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Fixed point theorem for F-contraction mappings in partial symmetric space with some applications to chemical reactor integral equations

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Abstract

This paper proves a fixed point theorem for F-contraction mappings in partial symmetric spaces. In doing so, we extended and generalized the results in the literature by employing a rational-type contraction condition. We also provided an illustrative example to support the results. Finally, we demonstrate the results by the applications to Volterra integral equation inclusion and chemical reactor integral equations.

Keywords: Fixed point, partial symmetric space, F-contraction mapping, integral equation
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1 Introduction

In 1994, Matthew [22] introduced non-zero self-distance, which is extensively applied in computer networking, data structure, and Computer programming languages. By using the concept of non-self distance, Matthew [22] generalizes the axioms of metric to partial metric and also explained the metric and topological properties for the new space. Some of these properties are complete spaces, the Cauchy sequence and the contraction fixed point theorem, generalizing the Banach contraction principle.

Wardowski [33] introduced a new contraction called F-contraction in metric spaces and proved fixed point results as a generalization of the Banach contraction principle. Wardowski and Van Dung [34] established weak F-contraction in metric space and proved fixed point results as an extension of the Banach contraction principle. Also, Cosentino et al. [11] improved the results due to Wardowski [33] by using the concept of b-metric space and proved some fixed point results. Aydi et al. [7] modified F-contractions via α-admissible mappings and application to integral equations. Nazam et al. [26] proved the results for some F-contraction mappings in a dualistic partial metric space which provide sufficient related conditions for the existence of a fixed point. Wangwe and Kumar [30, 31, 32] used the concept of F-contraction to prove the fixed point and common fixed point theorem in ordered partial metric spaces, weak partial metric spaces and generalized metric spaces.

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Recently, Acar [2] proved a fixed point theorem for multivalued almost F-δ-contraction. Kadelburg and Radenović [19] gave notes on some recent papers concerning F-contractions in b-metric spaces. Nazam and Acar [25] proved the results on common fixed points theorems for ordered F-contractions with application. Acar [1] proved a fixed point theorems for rational type F-contraction in complete metric space. Vetro [28] proved a fixed-point problem with mixed-type on F-H contractive condition.

On the other hand, Asim et al. [5] initiated the study of partial symmetric space by combining the concept of partial metric space due to Matthew [22] and symmetric space concept due to Wilson [35]. Using this space, they proved some related fixed point results for single-valued and multi-valued mappings. Since then, several researchers have been motivated to do their research in this direction. In 2021, Asim and Imdad [4] proved a common fixed point result in partial symmetric space. Furthermore, Asim et al. [6] proved a multi-valued result using Suzuki and Wardowski-type contraction mapping in partial symmetric space. Wangwe and Kumar [29] proved the fixed point theorem for multi-valued non-self mappings in partial symmetric spaces.

In this paper, we attempted to prove the fixed point theorem for F-contraction mappings in partial symmetric spaces by combining the concept of partial symmetric space [5, 22, 35], and F-contraction notions [33]. We generalize the theorem due to Dass and Gupta [12], and Jaggi [18]. Also, we give an example and demonstrate the results with an application to Volterra integral equation inclusion. With this, we have denoted $(\mathcal{X}, p_s)$ as a partial symmetric space.

2 Preliminaries

The following preliminaries and results will be helpful to develop the new theorem for this paper. Wilson [35] introduced fixed point results in symmetric spaces (also called E-space, in the terminology of Fréchet) which did not require triangular inequality, which is as follows:

Definition 2.1. [35] A symmetric space is a pair $(\mathcal{X}, d)$ consisting of a non-empty set $\mathcal{X}$ together with a function $d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ called the symmetric metric, if and only if it satisfies:

(W1) $d(\xi, \eta) = 0$ if and only if $\xi = \eta$;
(W2) $d(\xi, \eta) = d(\eta, \xi)$, for all $\xi = \eta$.

Then the pair $(\mathcal{X}, d)$ is called a symmetric space.

Matthews [22] replaced self distance by non-zero value to use it in computer semantics which is as follows:

Definition 2.2. [22] A partial metric space is a pair $(\mathcal{X}, p)$ consisting of a non-empty set $\mathcal{X}$ together with a function $p : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$, called the partial metric, such that for all $\xi, \eta, \theta \in \mathcal{X}$ we have the following properties:

(P1) $\xi = \eta$ if and only if $p(\xi, \xi) = p(\xi, \eta) = p(\eta, \eta)$;
(P2) $p(\xi, \xi) \leq p(\xi, \eta)$;
(P3) $p(\xi, \eta) \leq p(\eta, \xi)$; and
(P4) $p(\xi, \eta) \leq p(\xi, \theta) + p(\theta, \eta) - p(\theta, \theta)$.

Then the pair $(\mathcal{X}, p)$ is called a partial metric space.

In partial metric space, it is not necessary that $p(\xi, \xi) = 0$, for every $\xi = \eta$, while in metric if $\xi = \eta$, then $p(\xi, \xi) = 0$. Asim et al. [5] by combining the concept of symmetric space [35] and partial metric space [22] introduced the partial symmetric space notions and gave some of its properties such as convergence, Cauchy sequence and completeness.

Definition 2.3. [5] Let $\mathcal{X}$ be a non-empty set. A mapping $p_s : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ is said to be partial symmetric if for all $\xi, \eta, \theta \in \mathcal{X}$, we have the following properties:

(PS1) $\xi = \eta$ if and only if $p_s(\xi, \eta) = p_s(\xi, \xi) = p_s(\eta, \eta)$,
(PS2) $p_s(\xi, \xi) \leq p_s(\xi, \eta)$,
(PS3) $p_s(\xi, \eta) = p_s(\eta, \xi)$.

Then the pair $(\mathcal{X}, p_s)$ is said to be partial symmetric space.

From $(PS1)$ and $(PS2)$ we have

$$p_s(\xi,\eta)=0 \Rightarrow p_s(\xi,\xi)=p_s(\eta,\eta) \Rightarrow \xi=\eta. \tag{2.1}$$

A partial symmetric space $(\mathcal{X},p_s)$ reduces to a symmetric space if $p_s(\xi,\xi)=0$, for all $\xi\in\mathcal{X}$. Every symmetric space is partial symmetric space, but not conversely.

Let $(\mathcal{X},p_s)$ be a partial symmetric space. Then, the p_s-open ball, with center $\xi\in\mathcal{X}$ and radius $\varepsilon>0$, is defined by: $B_{p_s}(\xi,\varepsilon)=\{\eta\in\mathcal{X}: p_s(\xi,\eta)<p_s(\xi,\xi)+\varepsilon\}$.

Similarly, the p_s-closed ball, with center $\xi\in\mathcal{X}$ and radius $\varepsilon>0$, is defined by $B_{p_s}[\xi,\varepsilon]=\{\eta\in X: p_s(\xi,\eta)\leq (\xi,\xi)+\varepsilon\}$.

The family of p_s-open balls for all $\xi\in\mathcal{X}$ *and* $\varepsilon>0$, $U_{p_s}=\{B_{p_s}(\xi,\varepsilon):\xi\in\mathcal{X},\varepsilon>0\}$, forms basis of some topology τ_{p_s} on $\mathcal{X}$.

Definition 2.4. [5] Let $(\mathcal{X},p_s)$ be a partial symmetric space. Then,

(i) a sequence $\{\xi_n\}$ in $(\mathcal{X},p_s)$ is said to be p_s-convergent to $\xi\in\mathcal{X}$, with respect to τ_{p_s} if $p_s(\xi,\xi)=\lim\limits_{n\to\infty}p_s(\xi_n,\xi)$.

(ii) a sequence $\{\xi_n\}$ in $(\mathcal{X},p_s)$ is called a p_s-Cauchy sequence if only if $\lim\limits_{n,m\to\infty}p_s(\xi_n,\xi_m)$ exists and is finite.

