_1(1),u_2(1)) = \lim_{x\to 1}\Big(c + x\widehat{h}(x)\Big) \\ F_2(u_1(1),u_2(1)) = \frac{(1-b)}{(\delta\alpha-\gamma\beta)}\lim_{x\to 1} g(x) \end{cases}$$

So the proof is completed. □

4 Numerical Results

In this section, we test the proposed method by solving two problems and compare obtained results with the results of other methods. Numerical results show the efficiency and superiority of the proposed method. All computations have been done by programming in Maple 2016, with Digits=20. Here, we take the uniform mesh $x_i := \frac{i}{N},\ i = 0, ..., N$ and approximate solution in the space $S^{(-1)}_{m-1}(I_h)$. We also consider the collocation parameters as (**a**) for the space $S^{(-1)}_1(I_h)$

$$c_1 = \frac{1}{3},\quad c_2 = \frac{2}{3},$$

(**b**) for the space $S^{(-1)}_2(I_h)$

$$c_1 = \frac{3-\sqrt{3}}{6},\quad c_2 = \frac{1}{2},\quad c_3 = \frac{3+\sqrt{3}}{6},$$

(**c**) for the space $S_3^{(-1)}(I_h)$

$$c_1 = 0 \quad c_2 = \frac{1}{3}, \quad c_3 = \frac{2}{3}, \quad c_4 = 1.$$

The maximum absolute error is obtained as

$$||e_h||_\infty := \max_{0\leq i\leq N} |u_1(x_i) - u_h(x_i)|,$$
$$||\overline{e}_h||_\infty := \max_{0\leq i\leq N} |u_2(x_i) - \overline{u}_h(x_i)|,$$

Example 4.1. Consider the following system of generalized Abel integral equations [19]:

$$\begin{cases} \int_0^x \frac{u_1^2(t)dt}{(x-t)^{\frac{1}{3}}} + 3\int_x^1 \frac{u_2^3(t)dt}{(t-x)^{\frac{1}{3}}} = f_1(x), \\ 2\int_0^x \frac{u_1^2(t)dt}{(t-x)^{\frac{1}{3}}} + \int_x^1 \frac{u_2^3(t)dt}{(x-t)^{\frac{1}{3}}} = f_2(x), \end{cases} \tag{4.1}$$

with

$$f_1(x) = \frac{729}{1540}x^{\frac{14}{3}} + \frac{9}{440}(1-x)^{\frac{2}{3}}(40 + 9x(5 + 6x + 9x^2)),$$

$$f_2(x) = -\frac{729}{770}(-x)^{\frac{14}{3}} - \frac{3}{440}(-1+x)^{\frac{2}{3}}(40 + 9x(5 + 6x + 9x^2)).$$

The exact solution of this system is $u_1(x) = x^2$, $u_2(x) = x$. The solution of (4.1) is approximated in the space $S_1^{(-1)}(I_h)$. The maximum absolute errors for different values of h are reported in table 1 and compared with results obtained by the Legendre method of [19].

Table 1: Numerical results for Example 4.1.

	N	$\|\|e_h\|\|_\infty$	$\|\|\overline{e}_h\|\|_\infty$
	2	9.6066e-2	6.2954e-5
The proposed method	4	2.7982e-2	2.6206e-9
	8	7.4606e-3	1.8861e-9
The Legendre method		1.2575e-3	7.1092e-3

Also, the collocation approximations of solutions of the equation (4.1) in the collocation space $S_2^{(-1)}(I_h)$ are

$$u_1(x) = x^2 - 3.5830 \times 10^{-10}x - 5.8497 \times 10^{-12}$$
$$u_2(x) = x - 1.8219 \times 10^{-10}$$

which are approximately the exact solutions.

Example 4.2. As second example, consider the following system of generalized Abel integral equations:

$$\begin{cases} \frac{1}{4}\int_0^x \frac{u_1(t)u_2(t)dt}{(x-t)^{\frac{1}{7}}} + \frac{1}{3}\int_x^1 \frac{u_2(t)dt}{(t-x)^{\frac{1}{7}}} = f_1(x), \\ \frac{1}{7}\int_0^x \frac{u_1(t)u_2(t)dt}{(t-x)^{\frac{1}{7}}} + \frac{1}{2}\int_x^1 \frac{u_2(t)dt}{(x-t)^{\frac{1}{7}}} = f_2(x), \end{cases} \tag{4.2}$$

with

$$f_1(x) = \frac{823543}{15657408}x^{\frac{48}{7}} + \frac{1}{234861120}(1-x)^{\frac{6}{7}} \times (26775952x^3 + 22950816x^2 + 21311472x + 20296640),$$
$$f_2(x) = -\frac{117646}{3914352}(-x)^{\frac{48}{7}} - \frac{1}{19571760}(x-1)^{\frac{6}{7}} \times (3346994x^3 + 2868852x^2 + 2663934x + 2537080).$$

Table 2: Numerical results for Example 4.2 for $u_1(x), u_2(x)$.

	$\|\|e_h\|\|_\infty$		$\|\|\bar{e}_h\|\|_\infty$	
N	$m=2$	$m=3$	$m=2$	$m=3$
2	5.9964e-1	2.7865e-2	1.7832e-1	9.7969e-3
4	1.4285e-1	3.2885e-3	5.1856e-2	1.5737e-3
8	6.1796e-2	4.0207e-4	1.0667e-2	2.1803e-4

and the exact solution $u_1(x) = u_2(x) = x^3$. we approximate the solution of (4.2) in spaces $S_1^{(-1)}(I_h)$ and $S_2^{(-1)}(I_h)$. The maximum absolute errors are reported in table 2. The results show that the error improves when $h \to 0$ or m increases.

Also approximate solution of (4.2) in $S_3^{(-1)}$ is

$$u_1(x) = u_2(x) = 1.000000000000003500x^3 - 4.5 \times 10^{-15}x^2 + 1.0 \times 10^{-15}x.$$

which are again approximately the exact solutions.

5 Conclusion

In this paper, a numerical scheme was constructed to solve the non-linear system of Abel integral equations by collocation method. The solution was approximated in piecewise polynomial space $S_{m-1}^{(-1)}$. The existence and uniqueness of the solution were discussed. Two examples were solved by the proposed method. Numerical results show that approximation by the proposed method is more accurate than the results of the method of [19]. It seems that the following items can be done as future works.

1. The proposed method of this paper can be extended to solve the non-linear system of Volterra integral equations with weakly singular kernel.

2. The spline collocation method [5], Chebyshev collocation and LDE methods [9] and Haar wavelet collocation method [15] can be applied to solve the equations of the form (1.1).

3. The system (1.1) can be studied on Sobolev spaces.

4. The singularity of solution can be overcome by appropriate changing of variables. This increases the accuracy of the collocation method for approximating the solution.

Acknowledgements

The authors would like to thank the anonymous referees and editors whose comments improved the paper significantly.

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