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Growth of the s^{th} derivative of a polynomial with restricted zeros

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Abstract

In this paper, we establish some upper bound estimates for the maximum modulus of the higher order derivative of a polynomial with restricted zeros on a circle of $|z| = R$, $R \geq 1$, in terms of the maximum modulus of the same polynomial on the unit circle $|z| = 1$. These results generalize and sharpen some known results in this direction.

Keywords: Polynomials, Zeros, Inequalities, Maximum modulus
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1 Introduction

The study of extremal problems of functions and the results where some approaches to obtaining polynomial inequalities for various norms and various constraints on using different methods of the geometric function theory is a classical topic in analysis. Let $p(z)$ be a polynomial of degree n and for $R > 0$, set $M(p, R) = \max\limits_{|z|=R} |p(z)|$. We also denote $M(p, 1)$ by $\|p\|$, the uniform norm of a polynomial p on the unit circle $|z| = 1$. The study of inequalities that relate the norm of a polynomial on a larger circle to that of its norm on the unit circle and their various versions play a key role in the literature having its own intrinsic interest. Over a period, these inequalities have been generalized in different domains and in different norms. These inequalities for polynomials have been the subject of many research papers which is witnessed by many recent articles (for example, see [11], [10]). It is a simple deduction from the Maximum Modulus Principle (see [12], p. 158) that for $R \geq 1$,

$$M(p, R) \leq R^n \|p\|. \tag{1.1}$$

Equality holds in (1.1) for $p(z) = \lambda z^n$, $\lambda \neq 0$ being a complex number. It is worth to note that these extremal polynomials have all zeros at the origin. It is natural to seek improvements under appropriate conditions on the zeros of $p(z)$. It is shown by Ankeny and Rivlin [1] that if $p(z)$ is a polynomial of degree n having no zero in $|z| < 1$, then for $R \geq 1$,

$$M(p, R) \leq \left(\frac{R^n + 1}{2}\right) \|p\|. \tag{1.2}$$

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Inequality (1.2) becomes equality for $p(z) = \lambda + \mu z^n$, where $|\lambda| = |\mu|$. Govil [7] observed that since equality in (1.2) holds only for polynomials $p(z) = \lambda + \mu z^n$, $|\lambda| = |\mu|$, which satisfy

$$|\text{coefficient } of \ z^n| = \frac{1}{2}\|p\|, \tag{1.3}$$

it should be possible to improve upon the bound in (1.2) for polynomials not satisfying (1.3). In an attempt to solve this problem, Govil [7] obtained that if $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having no zero in $|z| < 1$, then for $R \geq 1$,

$$M(p,R) \leq \frac{(R^n+1)}{2}\|p\| - \frac{n}{2}\left(\frac{\|p\|^2 - 4|a_n|^2}{\|p\|}\right) \times \left[\frac{(R-1)\|p\|}{\|p\|+2|a_n|} - \ln\left\{1 + \frac{(R-1)\|p\|}{\|p\|+2|a_n|}\right\}\right].$$

Further, as a generalization and improvement of inequality (1.4), Govil and Nyuydinkong [9] proved that if $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then for $R \geq 1$,

$$\begin{aligned} M(p,R) \leq\ & \left(\frac{R^n+k}{1+k}\right)\|p\| - \left(\frac{R^n-1}{1+k}\right)m - \frac{n}{1+k}\left\{\frac{(\|p\|-m)^2 - (1+k)^2|a_n|^2}{\|p\|-m}\right\} \\ & \times \left[\frac{(R-1)(\|p\|-m)}{\|p\|-m+(1+k)|a_n|} - \ln\left\{1 + \frac{(R-1)(\|p\|-m)}{\|p\|-m+(1+k)|a_n|}\right\}\right], \end{aligned} \tag{1.4}$$

where here and throughout the paper $m = \min_{|z|=k} |p(z)|$.

2 Preliminaries

2.1 Main results

In this paper, we prove a result for s^{th} derivative of the polynomial $p(z)$ which as a special case gives inequality (1.4). More precisely, we prove the following.