(iii) a partial symmetric space $(\mathcal{X},p_s)$ is said to be p_s-complete if every p_s- Cauchy sequence $\{\xi_n\}$ in $\mathcal{X}$ is p_s convergent, with respect to τ_{p_s} to a point $\xi\in\mathcal{X}$, such that

$$p_s(\xi,\xi)=\lim_{n\to\infty}p_s(\xi_n,\xi)=\lim_{n,m\to\infty}p_s(\xi_n,\xi_m).$$

Definition 2.5. [5] Let $(\mathcal{X},p_s)$ be a partial symmetric space. Then

(A_1) $\lim\limits_{n\to\infty}p_s(\xi_n,\xi)=p_s(\xi,\xi)$ *and* $\lim\limits_{n\to\infty}p_s(\xi_n,\eta)=p_s(\xi,\eta)$ imply that $\xi=\eta$, for a sequence $\{\xi_n\}$, $\xi,\eta\in\mathcal{X}$.

(A_2) a partial symmetric p_s is said to be 1-continuous if $\lim\limits_{n\to\infty}p_s(\xi_n,\xi)=p_s(\xi,\xi)$ implies that $\lim\limits_{n\to\infty}p_s(\xi_n,\eta)=p_s(\xi,\eta)$, where $\{\xi_n\}$ is a sequence in $\mathcal{X}$ and $\xi,\eta\in\mathcal{X}$.

(A_3) a partial symmetric p_s is said to be continuous if $\lim\limits_{n\to\infty}p_s(\xi_n,\xi)=p_s(\xi,\xi)$ and $\lim\limits_{n\to\infty}p_s(\xi_n,\eta)=p_s(\xi,\eta)$ imply that $\lim\limits_{n\to\infty}p_s(\xi_n,\eta_n)=p_s(\xi,\eta)$ where $\{\xi_n\}$ and $\{\eta_n\}$ are sequences in $\mathcal{X}$ and $\xi,\eta\in\mathcal{X}$.

(A_4) $\lim\limits_{n\to\infty}p_s(\xi_n,\xi)=p_s(\xi,\xi)$ and $\lim\limits_{n\to\infty}p_s(\xi_n,\eta_n)=p_s(\xi,\xi) \implies \lim\limits_{n\to\infty}p_s(\eta_n,\xi)=p_s(\xi,\xi)$, for sequences $(\xi_n),(\eta_n)$, and ξ in $\mathcal{X}$.

(A_5) $\lim\limits_{n\to\infty}p_s(\xi_n,\eta_n)=p_s(\xi,\xi)$ and $\lim\limits_{n\to\infty}p_s(\eta_n,\theta_n)=p_s(\xi,\xi)\implies \lim\limits_{n\to\infty}p_s(\xi_n,\theta_n)=p_s(\xi,\xi)$, for sequences $(\xi_n),(\eta_n),(\theta_n)$, and ξ in $\mathcal{X}$.

We recall the following examples in [5] which satisfy the above axioms of partial symmetric space as follows:

(1) Let $\mathcal{X}=\mathbb{R}$. Define a mapping $p_s:\mathcal{X}\times\mathcal{X}\to\mathbb{R}_+$ for all $\xi,\eta\in\mathcal{X}$ and $p,q>1$, as follows:

$$p_s(\xi,\eta) = |\xi-\eta|^p+|\xi-\eta|^q.$$

Then the pair $(\mathcal{X},p_s)$ is a partial symmetric space.

(2) Let $\mathcal{X}=\mathbb{R}$. Define a mapping $p_s:\mathcal{X}\times\mathcal{X}\to\mathbb{R}_+$ for all $\xi,\eta\in\mathcal{X}$ and $p,q>1$, as below:

$$p_s(\xi,\eta) = (\max\{\xi,\eta\})^p+(\max\{\xi,\eta\})^q.$$

(3) Let $\mathcal{X}=\mathbb{R}$. Define a mapping $p_s:\mathcal{X}\times\mathcal{X}\to[0,\infty)$ for all $\xi,\eta\in\mathcal{X}$, $p,q>1$ and $\alpha\geq 0$, as follows:

$$p_s(\xi,\eta) = |\xi-\eta|^p+|\xi-\eta|^q+\beta.$$

(4) Let $\mathcal{X}=[0,\infty)$. Define a mapping $p_s:\mathcal{X}\times\mathcal{X}\to[0,\infty)$ for all $\xi,\eta\in\mathcal{X}$ and $p,q>1$, as follows:

$$p_s(\xi,\eta) = (\max\{\xi,\eta\})^p+|\xi-\eta|^q.$$

(5) Let $\mathcal{X} = [0, \pi)$ and define a mapping $p_s : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ for all $\xi, \eta \in \mathcal{X}$ and $p, q > 1$, as follows:

$$p_s(\xi, \eta) = (\max\{\xi, \eta\})^p + e^{|\xi-\eta|^q}.$$

(6) Let $\mathcal{X} = [0, 1)$ and define a mapping $p_s : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ for all $\xi, \eta \in \mathcal{X}$, $p, q > 1$ and $\alpha > 0$, as follows:

$$p_s(\xi, \eta) = \sin|\xi - \eta| + \alpha.$$

Wardowski [33] introduced a generalization of the Banach contraction principle in metric spaces as follows.

Definition 2.6. [33] Let $(\mathcal{X}, d)$ be a metric space. A self-mapping $\mathcal{T}$ on $\mathcal{X}$ is called an F-contraction mapping if there exists $F \in \mathcal{F}$ and $\Gamma \in \mathbb{R}^+$ such that for all $\xi, \eta \in \mathcal{X}$,

$$d(\mathcal{T}\xi, \mathcal{T}\eta) > 0 \Rightarrow \Gamma + F(d(\mathcal{T}\xi, \mathcal{T}\eta)) \leq F(d(\xi, \eta)).$$

Wardowski [33] proved the following fixed point theorem:

Theorem 2.7. [33] Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a F-contraction mapping. If there exist $\Gamma > 0$ such that for all $\xi, \eta \in \mathcal{X}$,

$$d(\mathcal{T}\xi, \mathcal{T}\eta) > 0 \implies \Gamma + F(d(\mathcal{T}\xi, \mathcal{T}\eta)) \leq F(d(\xi, \eta)), \tag{2.2}$$

then $\mathcal{T}$ has a unique fixed point.

The following explanations for developing the F-contraction definition were obtained from Wardowski [33], Wardowski and Van Dung [34], and Cosentino et al. [11].

Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying:

(F1) F is strictly increasing, i.e. for all $\xi, \eta \in \mathbb{R}^+$, $\xi < \eta$ implies $F(\xi) < F(\eta)$;
(F2) For each sequence $\{\xi_n\}_{n\in\mathbb{N}}$ of positive numbers, $\lim_{n\to\infty} \xi_n = 0$ if and only if $\lim_{n\to\infty} F(\xi_n) = -\infty$;
(F3) There exists $k \in (0, 1)$ satisfying $\lim_{n\to\infty}(\xi_n)^k F(\xi_n) = 0$, for all $n \in \mathbb{N}$, and some $\Gamma \in \mathbb{R}^+$.

We denote the family of all functions F satisfying conditions $((F1) - (F3))$ by $\mathcal{F}$. Some examples of functions $F \in \mathcal{F}$ are:

(1) $F_1(z) = \ln z \implies \frac{d(\mathcal{T}\xi,\mathcal{T}\eta)}{d(\xi,\eta)} \leq e^{-\Gamma}$;
(2) $F_2(z) = z + \ln z \implies \frac{d(\mathcal{T}\xi,\mathcal{T}\eta)}{d(\xi,\eta)} \leq e^{-\Gamma+d(\xi,\eta)-d(\mathcal{T}\xi,\mathcal{T}\eta)}$;
(3) $F_3(z) = -\frac{1}{\sqrt{z}} \implies \frac{d(\mathcal{T}\xi,\mathcal{T}\eta)}{d(x,y)} \leq \frac{1}{(1+\Gamma\sqrt{d(\xi,\eta)})^2}$;
(4) $F_4(z) = \ln(z^2 + z) \implies \frac{d(\mathcal{T}\xi,\mathcal{T}\eta)(1+d(\mathcal{T}\xi,\mathcal{T}\eta))}{d(\xi,\eta)(1+d(\xi,\eta))} \leq e^{-\Gamma}$.

In 1975, Dass and Gupta [12] proved the following results in complete metric space.

