Int. J. Nonlinear Anal. Appl. 14 (2023) 10, 247–256
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.27512.3632

On the zonal function and the Faraut-Korányi hypergeometric function in rank two

Fouziz El Wassouli, Daoud Oukacha

Department of Mathematics, Faculty of Sciences, University Hassan II, Casablanca, Morocco

(Communicated by Mugur Alexandru Acu)

Abstract

In this paper we give a complete description of the generalized hypergeometric functions, introduced by Faraut and Korányi on the Cartan domains. We establish some recursive relations with different arguments of zonal functions and some Gauss type contiguous relations between the Faraut-Korányi hypergeometric functions on the domains of rank two. Finally, we give an infinite sums involving classical hypergeometric functions.

Keywords: Gindikin Gamma function, generalized hypergeometric functions, zonal polynomials, Cartan domains
2020 MSC: 33C70, 33D70, 33C67, 33C45, 33C50

1 Introduction and notations

Let $\mathcal{D} \subset \mathbb{C}^d$ be a Cartan domain, i.e. $\mathcal{D}$ is an irreducible bounded symmetric domain in the Harish-Chandra realization. This is equivalent to saying that $\mathcal{D}$ is the open unit ball of $\mathbb{C}^d$ with respect to a certain norm $||.||$ such that the group $G := Aut(\mathcal{D})$ of all biholomorphic automorphisms of $\mathcal{D}$ acts transitively on $\mathcal{D}$.

By [8] there exists a triple product $\{.,.,.\} : \quad \mathbb{C}^d \times \mathbb{C}^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d$, so that $V^{\mathbb{C}} := (\mathbb{C}^d, ||.||, \{.,.,.\})$ is a Jordan-Banach*-triple(JB*-triple).

The maximal compact subgroup of G is $K := \{g \in G; \quad g(0) = 0\} = G \cap GL(V^{\mathbb{C}})$, and $\mathcal{D} = G/K$. We denote by r, a, b, d and p the rank, the characteristic multiplicities, the dimension and the genus of $\mathcal{D}$, respectively

$$d = r + \frac{r(r-1)}{2}a + rb, \quad p = 2 + (r-1)a + b. \tag{1.1}$$

In Table 1, $\mathbb{O}$ is the 8-dimensional Cayley algebra. Under the action $f \longmapsto f \circ k \, (k \in K)$ of K, the space $\mathcal{P}$ of holomorphic polynomials has the following Peter-Weyl decomposition (see [4])

$$\mathcal{P} = \bigoplus_{\mathbf{m} \in \Lambda} \mathcal{P}_{\mathbf{m}}$$

where Λ is the set of all r-tuples, $\mathbf{m} = (m_1, m_2, ..., m_r) \in \mathbb{Z}^r$ with $m_1 \geq m_2 \geq ... \geq m_r \geq 0$. For each $\mathbf{m}$, the elements of $\mathcal{P}_{\mathbf{m}}$ are homogeneous polynomials of degree $|\mathbf{m}| : m_1 + m_2 + ... + m_r$. Equipped with the Fischer (or Fock) scalar

Email addresses: elwassouli@gmail.com (Fouziz El Wassouli), daoudoukacha@gmail.com (Daoud Oukacha)

Received: July 2022 *Accepted:* August 2023

$\mathcal{D}$	$V^{\mathbb{C}}$	$\dim V$	(r, a, b)
$I_{n,m}(n \leq m)$	$M_{n,m}(\mathbb{C})$	nm	(n, 2, m-n)
$II_n(n\ even)$	$\{z \in M_n(\mathbb{C});\quad {}^tz = -z\}$	$\frac{n}{2}(n-1)$	$(\frac{n}{2}, 4, 0)$
$II_n(n\ odd)$	$\{z \in M_n(\mathbb{C});\quad {}^tz = -z\}$	$n(\frac{n-1}{2})$	$(\frac{n-1}{2}, 4, 2)$
III_n	$\{z \in M_n(\mathbb{C});\quad {}^tz = z\}$	$\frac{1}{2}n(n+1)$	(n, 1, 0)
IV_n	$\mathbb{C}^n$	n	(2, n-2, 0)
V	$M_{1,2}(\mathbb{O})$	16	(2, 6, 4)
VI	$\{z \in M_{3,3}(\mathbb{O});\quad {}^t\bar{z} = z\}$	27	(3, 8, 0)