Theorem 2.1. If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then for $0 \leq s < n$ and $R \geq 1$ and any positive integer $N \leq n$,

$$\begin{aligned} M(p^{(s)},R) \leq\ & \|p^{(s)}\| + \frac{n(n-1)(n-2)...(n-s)(R^{n-s}-1)}{(1+k^{s+1})(n-s)}(\|p\|-m) \\ & - \left\{\frac{n(n-1)(n-2)....(n-s)}{(1+k^{s+1})R^s}\right\}(\|p\|-m)\left\{1 - \frac{|a_n|(1+k^{s+1})}{\|p\|-m}\right\} I(N), \end{aligned} \tag{2.1}$$

where

$$I(N) = \left(R-1\right) - \left\{1 + \frac{(1+k^{s+1})|a_n|}{\|p\|-m}\right\} \times \ln\left\{1 + \frac{(R-1)(\|p\|-m)}{(\|p\|-m)+(1+k^{s+1})|a_n|}\right\} \quad \text{for } N = 1, \tag{2.2}$$

$$\begin{aligned} I(N) =& \left(\frac{R^N-1}{N}\right) + \sum_{v=1}^{N-1}\left(\frac{R^{N-v}-1}{N-v}\right)(-1)^v\left\{1 + \frac{(1+k^{s+1})|a_n|}{\|p\|-m}\right\}\left\{\frac{(1+k^{s+1})|a_n|}{\|p\|-m}\right\}^{v-1} \\ & + (-1)^N\left\{1 + \frac{(1+k^{s+1})|a_n|}{\|p\|-m}\right\}\left\{\frac{(1+k^{s+1})|a_n|}{\|p\|-m}\right\}^{N-1} \times \ln\left\{1 + \frac{(R-1)(\|p\|-m)}{(\|p\|-m)+(1+k^{s+1})|a_n|}\right\} \quad \text{for } N \geq 2. \end{aligned} \tag{2.3}$$

Remark 2.2. If $s = 0$, Theorem 2.1 reduces to the following result which is a generalization of a result of Mir et al. [11].

Corollary 2.3. If $p(z)=\sum_{v=0}^{n}a_vz^v$ is a polynomial of degree n having no zero in $|z|<k$, $k\geq 1$, then for $R\geq 1$ and any positive integer $N\leq n$,

$$M(p,R) \leq \left\{\frac{R^n+k}{1+k}\right\}\|p\|-\frac{(R^n-1)}{1+k}m-\left\{\frac{n(\|p\|-m)}{1+k}\right\}\left\{1-\frac{|a_n|(1+k)}{\|p\|-m}\right\}H(N), \tag{2.4}$$

where

$$H(N) = \Big(R-1\Big)-\left\{1+\frac{(1+k)|a_n|}{\|p\|-m}\right\}\times\ln\left\{1+\frac{(R-1)(\|p\|-m)}{(\|p\|-m)+(1+k)|a_n|}\right\} \quad \text{for } N=1, \tag{2.5}$$

$$\begin{aligned}H(N)=&\left(\frac{R^N-1}{N}\right)+\sum_{v=1}^{N-1}\left(\frac{R^{N-v}-1}{N-v}\right)(-1)^v\left\{1+\frac{(1+k)|a_n|}{\|p\|-m}\right\}\left\{\frac{(1+k)|a_n|}{\|p\|-m}\right\}^{v-1}\\&+(-1)^N\left\{1+\frac{(1+k)|a_n|}{\|p\|-m}\right\}\left\{\frac{(1+k)|a_n|}{\|p\|-m}\right\}^{N-1}\times\ln\left\{1+\frac{(R-1)(\|p\|-m)}{(\|p\|-m)+(1+k)|a_n|}\right\} \quad \text{for } N\geq 2.\end{aligned} \tag{2.6}$$

Remark 2.4. If $N=n$, Corollaray 2.3 reduces to the result of Mir et al. [11, Theorem 2].

Remark 2.5. From Lemma 2.15, we have $H(N)$ is a non-negative increasing function of N for $1\leq N$ and hence $H(1)\leq H(N)$, $1\leq N$. Using this and Lemma 2.18, Corollary 2.3 reduces to inequality (1.4) of Govil and Nyuydinkong [9].