Theorem 2.8. [12] Suppose $(\mathcal{X}, d)$ is a complete metric space. Let $\mathcal{T} : \mathcal{X} \longrightarrow \mathcal{X}$ be a mapping such that there exists $\delta_1, \delta_2 \in [0, 1)$ with $\delta_1 + \delta_2 < 1$ satisfying

$$d(\mathcal{T}\xi, \mathcal{T}\eta) \leq \frac{\delta_1 d(\xi, \mathcal{T}\xi)[1 + d(\eta, \mathcal{T}\eta)]}{1 + d(\xi, \eta)} + \delta_2 d(\xi, \eta), \tag{2.3}$$

for any distinct $\xi, \eta \in \mathcal{X}$. Then $\mathcal{T}$ has a unique fixed point in $\mathcal{X}$.

The above results have been generalized in different directions employing various abstract spaces. For more details, we refer the reader in [8, 9, 10, 14, 15, 18, 21] and the references therein.

3 Main Results

We begin this section by introducing a definition of partial symmetric space.

Definition 3.1. Let $(\mathcal{X}, p_s)$ be a partial symmetric space. A mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is said to be a Dass-Gupta type F-contraction if, for all $\xi, \eta \in \mathcal{X}$,

$$\Gamma + F(p_s(\mathcal{T}\xi, \mathcal{T}\eta)) \leq F\Bigg(\frac{\delta_1 p_s(\xi, \mathcal{T}\xi)[1 + p_s(\eta, \mathcal{T}\eta)]}{1 + p_s(\mathcal{T}\xi, \mathcal{T}\eta)} + \delta_2 p_s(\xi, \eta)\Bigg),$$

where $\delta_1, \delta_2 \in [0, 1)$ with $\delta_1 + \delta_2 < 1$.

We prove our main theorem as given below:

Theorem 3.2. Let $(\mathcal{X}, p_s)$ be a complete partial symmetric space and $\mathcal{T} : \mathcal{X} \longrightarrow \mathcal{X}$. Assume that the following conditions hold:

(i) there is $\xi_0 \in \mathcal{X}$ such that $\xi_0 \in \mathcal{T}\xi_0$,
(ii) T is continuous;
(ii) there exists $F \in \mathcal{F}$ and $\Gamma > 0$ with $p_s(\mathcal{T}\xi, \mathcal{T}\eta) > 0$; such that

$$\Gamma + F(p_s(\mathcal{T}\xi, \mathcal{T}\eta)) \leq F\Bigg(\frac{\delta_1 p_s(\xi, \mathcal{T}\xi)[1 + p_s(\eta, \mathcal{T}\eta)]}{1 + p_s(\mathcal{T}\xi, \mathcal{T}\eta)} + \delta_2 p_s(\xi, \eta)\Bigg), \tag{3.1}$$

where $\delta_1, \delta_2 \in [0, 1)$ with $\delta_1 + \delta_2 < 1$.

Then, $\mathcal{T}$ has a unique fixed point, if there exists $\xi \in \mathcal{X}$ such that $p_s(\xi, \xi) = 0$.

Proof . Suppose that ξ_0 is an arbitrary point in $\mathcal{X}$. By given assumption, there exists ξ_0 and ξ_1 in $\mathcal{X}$ such that $\xi_0 \in \mathcal{T}\xi_0$ and $\xi_1 \in \mathcal{T}\xi_1$. If $\xi_0 = \mathcal{T}\xi_0$ or $\xi_1 \in \mathcal{T}\xi_1$, then, either ξ_0 or ξ_1 is a fixed point of $\mathcal{T}$. Hence, our proof is complete. On contrary, assume that $\xi_0 \notin \mathcal{T}\xi_0$ and $\xi_1 \notin \mathcal{T}\xi_1$, we can construct a sequence $\{\xi_n\}$ in $\mathcal{X}$ such that $p_s(\xi_n, \xi_{n+1}) > 0$ which implies that

$$p_s(\xi_n, \xi_{n+1}) = p_s(\mathcal{T}\xi_{n-1}, \mathcal{T}\xi_n), \forall n \in \mathbb{N}. \tag{3.2}$$

Now, taking $\xi = \xi_{n-1}$ and $\eta = \xi_n$ in (3.1) and using (3.2) we have

$$\Gamma + F(p_s(\mathcal{T}\xi_{n-1}, \mathcal{T}\xi_n)) \leq F\Bigg(\frac{\delta_1 p_s(\xi_{n-1}, \mathcal{T}\xi_{n-1})[1 + p_s(\xi_n, \mathcal{T}\xi_n)]}{1 + p_s(\mathcal{T}\xi_{n-1}, \mathcal{T}\xi_n)} + \delta_2 p_s(\xi_{n-1}, \xi_n)\Bigg).$$

It follows that

$$\Gamma + F(p_s(\xi_n, \xi_{n+1})) \leq F\Bigg(\frac{\delta_1 p_s(\xi_{n-1}, \xi_n)[1 + p_s(\xi_n, \xi_{n+1})]}{1 + p_s(\xi_n, \xi_{n+1})} + \delta_2 p_s(\xi_{n-1}, \xi_n)\Bigg).$$

Consequently,

$$\Gamma + F(p_s(\xi_n, \xi_{n+1})) \leq F\Big(\delta_1 p_s(\xi_{n-1}, \xi_n) + \delta_2 p_s(\xi_{n-1}, \xi_n)\Big),$$

which is equivalent to

$$\Gamma + F(p_s(\xi_n, \xi_{n+1})) \leq F\Big((\delta_1 + \delta_2) p_s(\xi_{n-1}, \xi_n)\Big). \tag{3.3}$$

Since $\delta_1 + \delta_2 < 1$, from (3.3) we obtain

$$\Gamma + F(p_s(\xi_n, \xi_{n+1})) \leq F(p_s(\xi_{n-1}, \xi_n). \tag{3.4}$$

Thus, we have

$$F(p_s(\xi_n, \xi_{n+1})) \leq F(p_s(\xi_{n-1}, \xi_n) - \Gamma. \tag{3.5}$$

Similarly, we obtain

$$F(p_s(\xi_{n+1}, \xi_{n+2})) \leq F(p_s(\xi_{n-1}, \xi_n) - 2\Gamma. \tag{3.6}$$

Continuing with this process through induction, for $n \in \mathbb{N}$, we get

$$F(p_s(\xi_n, \xi_{n+1})) \leq F(p_s(\xi_{n-1}, \xi_n) - n\Gamma. \tag{3.7}$$

By letting $\xi_n = p_s(\xi_{n+1}, \xi_{n+2})$, for every $n \in \mathbb{N}$ and using (3.1), (3.7) with $(F1)$, the following hold for every $n \in \mathbb{N}$:

$$F(\xi_n) \leq F(\xi_{n-1}) - \Gamma \leq F(\xi_{n-1}) - 2\Gamma \cdots \leq F(\xi_0) - n\Gamma. \tag{3.8}$$

By (3.7), $(F2)$ and letting $n \to \infty$ gives $\lim_{n\to\infty} F(\xi_n) = -\infty$. So,

$$\lim_{n\to\infty} \xi_n = 0. \tag{3.9}$$

From condition $(F3)$ there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} (\xi_n)^k F(\xi_n) = 0. \tag{3.10}$$

By (3.8), the following hold for all $n \in \mathbb{N}$:

$$\begin{aligned} (\xi_n)^k F(\xi_n) &\leq (\xi_n)^k (F(\xi_{n-1}) - \Gamma) \leq (\xi_n)^k (F(\xi_{n-1}) - 2\Gamma) \dots \\ &\leq (\xi_n)^k F(\xi_0) - (\xi_n)^k n\Gamma. \end{aligned} \tag{3.11}$$

Using (3.9) in (3.11) we get

$$0 \leq -(\xi_n)^k \Gamma \leq -(\xi_n)^k 2\Gamma \cdots \leq -(\xi_n)^k n\Gamma. \tag{3.12}$$