Table 1: Irreducible bounded symmetric domains of non-compact type.

product

$$\begin{aligned}\langle f, g\rangle : &= f(\partial)\overline{g(\bar{z})}\ |_{z=0} \\ &= \pi^{-d}\int_{\mathbb{C}^d} f(x)\overline{g(x)}e^{-|x|^2}dm(x),\end{aligned}$$

each space $\mathcal{P}_{\mathbf{m}}$ becomes a finite-dimensional Hilbert space of function on $\mathbb{C}^d$, and thus has a reproducing kernel $K^{\mathbf{m}}(x, y)$, holomorphic in x and anti-holomorphic in y.

The Faraut-Korányi functions [3] on $\mathcal{D}$ are defined by

$$_2F_1^{(a)}(\alpha, \beta; \gamma; x) \quad := \quad \sum_{\mathbf{m}\geq 0} \frac{(\alpha)_{\mathbf{m}}(\beta)_{\mathbf{m}}}{(\gamma)_{\mathbf{m}}} K^{\mathbf{m}}(x, e). \tag{1.2}$$

Here $(.)_{\mathbf{m}}$ is the generalized Pochhammer symbol

$$(\alpha)_{\mathbf{m}} := \prod_{j=1}^{r}(\alpha - \frac{j-1}{2}a)_{m_j}, \quad where \quad (\alpha)_k := \alpha(\alpha+1)...(\alpha+k-1). \tag{1.3}$$

We fix a Jordan frame $e_1, e_2, ..., e_r \in \mathbb{C}^d$, so that each $x \in \mathbb{C}^d$ has the polar decomposition

$$x = k\left(\sum_{i=1}^{r} t_i e_i\right),\ k \in K,\ t_1 \geq t_2 \geq ... \geq t_r \geq 0.$$

The point $e = e_1 + e_2 + ... + e_r$ belongs to the Shilov boundary S of $\mathcal{D}$. The group K acts transitively on S, so that $S = \{ke,\ k \in K\} \simeq K/L$, where L is the stabilizer of e in K. Each Peter-Weyl space $\mathcal{P}_{\mathbf{m}}$ contains a unique L-invariant polynomial $\phi_{\mathbf{m}}^{(a)}$ satisfying the normalization condition $\phi_{\mathbf{m}}^{(a)}(e) = 1$. We will usually write $\phi_{\mathbf{m}}^{(a)}(t_1, ..., t_r)$ instead of $\phi_{\mathbf{m}}^{(a)}(t_1e_1 + ... + t_re_r)$. The polynomials $\phi_{\mathbf{m}}^{(a)}$ satisfy

$$\phi_{(0,0,...,0)}^{(a)} = 1, \quad \phi_{(m_1+1, m_2+1, ..., m_r+1)}^{(a)}(t_1, t_2, ..., t_r) = t_1t_2...t_r\phi_{\mathbf{m}}^{(a)}(t_1, ..., t_r),$$

and are related to the reproducing kernels $K^{\mathbf{m}}$ by the formula we have

$$K^{\mathbf{m}}(x, e) = \frac{d_{\mathbf{m}}}{(\frac{d}{r})_{\mathbf{m}}}\phi_{\mathbf{m}}^{(a)}(x), \tag{1.4}$$

where $d_{\mathbf{m}} := \dim \mathcal{P}_{\mathbf{m}}$ (see [12]). It is known that the last dimension is given by the formula

$$d_{\mathbf{m}} = \frac{(\frac{d}{r})_{\mathbf{m}}}{(q)_{\mathbf{m}}}\pi_{\mathbf{m}}$$

where

$$q := \frac{r-1}{2}a + 1$$

and

$$\pi_{\mathbf{m}} := \prod_{1\le i<j\le r} \frac{m_i - m_j + \frac{j-1}{2}a}{\frac{j-1}{2}a} \frac{(\frac{j-i+1}{2}a)_{m_i-m_j}}{(\frac{j-i-1}{2}a+1)_{m_i-m_j}}.$$