Remark 2.6. For $k=1$, Corollary 2.3 becomes the following result which is a generalization and improvement of the theorem due to Dewan and Bhat [5]. Further, it sharpens a result due to Dalal and Govil [3, Theorem 2.1]

Corollary 2.7. If $p(z)=\sum_{v=0}^{n}a_vz^v$ is a polynomial of degree n having no zero in $|z|<k$, $k\geq 1$, then for $R\geq 1$ and any positive integer $N\leq n$,

$$M(p,R)\leq\left(\frac{R^n+1}{2}\right)\|p\|-\left(\frac{R^n-1}{2}\right)m-n\left(\frac{\|p\|-m}{2}-|a_n|\right)f(N), \tag{2.7}$$

where

$$\begin{aligned}f(N) =& \left(\frac{R^N-1}{N}\right)+\sum_{v=1}^{N-1}\left(\frac{R^{N-v}-1}{N-v}\right)(-1)^v\left\{1+\frac{2|a_n|}{\|p\|-m}\right\}\left\{\frac{2|a_n|}{\|p\|-m}\right\}^{v-1}\\&+(-1)^N\left\{1+\frac{2|a_n|}{\|p\|-m}\right\}\left\{\frac{2|a_n|}{\|p\|-m}\right\}^{N-1}\times\ln\left\{1+\frac{(R-1)(\|p\|-m)}{(\|p\|-m)+2|a_n|}\right\}.\end{aligned} \tag{2.8}$$

Further, we also prove the following result involving certain co-efficients of $p(z)$. In fact, we prove

Theorem 2.8. If $p(z)=\sum_{v=0}^{n}a_vz^v$ is a polynomial of degree n having no zero in $|z|<k$, $k\geq 1$, then for $0\leq s<n$ and $R\geq 1$ and any positive integer $N\leq n$,

$$\begin{aligned}M(p^{(s)},R) \leq& \|p^{(s)}\|+\left[\frac{n(n-1)(n-2)....(n-s)(R^{n-s}-1)}{(n-s)(1+\gamma_{k,s})}\right]\|p\|\\&-\left\{\frac{n(n-1)(n-2)....(n-s)}{(1+\gamma_{k,s})R^s}\right\}\|p\|\left\{1-\frac{|a_n|(1+\gamma_{k,s})}{\|p\|}\right\}J(N),\end{aligned} \tag{2.9}$$

where

$$J(N) = \Big(R-1\Big)-\left\{1+\frac{(1+\gamma_{k,s})|a_n|}{\|p\|}\right\}\times\ln\left\{1+\frac{(R-1)\|p\|}{\|p\|+(1+\gamma_{k,s})|a_n|}\right\} \quad \text{for } N=1, \tag{2.10}$$

$$J(N) = \left(\frac{R^N-1}{N}\right) + \sum_{v=1}^{N-1}\left(\frac{R^{N-v}-1}{N-v}\right)(-1)^v\left\{1+\frac{(1+\gamma_{k,s})|a_n|}{\|p\|}\right\}\left\{\frac{(1+\gamma_{k,s})|a_n|}{\|p\|}\right\}^{v-1}$$
$$+(-1)^N\left\{1+\frac{(1+\gamma_{k,s})|a_n|}{\|p\|}\right\}\left\{\frac{(1+\gamma_{k,s})|a_n|}{\|p\|}\right\}^{N-1}\times\ln\left\{1+\frac{(R-1)\|p\|}{\|p\|+(1+\gamma_{k,s})|a_n|}\right\} \quad \text{for } N\geq 2, \tag{2.11}$$

and

$$\gamma_{k,s} = \frac{c(n,s+1)|a_0|k^{s+2}+|a_{s+1}|k^{2(s+1)}}{c(n,s+1)|a_0|+|a_{s+1}|k^{s+2}}. \tag{2.12}$$

and here and throughout the paper $c(n,s) = \dfrac{n!}{s!(n-s)!}$.

Remark 2.9. For $k=1$ and $s=0$, Theorem 2.8 reduces to a result of Dalal and Govil [3, Theorem 2.1].