By taking limit $n \to \infty$ in (3.12), we obtain

$$\lim_{n\to\infty} (\xi_n)^k n\Gamma = 0. \tag{3.13}$$

From (3.13) there exists $N \in \mathbb{N}$ such that $(\xi_n)^k n \leq 1$ for all $n \geq N$. As a result, we have

$$\begin{aligned} n(\xi_n)^k &\leq 1, \\ \xi_n &\leq \frac{1}{n^{\frac{1}{k}}}, \forall n \geq N. \end{aligned} \tag{3.14}$$

Therefore, $\sum_{n=0}^{\infty} \xi_n = \sum_{n=0}^{\infty} \frac{1}{n^{\frac{1}{k}}}$ converges. By (3.14) and (2.4), we prove that $\{\xi_n\}$ is a Cauchy sequence. A partial symmetric space is said to be p_s-complete if every p_s-Cauchy sequence $\{\xi_n\}$ in $\mathcal{X}$ is p_s-convergent.

$$p_s(\xi, \xi) = \lim_{n\to\infty} p_s(\xi_n, \xi) = \lim_{n,m\to\infty} p_s(\xi_n, \xi_m). \tag{3.15}$$

For all $n \geq N$ and $n, m \geq \mathbb{N}$, we have

$$\begin{aligned} p_s(\xi_n, \xi_m) &\leq p_s(\xi_n, \xi_{n+1}) + p_s(\xi_{n+1}, \xi_{n+2}) + \cdots + p_s(\xi_{m-1}, \xi_{m+n}), \\ &\leq \xi_n + \xi_{n+1} + \xi_{n+2} + \cdots + \xi_{m-1}, \\ &= \frac{1}{n^{\frac{1}{k}}} + \frac{1}{(n+1)^{\frac{1}{k}}} + \frac{1}{(n+2)^{\frac{1}{k}}} + \cdots + \frac{1}{(m-1)^{\frac{1}{k}}}, \\ &\leq \sum_{n=0}^{\infty} \frac{1}{n^{\frac{1}{k}}}. \end{aligned}$$

This shows that the series $\sum_{n=0}^{\infty} \frac{1}{n^{\frac{1}{k}}}$ converges. Thus, $\lim_{n\to\infty} p_s(\xi_n, \xi_m) = 0$. So, $\xi_n = \xi_m$ for every $m \geq n$ in $\mathcal{X}$. Hence, ξ_n is a Cauchy sequence in $\mathcal{X}$. Thus, we proved that $\{\xi_n\}$ is Cauchy sequence. Also, since $(\mathcal{X}, p_s)$ is a complete partial symmetric space, there exists $\xi^* \in \mathcal{X}$ such that $\xi_n \to \xi^*$, and $p(\xi^*, \xi^*) = 0$. Now we will show that ξ is a fixed point of $\mathcal{T}$. We do this by showing that $p_s(\xi^*, \mathcal{T}\xi^*) = p_s(\xi^*, \xi^*) = 0$. Suppose $p_s(\xi^*, \mathcal{T}\xi^*) > 0$. Then there exists some $N \in \mathbb{N}$ such that $p_s(\xi_n, \mathcal{T}\xi^*) > 0$ for all $n > N$. Letting $\xi = \xi_n$ and $\eta = \xi^*$ in (3.1), we have

$$\begin{aligned} \Gamma + F(p_s(\mathcal{T}\xi_n, \mathcal{T}\xi^*)) &\leq F\left(\frac{\delta_1 p_s(\xi_n, \mathcal{T}\xi_n)[1 + p_s(\xi^*, \mathcal{T}\xi^*)]}{1 + p_s(\mathcal{T}\xi_n, \mathcal{T}\xi^*)} + \delta_2 p_s(\xi_n, \xi^*)\right), \\ \Gamma + F(p_s(\mathcal{T}\xi_n, \mathcal{T}\xi^*)) &\leq F\Big(\delta_1 p_s(\xi_n, \mathcal{T}\xi_n) + \delta_2 p_s(\xi_n, \xi^*)\Big). \end{aligned} \tag{3.16}$$

Taking $n \to \infty$ in (3.16) and applying the fact that T is continuous, we get

$$\begin{aligned} \Gamma + F(p_s(\mathcal{T}\xi^*, \mathcal{T}\xi^*)) &\leq F\Big(\delta_1 p_s(\xi^*, \mathcal{T}\xi^*) + \delta_2 p_s(\xi^*, \xi^*)\Big), \\ \Gamma &\leq F\Big(\delta_1 p_s(\xi^*, \mathcal{T}\xi^*)\Big). \end{aligned}$$

The above inequality satisfies if $p_s(\xi^*, \mathcal{T}\xi^*) = 0$. By the continuity of $\mathcal{T}$, we get $\Gamma \leq p_s(\xi^*, \mathcal{T}\xi^*)$. Therefore, $\Gamma \leq 0$, which is a contradiction. Hence, ξ^* is a fixed point of $\mathcal{T}$. For the uniqueness of ξ^*, we claim that η^* is another element in $\mathcal{X}$ such that $\mathcal{T}\xi^* = \xi^*$ and $\mathcal{T}\eta^* = \eta^*$. Let $\xi = \xi^*$ and $\eta = \eta^*$ in (3.1), we have

$$\begin{aligned} \Gamma + F(p_s(\mathcal{T}\xi^*, \mathcal{T}\eta^*)) &\leq F\left(\frac{\delta_1 p_s(\xi^*, \mathcal{T}\xi^*)[1 + p_s(\eta^*, \mathcal{T}\eta^*)]}{1 + p_s(\mathcal{T}\xi^*, \mathcal{T}\eta^*)} + \delta_2 p_s(\xi^*, \eta^*)\right), \\ \Gamma + F(p_s(\xi^*, \eta^*)) &\leq F\left(\frac{\delta_1 p_s(\xi^*, \xi^*)[1 + p_s(\eta^*, \eta^*)]}{1 + p_s(\xi^*, \eta^*)} + \delta_2 p_s(\xi^*, \eta^*)\right). \end{aligned} \tag{3.17}$$

Using (2.1) in (3.17), we obtain

$$\Gamma + F(p_s(\xi^*, \eta^*)) \leq F\Big(\delta_2 p_s(\xi^*, \eta^*)\Big).$$

The above inequality satisfy if $p_s(\xi^*, \eta^*) = 0$, for $\Gamma \leq 0$ which is a contradiction. By$(F1)$ implies that

$$\begin{aligned} p_s(\xi^*, \eta^*) &< \delta_2 p_s(\xi^*, \eta^*), \\ (1 - \delta_2) p_s(\xi^*, \eta^*) &< 0, \\ p_s(\xi^*, \eta^*) &< 0, \end{aligned}$$

which is a contradiction. Hence, the fixed point is unique, that is to say, $\xi^* = \eta^*$. Thus, ξ^* is a unique fixed point of $\mathcal{T}$. $\square$

We give the following corollary:

Corollary 3.3. Let $(\mathcal{X}, p_s)$ be a complete partial symmetric space and suppose $\mathcal{D}$ is a non-empty subset of $\mathcal{X}$. Let $\mathcal{T} : \mathcal{D} \to \mathcal{D}$ be a mapping such that $\mathcal{T}\xi \neq 0$ for each $\xi, \eta \in \mathcal{D}$, $\Gamma > 0$ and $q, r \geq 2$ with $p_s(\mathcal{T}\xi, \mathcal{T}\eta) \geq 0$, we have

$$\Gamma + F(p_s(\mathcal{T}\xi, \mathcal{T}\eta)) \leq F(\mathcal{M}_{p_s}(\xi, \eta)),$$

where

$$\mathcal{M}_{p_s}(\xi, \eta) = \max\left\{p_s(\xi, \eta), \frac{p_s(\xi, \mathcal{T}\xi) + p_s(\eta, \mathcal{T}\eta)}{q}, \frac{p_s(\xi, \mathcal{T}\eta) + p_s(\eta, \mathcal{T}\xi)}{r}\right\}, \tag{3.18}$$

and F is an increasing function in $\mathcal{F}$. Also, assume that the following conditions hold:

(i) there exist $\xi_0 \in \mathcal{D}$ and $\xi_1 \in \mathcal{T}\xi_0$ such that $\mathcal{T}\xi_0 \in \mathcal{D}$,

(ii) F is continuous,

(iiii) for any sequence $\{\xi_n\}$ in $\mathcal{D}$ such that $\xi_n \to \xi$ as $n \to \infty$ and $\xi_n \preceq \xi_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$, $p_s(\xi_n, \xi) \to p_s(\xi, \xi)$ as $n \to \infty$.