Therefore, the Faraut-Korányi hypergeometric functions on $\mathcal{D}$ can be written as

$${}_2F_1^{(a)}(\alpha,\beta;\gamma;x) \quad := \quad \sum_{\mathbf{m}\ge 0} \frac{(\alpha)_{\mathbf{m}}(\beta)_{\mathbf{m}}}{(\gamma)_{\mathbf{m}}} \frac{\pi_{\mathbf{m}}}{(q)_{\mathbf{m}}} \phi_{\mathbf{m}}^{(a)}(x), \tag{1.5}$$

and the Faraut-Korányi hypergeometric functions ${}_2F_1^{(a)}(\alpha,\beta;\gamma;x)$ is invariant under L acting on x. Then, if $x = k\left(\sum_{j=1}^{r} t_j c_j\right)$, $k \in L$ is the spectral decomposition of x, we have

$${}_2F_1^{(a)}(\alpha,\beta;\gamma;x) ={}_2 F_1^{(a)}(\alpha,\beta;\gamma;t_1,t_2,...,t_r). \tag{1.6}$$

The six classical hypergeometric functions ${}_2F_1(a\pm 1,b;c;z)$, ${}_2F_1(a,b\pm 1;c;z)$, ${}_2F_1(a,b;c\pm 1;z)$ associated with ${}_2F_1(a,b;c;z)$ are called contiguous to it. A relations between ${}_2F_1(a,b;c;z)$ and any two contiguous functions is called a contiguous relation. These relations were given by Gauss and he used them to derive basic formulas.

These contiguous hypergeometric functions have some interesting applications. Indeed, contiguous relations gives an intertwining correspondence between Lie algebras and special functions (see [9]). They are also very useful in the derivation of summation and transformation formulas for hypergeometric series.

Recently, many people studies the contiguous hypergeometric functions and gives some new properties, consequences and interesting applications (see [1, 5, 6, 7, 10, 11, 13]).

The aim of this paper is to give some Gauss type contiguous relations between the Faraut-Korányi hypergeometric functions on the Cartan domains of rank 2.

The proofs rely on an interesting integral representation for the zonal functions $\phi_{\mathbf{m}}^{(a)}$: Namely

$$\phi_{\mathbf{m}}^{(a)}(t_1,t_2) = \frac{\Gamma(a)}{\Gamma(\frac{a}{2})^2}\int_0^1 \left[t_1-(t_1-t_2)y\right]^{m_1-m_2}(t_1t_2)^{m_2}\left[y(1-y)\right]^{\frac{a}{2}-1}dy. \tag{1.7}$$

The remainder of the paper is organized as follows. Section2 is devoted to the recursive relations with different arguments of zonal functions. In Section3, we get some Gauss type contiguous relations between the Faraut-Korányi hypergeometric functions on the domain of rank two. Finally, we give an infinite sums involving classical hypergeometric functions in Section4.

2 Relations between contiguous zonal functions

In this section we give the zonal partial differential equation and we establish the recursive relations with different arguments of zonal functions. To this end, we need to introduce the associated Legendre differential equation and the following relations between contiguous associated Legendre functions of the 2^{nd} kind.

$$\begin{aligned}
(1-x^2)\frac{d^2Q_\nu^\mu(x)}{dx^2} &= 2x\frac{dQ_\nu^\mu(x)}{dx} - \left[\nu(\nu+1) - \frac{\mu^2}{(1-x^2)}\right]Q_\nu^\mu(x), && (2.1)\\
Q_{\nu-1}^\mu(x) &= Q_{\nu+1}^\mu(x) + (2\nu+1)(1-x^2)^{\frac{1}{2}}Q_\nu^{\mu-1}(x), && (2.2)\\
(2\nu+1)xQ_\nu^\mu(x) &= (\nu-\mu+1)Q_{\nu+1}^\mu(x) + (\nu+\mu)Q_{\nu-1}^\mu(x), && (2.3)\\
(1-x^2)\frac{dQ_\nu^\mu(x)}{dx} &= (\nu+1)xQ_\nu^\mu(x) - (\nu-\mu+1)Q_{\nu+1}^\mu(x), && (2.4)\\
&= -\nu xQ_\nu^\mu(x) + (\nu+\mu)Q_{\nu-1}^\mu(x). && (2.5)
\end{aligned}$$

The main result of this section can be stated as follows.