Remark 2.10. If $k=1$, $s=0$ and $N=1$, Theorem 2.8 reduces to inequality (1.4) of Govil [7].

Remark 2.11. It can be easily verified that Theorem 2.8 is a refinement and generalization of inequality (1.2) due to Ankeny and Rivlin [1].

2.2 Lemmas

We need the following lemmas to prove our results.

Lemma 2.12. If $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leq \mu \leq n$, is a polynomial of degree n, then for $|z| = R$, $R \geq 1$,

$$|p(z)| \leq R^n\left\{1-\frac{(\| p \| - |a_n|)(R-1)}{|a_n| + R \| p \|}\right\} \| p \| .$$

Lemmas 2.12 is due to Gardner et al. [6].

Lemma 2.13. If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n and $r \geq 1$, then the function

$$\left[1-\frac{\{y-n(n-1)....(n-s)|a_n|\}(r-1)}{ry+n(n-1)....(n-s)|a_n|}\right] y$$

is an increasing function of y for $y > 0$.

Proof . The proof simply follows by using the derivative and we omit it.□

Lemma 2.14. Let

$$h(N) = \int_1^R \frac{(r-1)r^{N-1}}{r+x}dr, \quad x > 0.$$

Then for $N \geq 2$,

$$h(N) \;=\; \left(\frac{R^N-1}{N}\right) + \sum_{v=1}^{N-1}\left(\frac{R^{N-v}-1}{N-v}\right)(-1)^v(x+1)x^{v-1} + (-1)^N(x+1)x^{N-1}\ln\left(\frac{R+x}{1+x}\right), \tag{2.13}$$

and for $N = 1$,

$$h(1) = (R-1) - (1+x)\ln\left(1+\frac{R-1}{1+x}\right). \tag{2.14}$$

Lemma 2.14 is due to Dalal and Govil [3, Lemma 3.6].

Lemma 2.15. The function $h(N)$ defined in Lemma *2.14* is a non-negative increasing function of N for $N \geq 1$.

Proof . The proof of this lemma was done by Dalal and Govil [3, Lemma 3.7] by using recurrence relation. Moreover, a simple alternative proof was also given recently by Devi et al. [4, Lemma 3] by applying the method of differentiation under integral sign and for the sake of reader's interest, a brief proof is given:
Using the method of differentiation under the integral sign, we have

$$\frac{d}{dN}h(N) = \int_1^R \frac{(r-1)(r^{N-1})}{r+x}\ln r dr. \tag{2.15}$$

Since, for $r \in [1, R]$, $\frac{(r-1)r^{N-1}}{r+x}\ln r \geq 0$, we have

$$\int_1^R \frac{(r-1)r^{N-1}}{r+x}\ln r dr \geq 0.$$

From equality (2.15),

$$\frac{d}{dN}h(N) \geq 0, \quad \text{for } N \geq 1.$$

Therefore, $h(N)$ is an increasing function of N for $N \geq 1$. Further, since $\frac{(r-1)r^{N-1}}{r+x} \geq 0$, for $N \geq 1$, $h(N) \geq 0$ for $N \geq 1$. This completes the proof of Lemma 2.15. □

Lemma 2.16. If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then for $1 \leq s \leq n$,

$$\|p^{(s)}\| \leq \frac{n(n-1)....(n-s+1)}{1+k^s}\left\{\|p\| - m\right\}. \tag{2.16}$$

Lemma 2.16 is due to Govil [8]. Next lemma is due to Aziz and Rather [2].