Then $\mathcal{T}$ has a unique fixed point.

Proof . Let $\xi_0 \in \mathcal{D}$, since $\mathcal{T}\xi_0 \in \mathcal{D}$ for every $\xi_0 \in \mathcal{X}$. Then there exists $\xi_1 \in \mathcal{D}$ such that $\xi_1 \in \mathcal{T}\xi_0$. Assume that $\xi_1 \notin \mathcal{T}\xi_0$, on contrary to that ξ_1 is a fixed point of $\mathcal{T}$. Then, since $\mathcal{T}\xi_0$ is closed in $\mathcal{D}$, $p_s(\xi_0, \mathcal{T}\xi_0) > 0$. Suppose that $\xi \in \mathcal{D}$ and $\{\xi_n\}_n$ be a sequence in $D \subseteq \mathcal{X}$ defined as $\xi_n = \mathcal{T}\xi_{n-1}$. Let $\xi = \xi_{n-1}$ and $\eta = \xi_n$ in (3.18), we have

$$\Gamma + F(p_s(\mathcal{T}\xi_{n-1}, \mathcal{T}\xi_n)) \leq F(\mathcal{M}_{p_s}(\xi_{n-1}, \xi_n)), \tag{3.19}$$

where

$$\begin{aligned}\mathcal{M}_{p_s}(\xi_{n-1}, \xi_n) &= \max\left\{p_s(\xi_{n-1}, \xi_n), \frac{p_s(\xi_{n-1}, \mathcal{T}\xi_{n-1}) + p_s(\xi_n, \mathcal{T}\xi_n)}{q}, \frac{p_s(\xi_{n-1}, \mathcal{T}\xi_n) + p_s(\xi_n, \mathcal{T}\xi_{n-1})}{r}\right\},\\ &\leq \max\left\{p_s(\xi_{n-1}, \xi_n), \frac{p_s(\xi_{n-1}, \xi_n) + p_s(\xi_n, \xi_{n+1})}{q}, \frac{p_s(\xi_{n-1}, \xi_{n+1}) + p_s(\xi_n, \xi_n)}{r}\right\}.\end{aligned}$$

Since $q, r \geq 2$, it follows that

$$\mathcal{M}_{p_s}(\xi_{n-1}, \xi_n) = p_s(\xi_{n-1}, \xi_n). \tag{3.20}$$

Using (3.20) in (3.19) yields

$$\Gamma + F(p_s(\xi_n, \xi_{n+1})) \leq F(p_s(\xi_{n-1}, \xi_n)).$$

Consequently,

$$F(p_s(\xi_n, \xi_{n+1})) \leq F(p_s(\xi_{n-1}, \xi_n)) - \Gamma.$$

The steps follow the similar proof of Theorem (3.2). This completes the proof. □

Corollary 3.4. Let $(\mathcal{X}, p_s)$ be a complete partial symmetric spaces, $\mathcal{C}$ a non empty closed subset of $\mathcal{X}$ and $\mathcal{T} : \mathcal{C} \longrightarrow \mathcal{C}$ be a self- mapping. Assume that the following conditions hold:

(i) $\mathcal{T}\xi_0 \in \mathcal{C}$ for each $\xi_0 \in \mathcal{C}$,

(ii) $\mathcal{T}$ is F-continuous,

(iii) There exists $k \in (0,1), \forall\, \xi, \eta \in\ \mathcal{C}$ and $\Gamma > 0$ with $p_s(\mathcal{T}\xi, \mathcal{T}\eta) > 0$; such that

$$\Gamma + F(p_s(\mathcal{T}\xi, \mathcal{T}\eta)) \leq F(k\mathcal{M}_{p_s}(\xi, \eta)), \tag{3.21}$$

where

$$\mathcal{M}_{p_s}(\xi, \eta) = \max\left\{p_s(\xi, \eta), \frac{p_s(\xi, \mathcal{T}\xi) + p_s(\eta, \mathcal{T}\eta)}{q}, \frac{p_s(\xi, \mathcal{T}\eta) + p_s(\eta, \mathcal{T}\xi)}{r}\right\}.$$

Then, $\mathcal{T}$ has a unique fixed point, if there exists $\xi \in \mathcal{X}$ such that $p_s(\xi, \xi) = 0$.

Proof . The proof of this corollary follows the similar steps of Theorem (3.2). This completes the proof. □

Now, we give an example to illustrate the use of Theorem 3.2.

Example 3.5. Let $\mathcal{X} = [0, \infty)$ be an Euclidean space. Denote the unit interval of real numbers, with the partial symmetric $p_s(\xi, \eta) = p_s : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ defined by $p_s(\xi, \eta) = |\xi - \eta|^p + |\xi - \eta|^q$, $\Gamma > 0$, for all $\xi, \eta \in \mathcal{X}$ and a mapping $\mathcal{T} : \mathcal{X} \longrightarrow \mathcal{X}$ given by

$$\mathcal{T}\xi = \frac{\xi}{2^{3\xi}}.$$

Let $F(z) = \ln(z^2 + z)$, using (3.1), we obtain

$$\frac{p_s(\mathcal{T}\xi, \mathcal{T}\eta)[1 + p_s(\mathcal{T}\xi, \mathcal{T}\eta)]}{\mathcal{M}_{p_s}(\xi, \eta)[1 + \mathcal{M}_{p_s}(\xi, \eta)]} \leq e^{-\Gamma}, \tag{3.22}$$

where

$$\mathcal{M}_{p_s}(\xi, \eta) = \frac{\delta_1 p_s(\xi, \mathcal{T}\xi)[1 + p_s(\eta, \mathcal{T}\eta)]}{1 + p_s(\mathcal{T}\xi, \mathcal{T}\eta)} + \delta_2 p_s(\xi, \eta),$$

where $\delta_1, \delta_2 \in [0, 1)$ with $\delta_1 + \delta_2 < 1$.

Equivalent to

$$\mathcal{M}_{p_s}(\xi, \eta) = \frac{\delta_1 p_s(\xi, \mathcal{T}\xi)[1 + p_s(\eta, \mathcal{T}\eta)] + \delta_2 p_s(\xi, \eta)[1 + p_s(\mathcal{T}\xi, \mathcal{T}\eta)]}{1 + p_s(\mathcal{T}\xi, \mathcal{T}\eta)}.$$

We prove that $\mathcal{T}$ satisfies (3.22). Now we calculate the following metrics:

$$\begin{aligned} p_s(\mathcal{T}\xi, \mathcal{T}\eta) &= |\mathcal{T}\xi - \mathcal{T}\eta|^p + |\mathcal{T}\xi - \mathcal{T}\eta|^q \\ &\leq \left|\frac{\xi}{2^{3\xi}} - \frac{\eta}{2^{3\eta}}\right|^p + \left|\frac{\xi}{2^{3\xi}} - \frac{\eta}{2^{3\eta}}\right|^q \\ &= \left|\frac{2^{3\eta}\xi - 2^{3\xi}\eta}{2^{3\xi+3\eta}}\right|^p + \left|\frac{2^{3\eta}\xi - 2^{3\xi}\eta}{2^{3\xi+3\eta}}\right|^q, \end{aligned} \tag{3.23}$$

$$\begin{aligned} p_s(\xi, \mathcal{T}\xi) &= |\xi - \mathcal{T}\xi|^p + |\xi - \mathcal{T}\xi|^q \\ &\leq \left|\xi - \frac{\xi}{2^{3\xi}}\right|^p + \left|\xi - \frac{\xi}{2^{3\xi}}\right|^q \\ &= \left|\frac{2^{3\xi}\xi - \xi}{2^{3\xi}}\right|^p + \left|\frac{2^{3\xi}\xi - \xi}{2^{3\xi}}\right|^q, \end{aligned} \tag{3.24}$$