Theorem 2.1. For given $(t_1,t_2)\in\mathbb{R}^2$ such that $t_1>t_2>0$, we have the following zonal partial differential equation

$$\begin{aligned}
&\sum_{j=1}^{2}\Big(\frac{\partial^2}{\partial t_j\partial t_{3-j}}-\frac{t_{3-j}}{t_j}\frac{\partial^2}{\partial t_{3-j}^2}-\Big[1+\frac{a}{t_j}\Big(\frac{t_j+t_{3-j}}{t_{3-j}-t_j}\Big)\Big]\frac{\partial}{\partial t_{3-j}}\Big)\phi^{(a)}_{m_1,m_2}(t_1,t_2)\\
=&\frac{1}{t_1t_2}\Big[\frac{a^2t_1t_2}{(t_1-t_2)^2}+\frac{a(t_1+t_2)}{2(t_1-t_2)}+\frac{a}{4(t_1-t_2)^2}\Big(a(t_1-t_2)^2-2(t_1^2+t_2^2)\Big)\\
&+\Big(m_1-m_2+\frac{a}{2}\Big)\Big(m_1-m_2+\frac{3a}{4}+1\Big)\Big]\phi^{(a)}_{m_1,m_2}(t_1,t_2).
\end{aligned}\tag{2.6}$$

To prove the above theorems we will need the two following lemmas.

Lemma 2.2. We have,

$$\frac{d}{d\Big(\frac{t_1+t_2}{t_1-t_2}\Big)}=\frac{(t_1-t_2)^2}{4}\Big[\frac{1}{t_1}\frac{\partial}{\partial t_2}-\frac{1}{t_2}\frac{\partial}{\partial t_1}\Big]\tag{2.7}$$

$$\frac{d^2}{d\Big(\frac{t_1+t_2}{t_1-t_2}\Big)^2}=\frac{(t_1-t_2)^3}{16}\Big[\frac{3t_1+t_2}{t_1t_2^2}\frac{\partial}{\partial t_1}-\frac{3t_2+t_1}{t_2t_1^2}\frac{\partial}{\partial t_2}+(t_1-t_2)\Big(\frac{1}{t_1^2}\frac{\partial^2}{\partial t_2^2}+\frac{1}{t_2^2}\frac{\partial^2}{\partial t_1^2}-\frac{2}{t_1t_2}\frac{\partial^2}{\partial t_1\partial t_2}\Big)\Big].\tag{2.8}$$

Proof . In order to prove the (2.7) we identify $f(\frac{t_1+t_2}{t_1-t_2})$ with the function $f\circ h(t_1,t_2)$ for all function which is two times differentiable $f:\ t\longmapsto f(t)$, where $h(t_1,t_2)=\frac{t_1+t_2}{t_1-t_2}$ and $\frac{df(t)}{dt}$ can be written as the form

$$\frac{df(t)}{dt}=\frac{df(\frac{t_1+t_2}{t_1-t_2})}{d(\frac{t_1+t_2}{t_1-t_2})}=f'(t)\quad with\quad t=\frac{t_1+t_2}{t_1-t_2}.$$

On the one hand

$$dt=d(\frac{t_1+t_2}{t_1-t_2})=\frac{-2t_2}{(t_1-t_2)^2}dt_1+\frac{2t_1}{(t_1-t_2)^2}dt_2.$$

Thus

$$df(t)=f'(t)dt=\frac{df(\frac{t_1+t_2}{t_1-t_2})}{d(\frac{t_1+t_2}{t_1-t_2})}\times\Big[\frac{-2t_2}{(t_1-t_2)^2}dt_1+\frac{2t_1}{(t_1-t_2)^2}dt_2\Big].$$

On the other hand

$$df(t)=df\circ h(t_1,t_2)=\frac{\partial f(\frac{t_1+t_2}{t_1-t_2})}{\partial t_1}dt_1+\frac{\partial f(\frac{t_1+t_2}{t_1-t_2})}{\partial t_2}dt_2.$$

Therefore,

$$\frac{df(\frac{t_1+t_2}{t_1-t_2})}{d(\frac{t_1+t_2}{t_1-t_2})}=-\frac{(t_1-t_2)^2}{2t_2}\frac{\partial f(\frac{t_1+t_2}{t_1-t_2})}{\partial t_1}=\frac{(t_1-t_2)^2}{2t_1}\frac{\partial f(\frac{t_1+t_2}{t_1-t_2})}{\partial t_2}.$$

Hence

$$\frac{df(\frac{t_1+t_2}{t_1-t_2})}{d(\frac{t_1+t_2}{t_1-t_2})}=\frac{(t_1-t_2)^2}{4}\Big[\frac{1}{t_1}\frac{\partial f(\frac{t_1+t_2}{t_1-t_2})}{\partial t_2}-\frac{1}{t_2}\frac{\partial f(\frac{t_1+t_2}{t_1-t_2})}{\partial t_1}\Big].$$

By straightforward computation, we get (2.8). □

As an immediate consequence of the Lemma 2.2 is the following.