Lemma 2.17. If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial having no zero in $|z| < k$, $k \geq 1$, then for $1 \leq s \leq n$,

$$\|p^{(s)}\| \leq \frac{n(n-1)....(n-s+1)}{1+\delta_{k,s}}\|p\|, \tag{2.17}$$

where

$$\delta_{k,s} = \frac{c(n,s)|a_0|k^{s+1} + |a_s|k^{2s}}{c(n,s)|a_0| + |a_s|k^{s+1}}.$$

Lemma 2.18. If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then for $0 \leq s < n$,

$$|a_n| \leq \frac{1}{1+k^{s+1}}\left(\|p\| - m\right). \tag{2.18}$$

Proof . Since $p(z) = \sum_{v=0}^{n} a_v z^v$, $p^{(s+1)}(z) = \sum_{v=s+1}^{n} v(v-1)...(v-s)a_v z^{v-(s+1)}$, by applying Cauchy's inequality to $p^{(s+1)}(z)$ on the unit circle $|z| = 1$, we have

$$\left|\frac{d^{\{n-(s+1)\}}}{dz^{\{n-(s+1)\}}}p^{(s+1)}(z)\right|_{z=0} \leq \{n-(s+1)\}! \max_{|z|=1}|p^{(s+1)}(z)|. \tag{2.19}$$

By Lemma 2.16, we have

$$\max_{|z|=1}|p^{(s+1)}(z)| \leq \frac{n(n-1)....(n-s)}{1+k^{s+1}}\left\{\max_{|z|=1}|p(z)| - m\right\}. \tag{2.20}$$

Using inequality (2.20) in the right hand side of (2.19), we have inequality (2.18) and this completes the proof of Lemma 2.18.□

Lemma 2.19. If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then for $0 \leq s < n$,

$$|a_n| \leq \frac{1}{1+\gamma_{k,s}} \|p\|, \tag{2.21}$$

where $\gamma_{k,s}$ is as defined in (2.12).

Proof . The proof of this lemma follows on the same lines as that of Lemma 2.18, but instead of applying inequality (2.16) of Lemma 2.16, we use inequality (2.17) of Lemma 2.17 to the polynomial $p(z)$ and we omit the details.□

2.3 Proofs of the theorems

We first prove Theorem 2.1.

Proof .[Proof of Theorem 2.1] For each θ, $0 \leq \theta < 2\pi$ and $1 \leq r \leq R$, we have for $0 \leq s < n$,

$$p^{(s)}(Re^{i\theta}) - p^{(s)}(e^{i\theta}) = \int_1^R e^{i\theta} p^{(s+1)}(re^{i\theta}) dr,$$

which implies

$$|p^{(s)}(Re^{i\theta}) - p^{(s)}(e^{i\theta})| \leq \int_1^R |p^{(s+1)}(re^{i\theta})| dr.$$

Now, applying Lemma 2.12 to the polynomial $p^{(s+1)}(z)$ which is of degree $\{n-(s+1)\}$, we get

$$|p^{(s)}(Re^{i\theta}) - p^{(s)}(e^{i\theta})| \leq \int_1^R r^{n-(s+1)} \left\{1 - \frac{(\|p^{(s+1)}\| - n(n-1)(n-2)...(n-s)|a_n|)(r-1)}{n(n-1)(n-2).....(n-s)|a_n| + r\|p^{(s+1)}\|}\right\} \|p^{(s+1)}\| dr. \tag{2.22}$$

By Lemma 2.13, the quantity $\left\{1 - \frac{(\|p^{(s+1)}\| - n(n-1)(n-2)....(n-s)|a_n|)(r-1)}{n(n-1)(n-2)....(n-s)|a_n| + r\|p^{(s+1)}\|}\right\} \|p^{(s+1)}\|$ occurring in the integrand of (2.22) is an increasing function of $\|p^{(s+1)}\|$, and using Lemma 2.16, we have for $0 \leq \theta < 2\pi$,