$$\begin{aligned} p_s(\eta, \mathcal{T}\eta) &= |\eta - \mathcal{T}\eta|^p + |\eta - \mathcal{T}\eta|^q \\ &\leq \left|\eta - \frac{\eta}{2^{3\eta}}\right|^p + \left|\eta - \frac{\eta}{2^{3\eta}}\right|^q \\ &= \left|\frac{2^{3\eta}\eta - \eta}{2^{3\eta}}\right|^p + \left|\frac{2^{3\eta}\eta - \eta}{2^{3\eta}}\right|^q, \end{aligned} \tag{3.25}$$

$$p_s(\xi, \eta) = |\xi - \eta|^p + |\xi - \eta|^q. \tag{3.26}$$

Using (3.23)-(3.26) in (3.22), we obtain

$$\frac{\left[\left|\frac{2^{3\eta}\xi - 2^{3\xi}\eta}{2^{3\xi+3\eta}}\right|^p + \left|\frac{2^{3\eta}\xi - 2^{3\xi}\eta}{2^{3\xi+3\eta}}\right|^q\right]\left[1 + \left|\frac{2^{3\eta}\xi - 2^{3\xi}\eta}{2^{3\xi+3\eta}}\right|^p + \left|\frac{2^{3\eta}\xi - 2^{3\xi}\eta}{2^{3\xi+3\eta}}\right|^q\right]}{\mathcal{M}_{p_s}(\xi, \eta)[1 + \mathcal{M}_{p_s}(\xi, \eta)]} \leq e^{-\Gamma}, \tag{3.27}$$

where

$$\mathcal{M}_{p_s}(\xi, \eta) = \frac{\begin{array}{c} \delta_1 \left[\left|\frac{2^{3\xi}\xi - \xi}{2^{3\xi}}\right|^p + \left|\frac{2^{3\xi}\xi - \xi}{2^{3\xi}}\right|^q\right]\left[1 + \left|\frac{2^{3\eta}\eta - \eta}{2^{3\eta}}\right|^p + \left|\frac{2^{3\eta}\eta - \eta}{2^{3\eta}}\right|^q\right] + \\ \delta_2 \left[|\xi - \eta|^p + |\xi - \eta|^q\right]\left[1 + \left|\frac{2^{3\eta}\xi - 2^{3\xi}\eta}{2^{3\xi+3\eta}}\right|^p + \left|\frac{2^{3\eta}\xi - 2^{3\xi}\eta}{2^{3\xi+3\eta}}\right|^q\right] \end{array}}{1 + \left|\frac{2^{3\eta}\xi - 2^{3\xi}\eta}{2^{3\xi+3\eta}}\right|^p + \left|\frac{2^{3\eta}\xi - 2^{3\xi}\eta}{2^{3\xi+3\eta}}\right|^q}.$$

By choosing $\delta_1 = \frac{1}{2}, \delta_2 = \frac{1}{4}, p = q = 2,\ \xi = 1, \eta = 2$ and $\Gamma = \frac{1}{2}$ in (3.27) we get

$$\frac{\left[\left|\frac{2^{3\times2}\times1-2^{3\times1}\times2}{2^{3\times1+3\times2}}\right|^2 + \left|\frac{2^{3\times2}\times1-2^{3\times1}\times2}{2^{3\times1+3\times2}}\right|^2\right]\times \left[1+\left|\frac{2^{3\times2}\times1-2^{3\times1}\times2}{2^{3\times1+3\times2}}\right|^2 + \left|\frac{2^{3\times2}\times1-2^{3\times1}\times2}{2^{3\times1+3\times2}}\right|^2\right]}{\mathcal{M}_{p_s}(1,2)[1+\mathcal{M}_{p_s}(1,2)]} \leq e^{-\Gamma},$$

where

$$\mathcal{M}_{p_s}(1,2) = \frac{\frac{1}{2}\left[\left|\frac{2^{3\times1}\times1-1}{2^{3\times1}}\right|^2 + \left|\frac{2^{3\times1}\times1-1}{2^{3\times1}}\right|^2\right]\left[1+\left|\frac{2^{3\times2}\times2-2}{2^{3\times2}}\right|^2 + \left|\frac{2^{3\times2}\times2-2}{2^{3\times2}}\right|^2\right] + \frac{1}{4}\left[|1-2|^2+|1-2|^2\right]\left[1+\left|\frac{2^{3\times2}\times1-2^{3\times1}\times2}{2^{3\times1+3\times2}}\right|^2 + \left|\frac{2^{3\times2}\times1-2^{3\times1}\times2}{2^{3\times1+3\times2}}\right|^2\right]}{1+\left|\frac{2^{3\times2}\times1-2^{3\times1}\times2}{2^{3\times1+3\times2}}\right|^2 + \left|\frac{2^{3\times2}\times1-2^{3\times1}\times2}{2^{3\times1+3\times2}}\right|^2}.$$

Consequently,

$$\frac{\left[\left|\frac{2^6-2^4}{2^8}\right|^2 + \left|\frac{2^6-2^4}{2^8}\right|^2\right]\left[1+\left|\frac{2^6-2^4}{2^8}\right|^2 + \left|\frac{2^6-2^4}{2^8}\right|^2\right]}{\mathcal{M}_{p_s}(1,2)[1+\mathcal{M}_{p_s}(1,2)]} \leq e^{-\frac{1}{2}},$$

where

$$\begin{aligned}
\mathcal{M}_{p_s}(1,2) &= \frac{\frac{1}{2}\left[\left|\frac{2^3-1}{2^3}\right|^2 + \left|\frac{2^3-1}{2^3}\right|^2\right]\left[1+\left|\frac{2^7-2}{2^6}\right|^2 + \left|\frac{2^7-2}{2^6}\right|^2\right] + \frac{1}{4}\left[1+1\right]\left[1+\left|\frac{2^6-2^4}{2^9}\right|^2 + \left|\frac{2^6-2^4}{2^9}\right|^2\right]}{1+\left|\frac{2^6-2^4}{2^9}\right|^2 + \left|\frac{2^6-2^4}{2^9}\right|^2}, \\
&= \frac{6.700715786 + 0.508789062}{1.01758125}, \\
&= \frac{7.209504848}{1.01758125} = 7.084942699.
\end{aligned}$$

From the above inequality, follows that

$$\begin{aligned}
\frac{0.0752563476}{7.084942699[1+7.084942699]} &\leq e^{-\frac{1}{2}}, \\
\frac{0.0752563476}{57.28135575} &\leq e^{-\frac{1}{2}}, \\
0.001313801 &\leq 0.606530659,
\end{aligned}$$

which shows that the contraction (3.1) is satisfied. Therefore, the conditions of the Theorem 3.2 are satisfied; thus, $\mathcal{T}$ has a unique fixed point such that $\mathcal{T}0 = 0$. Hence, $p_s(0,0) = 0$.

4 Some Applications

In this section, we attempt to apply Theorem 3.2 and Corollary 3.3 to prove the existence and uniqueness of a solution of Volterra integral inclusion and the Hammerstein integral equation.