Corollary 2.3. The Legendre differential equation (2.1) can be written as

$$\begin{aligned}
&\frac{4}{(t_1-t_2)^2}\left[\nu(\nu+1)+\frac{(t_1-t_2)^2\mu^2}{4t_1t_2}\right]Q^{\mu}_{\nu}(\frac{t_1+t_2}{t_1-t_2})\\
=&\left(\frac{1}{t_1}\frac{\partial}{\partial t_2}+\frac{1}{t_2}\frac{\partial}{\partial t_1}\right)Q^{\mu}_{\nu}(\frac{t_1+t_2}{t_1-t_2})+\left[2\frac{\partial^2}{\partial t_1\partial t_2}-\frac{t_2}{t_1}\frac{\partial^2}{\partial t_2^2}-\frac{t_1}{t_2}\frac{\partial^2}{\partial t_1^2}\right]Q^{\mu}_{\nu}(\frac{t_1+t_2}{t_1-t_2}).
\end{aligned}\tag{2.9}$$

Lemma 2.4. We have,

$$\phi^{(a)}_{m_1,m_2}(t_1,t_2) = 2^{\frac{a}{2}}\frac{\Gamma(a)(t_1t_2)^{\frac{m_1+m_2+\frac{a}{2}}{2}}e^{-i\pi(m_1-m_2+\frac{a}{2})}}{(t_1-t_2)^{\frac{a}{2}}\Gamma(\frac{a}{2})\Gamma(m_1-m_2+\frac{a}{2})}Q^{(m_1-m_2+\frac{a}{2})}_{\frac{a}{2}}\Big(\frac{t_1+t_2}{t_1-t_2}\Big), \quad i=\sqrt{-1}$$

Proof . Observe, first of all, that

$$(1-t_1)^{-\frac{a}{2}}(1-t_2)^{-\frac{a}{2}} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{(\frac{a}{2})_{n_1}(\frac{a}{2})_{n_2}}{n_1!n_2!}t_1^{n_1}t_2^{n_2}$$

and

$$\begin{aligned}\int_0^1(yt_2+t_1(1-y))^n(y(1-y))^{\frac{a}{2}-1}dy &= \sum_{n_1+n_2=n}\frac{n!}{n_1!n_2!}t_1^{n_1}t_2^{n_2}\int_0^1 y^{n_2+\frac{a}{2}-1}(1-y)^{n_1+\frac{a}{2}-1}dy\\ &= \sum_{n_1+n_2=n}\frac{n!}{n_1!n_2!}t_1^{n_1}t_2^{n_2}\frac{\Gamma(n_1+\frac{a}{2})\Gamma(n_2+\frac{a}{2})}{\Gamma(n+a)}\\ &= \frac{n!\Gamma^2(\frac{a}{2})}{\Gamma(n+a)}\sum_{n_1+n_2=n}\frac{(\frac{a}{2})_{n_1}(\frac{a}{2})_{n_2}}{n_1!n_2!}t_1^{n_1}t_2^{n_2}.\end{aligned}$$

Consequently

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\Gamma(n+a)}{n!\Gamma^2(\frac{a}{2})}\int_0^1(yt_2+t_1(1-y))^n(y(1-y))^{\frac{a}{2}-1}dy &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{(\frac{a}{2})_{n_1}(\frac{a}{2})_{n_2}}{n_1!n_2!}t_1^{n_1}t_2^{n_2}\\ &= \prod_{j=1}^{2}(1-t_j)^{-\frac{a}{2}}.\end{aligned}$$