$$\begin{aligned}
|p^{(s)}(Re^{i\theta}) - p^{(s)}(e^{i\theta})| \leq & \int_1^R r^{n-(s+1)} \left[1 - \frac{\left\{\frac{n(n-1)(n-2)....(n-s)}{1+k^{s+1}}(\|p\| - m) - n(n-1)(n-2)...(n-s)|a_n|\right\}(r-1)}{n(n-1)(n-2)...(n-s)|a_n| + r(\frac{n(n-1)(n-2)...(n-s)}{1+k^{s+1}})(\|p\| - m)}\right] \\
& \times \frac{n(n-1)(n-2)...(n-s)}{1+k^{s+1}}(\|p\| - m) dr \\
= & \frac{n(n-1)(n-2)...(n-s)}{1+k^{s+1}}(\|p\| - m) \int_1^R r^{n-s-1} dr - \frac{n(n-1)(n-2)...(n-s)}{1+k^{s+1}}(\|p\| - m) \\
& \times \int_1^R r^{n-s-1} \left\{\frac{(\|p\| - m) - (1+k^{s+1})|a_n|}{(1+k^{s+1})|a_n| + r(\|p\| - m)}\right\}(r-1) dr \\
= & \frac{n(n-1)(n-2)...(n-s)(R^{n-s} - 1)}{(1+k^{s+1})(n-s)}(\|p\| - m) \\
& - \frac{n(n-1)(n-2)...(n-s)}{1+k^{s+1}}(\|p\| - m)\|(1-a) \int_1^R \frac{(r-1)r^{n-s-1}}{r+a} dr,
\end{aligned} \tag{2.23}$$

where $a = \frac{|a_n|(1+k^{s+1})}{(\|p\| - m)}$.

Note that from Lemma 2.15, the integral $\int_1^R \frac{(r-1)r^{N-s-1}}{r+a} dr$ is a non-negative and increasing function of N for $1 \leq N \leq n$ therefore, for $0 \leq s < n$, we have

$$\int_1^R \frac{(r-1)r^{N-s-1}}{r+a} dr \leq \int_1^R \frac{(r-1)r^{n-s-1}}{r+a} dr. \tag{2.24}$$

Noting from Lemma 2.18 that $(1-a) \geq 0$ and using inequality (2.24) to (2.23), we have for every N, $1 \leq N \leq n$,

$$|p^{(s)}(Re^{i\theta}) - p^{(s)}(e^{i\theta})| \leq \frac{n(n-1)(n-2)...(n-s)(R^{n-s} - 1)}{(1+k^{s+1})(n-s)}(\|p\| - m) \tag{2.25}$$

$$- \frac{n(n-1)(n-2)...(n-s)}{1+k^{s+1}}(\|p\| - m)(1-a) \times \int_1^R \frac{(r-1)r^{N-s-1}}{r+a} dr. \tag{2.26}$$

For $1 \leq r \leq R$, we have $1 \geq \frac{1}{r^s} \geq \frac{1}{R^s}$. Therefore for $r^{N-s-1} = \frac{r^{N-1}}{r^s} \geq \frac{r^{N-1}}{R^s}$, we have

$$-\int_1^R \frac{(r-1)r^{N-s-1}}{r+a}dr \leq \frac{-1}{R^s}\int_1^R \frac{(r-1)r^{N-1}}{r+a}dr. \tag{2.27}$$

Using inequality (2.27) in (2.25), we have

$$\begin{aligned}|p^{(s)}(Re^{i\theta}) - p^{(s)}(e^{i\theta})| &\leq \frac{n(n-1)(n-2)...(n-s)(R^{n-s}-1)}{(1+k^{s+1})(n-s)}(\|p\|-m)\\ &\quad -\frac{n(n-1)(n-2)...(n-s)}{1+k^{s+1}}(\|p\|-m)\frac{(1-a)}{R^s}\times\int_1^R \frac{(r-1)r^{N-1}}{r+a}dr. \end{aligned}\tag{2.28}$$

Using Lemma 2.14 (on replacing x by a) for the value of integral in (2.28), we have for each $0 \leq \theta < 2\pi$,

$$\begin{aligned}|p^{(s)}(Re^{i\theta}) - p^{(s)}(e^{i\theta})| &\leq \frac{n(n-1)(n-2)...(n-s)(R^{n-s}-1)}{(1+k^{s+1})(n-s)}(\|p\|-m)\\ &\quad - \frac{n(n-1)(n-2)...(n-s)}{(1+k^{s+1})R^s}(\|p\|-m)(1-a)I(N),\end{aligned}\tag{2.29}$$

where $I(N)$ is as defined in (2.2) and (2.3). Now, substituting the value of a and using the obvious inequality