4.1 An Application to Volterra Integral Inclusion

This subsection proves the existence of Volterra integral equation inclusion, where Theorem 3.2 can be applied as an application to the fixed point theory in complete partial symmetric space. Motivated by [3, 4, 13, 27], we present an existence result for the following Volterra integral equation inclusion given by

$$\xi(t) \in \int_0^T G(t,s)\mathcal{H}(s,\xi(s))ds + h(t),\ t\in[0,T],\ h\in\mathcal{X}, \tag{4.1}$$

where $F : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous function with non-empty compact values. Here $T > 0$ is a constant. Throughout this section, the map $\xi \to F(t, \xi)$ is lower semi-continuous i.e. $t \in [0, T]$. Define $p_s : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ by

$$p_s(\xi, \eta) \quad = \quad \sup_{t\in[0,T]} |\xi - \eta|^p + \sup_{t\in[0,T]} |\xi - \eta|^q, p, q > 1. \tag{4.2}$$

Then $(\mathcal{X}, p_s)$ is a p_s-complete partial symmetric space. We recall the Hölder's inequality as follows:

Definition 4.1. [24] Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $f\xi$ and $g\xi$ are continuous real-valued functions on $[a, b]$, then the Hölder's inequality for integral states that

$$\int_a^b |f\xi, g\xi| ds \quad \leq \quad \left[\int_a^b |f\xi|^p ds\right]^{\frac{1}{p}} \left[\int_a^b |g\xi|^q ds\right]^{\frac{1}{q}}. \tag{4.3}$$

Now, we prove our results as follows:

Theorem 4.2. Suppose that, for all $\xi, \eta \in C([0, T], \mathbb{R})$, the following conditions hold:

(i) there exists a continuous function $\mathcal{H} \in [0, T] \times \mathbb{R}$ such that

$$|\mathcal{H}(t, s, \xi(s)) - \mathcal{H}(t, s, \eta(s))| \leq \frac{e^{-\Gamma}}{t} |\xi(s) - \eta(s)|,$$

where

$$p_s(\xi, \eta) = \mathcal{M}_{p_s}(\xi, \eta) \quad \leq \quad \frac{\delta_1 p_s(\xi, \mathcal{T}\xi)[1 + p_s(\eta, \mathcal{T}\eta)]}{1 + p_s(\mathcal{T}\xi, \mathcal{T}\eta)} + \delta_2 p_s(\xi, \eta),$$

and $\delta_1, \delta_2 \in [0, 1)$ with $\delta_1 + \delta_2 < 1$.

(ii) there exist $t, s \in [0, T]$ and $\Gamma > 0$ such that

$$\left|\int_0^t G(t, s) ds\right|^p = \left|\int_0^t G(t, s) ds\right|^q \quad \leq \quad 1. \tag{4.4}$$

Then, the Volterra integral equation inclusion (4.1) has a unique solution.

Proof . Using the Volterra integral equation inclusion (4.1), we can define an operator $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ as follows:

$$\mathcal{T}\xi(t) \in \int_0^T G(t, s)\mathcal{H}(s, \xi(s)) ds + h(t),\ t \in [0, T],\ h \in \mathcal{X}, \tag{4.5}$$

which shows that ξ is a fixed point of the operator $\mathcal{T}$ if and only if it is a solution of Equation (4.1). Now, for all $\xi, \eta \in \mathcal{X}$, using condition (i), (ii) and (4.3) we obtain

$$\begin{aligned}
|\mathcal{T}\xi - \mathcal{T}\eta|^p + |\mathcal{T}\xi - \mathcal{T}\eta|^q \quad &\leq \quad \left[\int_0^t G(t, s)(\mathcal{H}(s, \xi(s)) - \mathcal{H}(s, \eta(s))) ds\right]^p + \left[\int_0^t G(t, s)(\mathcal{H}(s, \xi(s)) - \mathcal{H}(s, \eta(s))) ds\right]^q \\
&\leq \quad \left[\left[\left|\int_0^t G(t, s) ds\right|^q\right]^{\frac{1}{q}} \left[\int_0^t \left|\mathcal{H}(s, \xi(s)) - \mathcal{H}(s, \eta(s))\right|^p ds\right]^{\frac{1}{p}}\right]^p \\
&\quad + \left[\left[\left|\int_0^t G(t, s) ds\right|^p\right]^{\frac{1}{p}} \left[\int_0^t \left|\mathcal{H}(s, \xi(s)) - \mathcal{H}(s, \eta(s))\right|^q ds\right]^{\frac{1}{q}}\right]^q \\
&\leq \quad \left|\int_0^t G(t, s) ds\right|^p \int_0^t \left|\mathcal{H}(s, \xi(s)) - \mathcal{H}(s, \eta(s))\right|^p ds \\
&\quad + \left|\int_0^t G(t, s) ds\right|^q \int_0^t \left|\mathcal{H}(s, \xi(s)) - \mathcal{H}(s, \eta(s))\right|^q ds,
\end{aligned}$$

$$\begin{aligned}
&\leq \left|\int_0^t G(t,s)ds\right|^p \left[\int_0^t ds\right] \Big|\mathcal{H}(s,\xi(s)) - \mathcal{H}(s,\eta(s))\Big|^p + \left|\int_0^t G(t,s)ds\right|^q \left[\int_0^t ds\right] \Big|\mathcal{H}(s,\xi(s)) - \mathcal{H}(s,\eta(s))\Big|^q \\
&\leq \sup_{t\in[0,t]} |\xi(s)-\eta(s)|^p \frac{e^{-\Gamma}}{t} \times t + \sup_{t\in[0,t]} |\xi(s)-\eta(s)|^q \frac{e^{-\Gamma}}{t} \times t, \\
&\leq \sup_{t\in[0,t]} |\xi(s)-\eta(s)|^p e^{-\Gamma} + \sup_{t\in[0,t]} |\xi(s)-\eta(s)|^q e^{-\Gamma}, \\
&\leq \left[\sup_{t\in[0,t]} |\xi(s)-\eta(s)|^p + \sup_{t\in[0,t]} |\xi(s)-\eta(s)|^q\right] e^{-\Gamma}, \\
&\leq p_s(\xi,\eta)e^{-\Gamma}.
\end{aligned}$$

So,

$$p_s(\mathcal{T}\xi,\mathcal{T}\eta) \quad \leq \quad p_s(\xi,\eta)e^{-\Gamma}. \tag{4.6}$$

By taking natural logarithms on both sides in (4.6) and the property of F, we obtain

$$F(p_s(\mathcal{T}\xi,\mathcal{T}\eta)) \quad \leq \quad F(p_s(\xi,\eta)) - \Gamma. \tag{4.7}$$

Consequently,

$$\Gamma + F(p_s(\mathcal{T}\xi,\mathcal{T}\eta)) \quad \leq \quad F(p_s(\xi,\eta)). \tag{4.8}$$

Equivalently,

$$\Gamma + F(p_s(\mathcal{T}\xi,\mathcal{T}\eta)) \quad \leq \quad F(\mathcal{M}_{p_s}(\xi,\eta)). \tag{4.9}$$

Hence, ξ is a fixed point of $\mathcal{T}$. Thus, (4.5) has a unique solution, which is also a solution of the Volterra integral equation inclusion (4.1). This completes our proof. □

4.2 An Application of Fixed Point Theorem to a Chemical Reactor Problem

This subsection covers the application of fixed point theorem to a chemical reactor problem using ordinary differential equations which can be converted to Hammerstein integral equation, where Corollary 3.3 is applied. The following differential equation was inspired by Heinemann and Poor [16, 17], Lovo and Balakotaiah [20] and McGhee et al. [23], which represents the mathematical model for an adiabatic tubular chemical reactor which processes an irreversible exothermic chemical reaction for steady-state solutions.

$$\begin{cases} \xi^{''} - \alpha\xi^{'} + \mathcal{W}(\alpha,\nu,\gamma,\xi) = 0, \\ \\ \xi^{'}(0) = \alpha\xi(0), \xi^{'}(1) = 0, \end{cases} \quad where,\ \mathcal{W}(\alpha,\nu,\gamma,\xi) = \alpha\nu(\gamma-\xi)e^{\xi}. \tag{4.10}$$

The unknown ξ represents the steady state temperature of the reaction, and the parameter α,ν and γ represent the Péclet number ($Pe = \frac{advective\ transport\ rate}{diffusive\ transport\ rate}$), Damkohler number ($Da = \frac{reaction\ rate}{convective\ mass\ transport\ rate}$) and the dimensionless adiabatic temperature rise respectively.

The differential equation (4.10) can be written in the form of a Hammerstein integral equation using the Green function technique.

$$\xi(t) = \nu\int_0^1 G(t,s)f(t,s,\xi(s))ds, \forall\ t\in[0,1], \tag{4.11}$$

where $\xi(t)$ is unknown function on $I=[0,1]$, The Green function associated with the Hammerstein integral Equation (4.11) is defined by

$$G(t,s) = \begin{cases} e^{\alpha(t-s)}, & 0\leq t\leq s\leq 1, \\ 1 & 0\leq s\leq t\leq 1, \end{cases} \tag{4.12}$$

and

$$f(t,s,\xi) = (\gamma-\xi)e^{\xi},$$

which we consider in the space $C[0,1]$ of continuous functions on the closed interval $[0,1]$. Throughout, we assume α and ν are positive, and γ is non-negative.