On the other hand, by the Faraut-Korànyi formula (see [3, 4]),

$$\begin{aligned}\prod_{j=1}^{2}(1-t_j)^{-\frac{a}{2}} &= \sum_{\mathbf{m}\geq 0}^{\infty}(\frac{a}{2})_{\mathbf{m}}\frac{\pi_{\mathbf{m}}}{(q)_{\mathbf{m}}}\phi^{(a)}_{\mathbf{m}}(t_1,t_2)\\ &= \sum_{n=0}^{\infty}(\frac{a}{2})_n\frac{\pi_{(n,0)}}{(q)_{(n,0)}}\phi^{(a)}_{n,0}(t_1,t_2)\end{aligned}$$

since $(\frac{a}{2})_{\mathbf{m}}=0$ if $m_2>0$. Since both $\phi^{(a)}_{n,0}$ and $\int_0^1(yt_2+t_1(1-y))^n(y(1-y))^{\frac{a}{2}-1}dy$ are homogeneous polynomials in t_1,t_2 of degree n, comparing the two expansion shows that

$$\phi^{(a)}_{n,0}(t_1,t_2) = \frac{\Gamma(a)}{\Gamma(\frac{a}{2})^2}\int_0^1\left[t_1-(t_1-t_2)y\right]^n\left[y(1-y)\right]^{\frac{a}{2}-1}dy.$$

By using the fact that the polynomial $\phi^{(a)}_{\mathbf{m}}$ satisfy $\phi^{(a)}_{m_1+1,m_2+1}(t_1,t_2)=t_1t_2\phi^{(a)}_{\mathbf{m}}(t_1,t_2)$, we get $\phi^{(a)}_{m_1,m_2}(t_1,t_2) = (t_1t_2)^{m_2}\phi^{(a)}_{m_1-m_2,0}(t_1,t_2)$. This leads to (1.7) and a change of variable to $y=\frac{1-\cos\theta}{2}$ eventually leads to

$$\begin{aligned}\phi^{(a)}_{\mathbf{m}}(t_1,t_2) &= \frac{\Gamma(a)(t_1t_2)^{m_2}}{\Gamma^2(\frac{a}{2})2^{m_1-m_2}}\int_0^1\Big[(t_1+t_2)+(t_1-t_2)(1-2y)\Big]^{m_1-m_2}[y(1-y)]^{\frac{a}{2}-1}dy\\ &= \frac{\Gamma(a)(t_1t_2)^{m_2}}{\Gamma^2(\frac{a}{2})2^{(m_1-m_2+\frac{a}{2}+1)}}\int_0^{\pi}\Big[(t_1+t_2)+(t_1-t_2)\cos\theta\Big]^{m_1-m_2}\sin^{a-1}\theta d\theta\\ &= \frac{\Gamma(a)(t_1t_2)^{m_2}(t_1-t_2)^{m_1-m_2}}{\Gamma^2(\frac{a}{2})2^{(m_1-m_2+\frac{a}{2})}}\int_0^{\pi}\Big[\frac{(t_1+t_2)}{(t_1-t_2)}+\cos\theta\Big]^{m_1-m_2}\sin^{a-1}\theta d\theta\\ &= 2^{\frac{a}{2}}\frac{\Gamma(a)(t_1t_2)^{\frac{m_1+m_2+\frac{a}{2}}{2}}e^{-i\pi(m_1-m_2+\frac{a}{2})}}{(t_1-t_2)^{\frac{a}{2}}\Gamma(\frac{a}{2})\Gamma(m_1-m_2+\frac{a}{2})}Q^{(m_1-m_2+\frac{a}{2})}_{\frac{a}{2}}\Big(\frac{t_1+t_2}{t_1-t_2}\Big).\end{aligned}$$

□

Proof .(Theorem 2.1) The Theorem 2.1 can be prove directly from Corollary 2.3 by applying again the Lemma 2.4. □

Now, we summaries some other results in the following.

Theorem 2.5. For given $(t_1, t_2) \in \mathbb{R}^2$ such that $t_1 > t_2 > 0$, we have the following recursive relations with different arguments

$$\phi^{(a)}_{m_1,m_2}(t_1,t_2) = \frac{4(m_1-m_2+a)}{(t_1+t_2)}\frac{a-1}{a+1}\phi^{(a-2)}_{m_1+1,m_2}(t_1,t_2) - \frac{(m_1-m_2+1)(t_1-t_2)^2}{4(a+1)^2(t_1+t_2)t_1t_2}\phi^{(a+2)}_{m_1,m_2+1}(t_1,t_2) \tag{2.10}$$

$$\phi^{(a)}_{m_1,m_2}(t_1,t_2) = \frac{16(a-3)(a-1)}{(t_1-t_2)^2}\phi^{(a-4)}_{m_1+2,m_2}(t_1,t_2) - 8\frac{(a-1)^2t_1t_2}{(m_1-m_2+\frac{a}{2}-1)(t_1-t_2)^2}\phi^{(a-2)}_{m_1,m_2}(t_1,t_2) \tag{2.11}$$