$$\begin{aligned}|p^{(s)}(Re^{i\theta})| &\leq |p^{(s)}(Re^{i\theta}) - p^{(s)}(e^{i\theta})| + |p^{(s)}(e^{i\theta})|\\ &\leq |p^{(s)}(Re^{i\theta}) - p^{(s)}(e^{i\theta})| + \|p^{(s)}\|\end{aligned}$$

in (2.29), we get for $0 \leq s < n$ and $R \geq 1$,

$$\begin{aligned}|p^{(s)}(Re^{i\theta})| &\leq \|p^{(s)}\| + \frac{n(n-1)(n-2)...(n-s)(R^{n-s}-1)}{(1+k^{s+1})(n-s)}(\|p\|-m)\\ &\quad -\frac{n(n-1)(n-2)...(n-s)}{(1+k^{s+1})R^s}(\|p\|-m)\left[1-\frac{a_n(1+k^{s+1})}{\|p\|-m}\right]I(N),\end{aligned}$$

which is equivalent to inequality (2.1) and hence the proof of Theorem 2.1 is completed.□

Proof .[Proof of Theorem 2.8] For each θ, $0 \leq \theta < 2\pi$ and $1 \leq r \leq R$, we have for $0 \leq s < n$,

$$p^{(s)}(Re^{i\theta}) - p^{(s)}(e^{i\theta}) = \int_1^R e^{i\theta}p^{(s+1)}(re^{i\theta})dr,$$

which implies

$$|p^{(s)}(Re^{i\theta}) - p^{(s)}(e^{i\theta})| \leq \int_1^R |p^{(s+1)}(re^{i\theta})|dr.$$

Now, applying Lemma 2.12 to the polynomial $p^{(s+1)}(z)$ which is of degree $\{n-(s+1)\}$, we get

$$|p^{(s)}(Re^{i\theta}) - p^{(s)}(e^{i\theta})| \leq \int_1^R r^{n-(s+1)}\left\{1-\frac{(\|p^{(s+1)}\| - n(n-1)(n-2)...(n-s)|a_n|)(r-1)}{n(n-1)(n-2).....(n-s)|a_n| + r\|p^{(s+1)}\|}\right\}\|p^{(s+1)}\|dr. \tag{2.30}$$

By Lemma 2.13, the quantity $\left\{1-\frac{(\|p^{(s+1)}\|-n(n-1)(n-2)....(n-s)|a_n|)(r-1)}{n(n-1)(n-2)....(n-s)|a_n|+r\|p^{(s+1)}\|}\right\}\|p^{(s+1)}\|$ occuring in the integrand of

(2.30) is an increasing function of $\|p^{(s+1)}\|$, and using Lemma 2.17, we have for $0 \leq \theta < 2\pi$,

$$|p^{(s)}(Re^{i\theta}) - p^{(s)}(e^{i\theta})| \leq \int_1^R r^{n-(s+1)} \left[1 - \frac{\left\{\frac{n(n-1)(n-2)....(n-s)}{1+\gamma_{k,s}}\|p\| - n(n-1)(n-2)...(n-s)|a_n|\right\}(r-1)}{n(n-1)(n-2)...(n-s)|a_n| + r(\frac{n(n-1)(n-2)...(n-s)}{1+\gamma_{k,s}})\|p\|}\right]$$

$$\times \frac{n(n-1)(n-2)...(n-s)}{1+\gamma_{k,s}}\|p\|dr$$

$$= \frac{n(n-1)(n-2)...(n-s)}{1+\gamma_{k,s}}\|p\| \int_1^R r^{n-s-1}dr - \frac{n(n-1)(n-2)...(n-s)}{1+\gamma_{k,s}}\|p\|$$

$$\times \int_1^R r^{n-s-1}\left\{\frac{\|p\| - (1+\gamma_{k,s})|a_n|}{(1+\gamma_{k,s})|a_n| + r\|p\|}\right\}(r-1)dr$$

$$= \frac{n(n-1)(n-2)...(n-s)(R^{n-s}-1)}{(1+\gamma_{k,s})(n-s)}\|p\|$$

$$- \frac{n(n-1)(n-2)...(n-s)}{1+\gamma_{k,s}}\|p\|(1-b)\int_1^R \frac{(r-1)r^{n-s-1}}{r+b}dr, \quad (2.31)$$