It is well known that $\xi \in C^2[0,1]$ is a solution of (4.10) which is equivalent to find a solution $\xi \in C[0,1]$ of the Hammerstein integral Equation (4.11).

Now, we will construct the following theorem.

Theorem 4.3. Suppose that $f:[0,1]\times[0,1]\times[0,1]\to\mathbb{R}$ is continuous for all $t,s,\xi,\eta \in C([0,1],\mathbb{R})$ and the following conditions are satisfied:

(i) for all $t,s \in [0,1]$ and $\xi,\eta \in \mathcal{D}$, we have

$$|f(t,s,\xi(s)) - f(t,s,\eta(s))| \leq e^{-\Gamma}|\xi(s)-\eta(s)| \tag{4.13}$$

where

$$\mathcal{L} = \gamma(e^{\xi}-e^{\eta}) + \eta e^{\eta} - \xi e^{\xi},$$
$$\int_0^1 G(t,s)ds = \mathcal{Z} = \frac{\alpha t + 1 - e^{\alpha(t-1)}}{\alpha},$$

and $\nu^p\mathcal{Z}\mathcal{L} \leq e^{-\Gamma}$ implies that $\nu^q\mathcal{Z}\mathcal{L} \leq e^{-\Gamma}, for\ p,q \geq 0$, with

$$\mathcal{M}_{p_s}(\xi,\eta) = \max\left\{p_s(\xi,\eta), \frac{p_s(\xi,\mathcal{T}\xi)+p_s(\eta,\mathcal{T}\eta)}{q}, \frac{p_s(\xi,\mathcal{T}\eta)+p_s(\eta,\mathcal{T}\xi)}{r}\right\}.$$

(ii) for any sequence $\{\xi_n\}$ in $\mathcal{D}$ such that $\xi_n \to \xi$ as $n\to\infty$ and $\xi_n \preceq \xi_{n+1}$ for each $n \in \mathbb{N}\cup\{0\}$, $p_s(\xi_n,\xi)\to p_s(\xi,\xi)$ as $n\to\infty$. Then there is a unique continuous functions $\xi:[0,1]\to\mathbb{R}$ which satisfies (4.11).

Then, the integral equation (4.11) has a unique solution $\xi \in \mathcal{X}$.

Proof . We prove this result by showing that, when inequality (4.13) is satisfied, the map $\mathcal{T}$ is a contraction on the normed space $C([0,1])$ with the uniform norm $\|.\|_\infty$. Now, define a map $\mathcal{T}: C[0,1]\to C[0,1]$ by

$$T\xi(t) = \nu\int_0^1 G(t,s)f(t,s,\xi(s))ds,\ t\in[0,1]. \tag{4.14}$$

Define $p_s : \mathcal{X}\times\mathcal{X}\to\mathcal{X}$ by

$$p_s(\xi,\eta) = \sup_{t\in[0,T]}|\xi-\eta|^p + \sup_{t\in[0,T]}|\xi-\eta|^q + \beta, \forall\ p,q > 1 and\ \beta \geq 0.$$

Then $(\mathcal{X},p_s)$ is a p_s-complete partial symmetric space. We prove that a mapping $\mathcal{T}$ defined in Equation (4.14) is a contraction for two continuous functions ξ and η on $C([0,1],\mathcal{X})$. By using the condition imposed in Theorem 4.3, (4.3) and (4.15) , for all $\xi,\eta \in \mathcal{X}$, we obtain

$$\begin{aligned}
|\mathcal{T}\xi-\mathcal{T}\eta|^p + |\mathcal{T}\xi-\mathcal{T}\eta|^q \leq & \left[\nu\int_0^1 G(t,s)(f(t,s,\xi(s)) - \mathcal{H}(s,\eta(s)))ds\right]^p \\
& + \left[\nu\int_0^1 G(t,s)(f(s,\xi(s)) - f(t,s,\eta(s)))ds\right]^q + \beta \\
\leq & \left[\left[\left|\int_0^1 G(t,s)ds\right|^q\right]^{\frac{1}{q}}\left[\nu^p\int_0^1\left|f(t,s,\xi(s)) - f(t,s,\eta(s))\right|^p ds\right]^{\frac{1}{p}}\right]^p \\
& + \left[\left[\left|\int_0^1 G(t,s)ds\right|^p\right]^{\frac{1}{p}}\left[\nu^q\int_0^1\left|f(t,s,\xi(s)) - f(t,s,\eta(s))\right|^q ds\right]^{\frac{1}{q}}\right]^q + \beta
\end{aligned}$$

$$
\begin{aligned}
&\leq \left|\int_0^t G(t,s)ds\right|^p \nu^p \int_0^1 \Big|f(t,s,\xi(s)) - f(t,s,\eta(s))\Big|^p ds + \left|\int_0^1 G(t,s)ds\right|^q \nu^q \int_0^t \Big|f(t,s,\xi(s)) - f(t,s,\eta(s))\Big|^q ds + \beta\\
&\leq \left|\int_0^1 G(t,s)ds\right|^p \left[\int_0^1 ds\right] \nu^p \Big|f(t,s,\xi(s)) - f(t,s,\eta(s))\Big|^p + \left|\int_0^1 G(t,s)ds\right|^q \left[\int_0^1 ds\right] \nu^q \Big|f(t,s,\xi(s)) - f(t,s,\eta(s))\Big|^q + \beta\\
&\leq \sup_{t\in[0,1]} |\xi(s)-\eta(s)|^p \nu^p \mathcal{Z}\mathcal{L} + \sup_{t\in[0,1]} |\xi(s)-\eta(s)|^q \nu^q \mathcal{Z}\mathcal{L}\\
&\leq \sup_{t\in[0,1]} |\xi(s)-\eta(s)|^p e^{-\Gamma} + \sup_{t\in[0,1]} |\xi(s)-\eta(s)|^q e^{-\Gamma}\\
&\leq \left[\sup_{t\in[0,t]} |\xi(s)-\eta(s)|^p + \sup_{t\in[0,t]} |\xi(s)-\eta(s)|^q\right] e^{-\Gamma}\\
&\leq p_s(\xi,\eta)e^{-\Gamma}.
\end{aligned}
$$

So,

$$p_s(\mathcal{T}\xi,\mathcal{T}\eta) \quad \leq \quad p_s(\xi,\eta)e^{-\Gamma}. \tag{4.15}$$

By taking natural logarithms on both sides in (4.15) and the property of F, we obtain

$$F(p_s(\mathcal{T}\xi,\mathcal{T}\eta)) \quad \leq \quad F(p_s(\xi,\eta)) - \Gamma. \tag{4.16}$$

As a result, we have

$$\Gamma + F(p_s(\mathcal{T}\xi,\mathcal{T}\eta)) \quad \leq \quad F(p_s(\xi,\eta)). \tag{4.17}$$

Equivalent to

$$\Gamma + F(p_s(\mathcal{T}\xi,\mathcal{T}\eta)) \quad \leq \quad F(\mathcal{M}_{p_s}(\xi,\eta)). \tag{4.18}$$

Operator $\mathcal{T}$ satisfies the condition of Equation 4.10. Hence by Theorem 4.3 we have shown that the operator $\mathcal{T}$ has a unique fixed point $\xi \in \mathcal{X}$, that is, the Hammerstein integral equation (4.11) has a solution in $\mathcal{X}$, which is the solution of differential equation (4.10). Therefore, $\mathcal{T}$ satisfies all the conditions in Corollary 3.3. □

5 Conclusion

The novelty of this study to fixed point theory is the fixed point result given in Theorem 3.2. This theorem provides the fixed points conditions for a substantial class of self-mappings on various abstract spaces. This paper, inspired by the results obtained by Wilson [35], Matthew [22], Wardowski [33] and Asim et al. [5] in partial symmetric space. We proved a fixed point theorem for a self-mapping involving F-contraction in partial symmetric space, which generalizes some well-known literature results. These results have some applications in many areas of applied mathematics, especially in integral equation inclusion.

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