$$\phi^{(a-2)}_{m_1+1,m_2}(t_1,t_2) = \frac{t_1-t_2}{4(m_1-m_2+a)(a-1)} \times \Big[t_1\frac{\partial\phi^{(a)}_{m_1,m_2}(t_1,t_2)}{\partial t_1} - t_2\frac{\partial\phi^{(a)}_{m_1,m_2}(t_1,t_2)}{\partial t_2}\Big] \tag{2.12}$$

$$\phi^{(a)}_{m_1,m_2}(t_1,t_2) = \frac{(t_1-t_2)}{(a+1)(t_1+t_2)} \times \Big[\frac{(m_1-m_2-1)(t_2-t_1)}{4(a+1)}\phi^{(a+2)}_{m_1,m_2+1}(t_1,t_2) + t_1\frac{\partial\phi^{(a)}_{m_1,m_2}(t_1,t_2)}{\partial t_1} - t_2\frac{\partial\phi^{(a)}_{m_1,m_2}(t_1,t_2)}{\partial t_2}\Big]. \tag{2.13}$$

Proof . From formulae (2.7) and Lemma 2.4 , we easily verify by straightforward computation that the equations (2.2), (2.3), (2.4), (2.5) implies the equations (2.10), (2.11), (2.12), (2.13) respectively. □

Some applications of the Theorem 2.1 obtained above will be discussed in the following section.

3 Gauss type contiguous relations between Faraut-Korányi hypergeometric functions

In this section, we establish the Gauss type contiguous relations between the Faraut-Korányi hypergeometric functions on the Cartan domains of rank two generalizing the classical contiguous relation

$$x(1-x)\frac{d}{dx}\,{}_2F_1(\alpha+1,\beta+1;\gamma+1;x) + (\gamma-(\alpha+\beta+1)x)\,{}_2F_1(\alpha+1,\beta+1;\gamma+1;x) = \gamma\,{}_2F_1(\alpha,\beta;\gamma;x),$$

where ${}_2F_1(\alpha,\beta;\gamma;x)$ is the classical Gauss hypergeometric function.

We start with the case when $(t_1,t_2) = (ue^t, ue^{-t})$, then by the formula (1.7), we have

$$\begin{aligned}
\phi^{(a)}_{\mathbf{m}}(ue^t,ue^{-t}) &= \frac{\Gamma(a)}{\Gamma^2(\frac{a}{2})}u^{m_1+m_2}\int_0^1\Big[\cosh t + (1-2y)\sinh t\Big]^{m_1-m_2}\Big(y(1-y)\Big)^{\frac{a}{2}-1}dy \\
&= \frac{\Gamma(a)}{2^{a-1}\Gamma^2(\frac{a}{2})}u^{m_1+m_2}\int_0^\pi(\cosh t+\sinh t\cos\theta)^{m_1-m_2}\sin^{a-1}\theta d\theta \\
&= \frac{\sqrt{\pi}\Gamma(a)}{8\Gamma(\frac{a}{2})\Gamma(\frac{a+1}{2})}u^{m_1+m_2}C^{\frac{a}{2}}_{m_1-m_2}(\cosh t),
\end{aligned}$$

where $C^{\frac{a}{2}}_k$ denotes the ultraspherical (Gegenbauer) polynomial of degree k with index $\frac{a}{2}$.

Notice that $(\alpha)_{\mathbf{m}} = (\alpha)_{(m_1,m_2)} = (\alpha)_{m_1}(\alpha-\frac{a}{2})_{m_2}$ and $(\alpha)_{\mathbf{1}}$ standard for $(\alpha)_{\mathbf{m}}$, with $\mathbf{m} = (1,1)$.