where $\gamma_{k,s}$ is as defined in (2.12) and $b = \frac{|a_n|(1+\gamma_{k,s})}{\|p\|}$. Note that from Lemma 2.15, the integral $\int_1^R \frac{(r-1)r^{N-s-1}}{r+b}dr$ is a non-negative and increasing function of N for $1 \leq N \leq n$ therefore, for $0 \leq s < n$, we have

$$\int_1^R \frac{(r-1)r^{N-s-1}}{r+b}dr \leq \int_1^R \frac{(r-1)r^{n-s-1}}{r+b}dr. \quad (2.32)$$

Noting from Lemma 2.19 that $(1-b) \geq 0$ and using inequality (2.32) to (2.31), we have for every N, $1 \leq N \leq n$,

$$|p^{(s)}(Re^{i\theta}) - p^{(s)}(e^{i\theta})| \leq \frac{n(n-1)(n-2)...(n-s)(R^{n-s}-1)}{(1+\gamma_{k,s})(n-s)}\|p\| - \frac{n(n-1)(n-2)...(n-s)}{1+\gamma_{k,s}}\|p\|(1-b)$$
$$\times \int_1^R \frac{(r-1)r^{N-s-1}}{r+b}dr. \quad (2.33)$$

For $1 \leq r \leq R$, we have $1 \geq \frac{1}{r^s} \geq \frac{1}{R^s}$. Therefore for $r^{N-s-1} = \frac{r^{N-1}}{r^s} \geq \frac{r^{N-1}}{R^s}$, we have

$$-\int_1^R \frac{(r-1)r^{N-s-1}}{r+b}dr \leq \frac{-1}{R^s}\int_1^R \frac{(r-1)r^{N-1}}{r+b}dr. \quad (2.34)$$

Using inequality (2.34) in (2.33), we have

$$|p^{(s)}(Re^{i\theta}) - p^{(s)}(e^{i\theta})| \leq \frac{n(n-1)(n-2)...(n-s)(R^{n-s}-1)}{(1+\gamma_{k,s})(n-s)}\|p\| - \frac{n(n-1)(n-2)...(n-s)}{1+\gamma_{k,s}}\|p\|\frac{(1-b)}{R^s}$$
$$\times \int_1^R \frac{(r-1)r^{N-1}}{r+b}dr. \quad (2.35)$$

Using Lemma 2.14 (on replacing x by b) for the value of integral in (2.35), we have for each $0 \leq \theta < 2\pi$,

$$|p^{(s)}(Re^{i\theta}) - p^{(s)}(e^{i\theta})| \leq \frac{n(n-1)(n-2)...(n-s)(R^{n-s}-1)}{(1+\gamma_{k,s})(n-s)}\|p\| - \frac{n(n-1)(n-2)...(n-s)}{(1+\gamma_{k,s})R^s}\|p\|(1-b)J(N), \quad (2.36)$$

where $J(N)$ is as defined in (2.10) and (2.11). Now, substituting the value of b and using the obvious inequality

$$|p^{(s)}(Re^{i\theta})| \leq |p^{(s)}(Re^{i\theta}) - p^{(s)}(e^{i\theta})| + |p^{(s)}(e^{i\theta})|$$
$$\leq |p^{(s)}(Re^{i\theta}) - p^{(s)}(e^{i\theta})| + \|p^{(s)}\|$$

in (2.36), we get for $0 \leq \theta < 2\pi$ and $R \geq 1$,

$$|p^{(s)}(Re^{i\theta})| \leq \|p^{(s)}\| + \frac{n(n-1)(n-2)...(n-s)(R^{n-s}-1)}{(1+\gamma_{k,s})(n-s)}\|p\| - \frac{n(n-1)(n-2)...(n-s)}{(1+\gamma_{k,s})R^s}\|p\|\left[1 - \frac{|a_n|(1+\gamma_{k,s})}{\|p\|}\right]J(N),$$

which is equivalent to inequality (2.9) of Theorem 2.8. □

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