We see that for $\mathbf{m} = (m_1,m_2)$ and $\mathbf{k} = (k,k) \in \mathbb{N}^2$ the following properties, which are useful in this and later sections, are easy to prove

$$(\alpha)_{\mathbf{m}+\mathbf{k}} = (\alpha+\mathbf{k})_{\mathbf{m}}(\alpha)_{\mathbf{k}}. \tag{3.1}$$

The Faraut-Korányi hypergeometric function is given by

$${}_2F_1^{(a)}(\alpha,\beta;\gamma;ue^t,ue^{-t}) = \frac{\sqrt{\pi}}{4a\Gamma(\frac{a}{2})\Gamma(\frac{a+1}{2})}\sum_{m_2=0}^\infty\sum_{m_1=m_2}^\infty\left(\frac{(\alpha)_{\mathbf{m}}(\beta)_{\mathbf{m}}}{(\gamma)_{\mathbf{m}}(\frac{a}{2}+1)_{\mathbf{m}}}\right)(m_1-m_2+\frac{a}{2})\frac{(a)_{m_1-m_2}}{(m_1-m_2)!}u^{m_1+m_2}C^{\frac{a}{2}}_{m_1-m_2}(\cosh t). \tag{3.2}$$

Theorem 3.1. It holds that

$$\begin{aligned}{}_2F_1^{(a)}(\alpha,\beta;\gamma;ue^t,ue^{-t}) =&\frac{(\gamma-1)_{\mathbf{1}}}{(\alpha-1)_{\mathbf{1}}(\beta-1)_{\mathbf{1}}}\Big[\frac{1}{4}\frac{d^2}{du^2}\ {}_2F_1^{(a)}(\alpha-1,\beta-1;\gamma-1;ue^t,ue^{-t})\\ &+\frac{a+1}{4u}\frac{d}{du}\,{}_2F_1^{(a)}(\alpha-1,\beta-1;\gamma-1;ue^t,ue^{-t})\\ &-\frac{a}{4}\frac{\cosh t}{\sinh t}\frac{d}{dt}\ {}_2F_1^{(a)}(\alpha-1,\beta-1;\gamma-1;ue^t,ue^{-t})-\frac{1}{4}\frac{d^2}{dt^2}\ {}_2F_1^{(a)}(\alpha-1,\beta-1;\gamma-1;ue^t,ue^{-t})\Big].\end{aligned}$$

Proof . By an elementary identity (3.1) ${}_2F_1^{(a)}(\alpha,\beta;\gamma;ue^t,ue^{-t})$ can be written as

$$\begin{aligned}{}_2F_1^{(a)}(\alpha,\beta;\gamma;ue^t,ue^{-t}) =&\frac{(\gamma-1)_{\mathbf{1}}}{(\alpha-1)_{\mathbf{1}}(\beta-1)_{\mathbf{1}}}\frac{\sqrt{\pi}}{4a\Gamma(\frac{a}{2})\Gamma(\frac{a+1}{2})}\sum_{m_2=0}^{\infty}\sum_{m_1=m_2}^{\infty}\left(\frac{(\alpha-1)_{\mathbf{m}}(\beta-1)_{\mathbf{m}}}{(\gamma-1)_{\mathbf{m}}(\frac{a}{2}+1)_{\mathbf{m}}}\right)\times\\ &m_2(m_1+\frac{a}{2})(m_1-m_2+\frac{a}{2})\prod_{j=1}^{a-1}(m_1-m_2+j)u^{m_1+m_2-2}C^{\frac{a}{2}}_{m_1-m_2}(\cosh t).\end{aligned}$$

Since

$$m_2(m_1+\frac{a}{2})=\frac{(m_1+m_2)(m_1+m_2-1)}{4}+\frac{(a+1)}{4}(m_1+m_2)=\frac{(m_1-m_2)(m_1-m_2+a)}{4},$$

we have

$$\begin{aligned}{}_2F_1^{(a)}(\alpha,\beta;\gamma;ue^t,ue^{-t}) =&\frac{(\gamma-1)_{\mathbf{1}}}{(\alpha-1)_{\mathbf{1}}(\beta-1)_{\mathbf{1}}}\Big[\frac{1}{4}\frac{d^2}{du^2}\ {}_2F_1^{(a)}(\alpha-1,\beta-1;\gamma-1;ue^t,ue^{-t})\\ &+\frac{a+1}{4u}\frac{d}{du}\,{}_2F_1^{(a)}(\alpha-1,\beta-1;\gamma-1;ue^t,ue^{-t})\\ &-\frac{\sqrt{\pi}}{16a\Gamma(\frac{a}{2})\Gamma(\frac{a+1}{2})}\sum_{m_2=0}^{\infty}\sum_{m_1=m_2}^{\infty}\left(\frac{(\alpha-1)_{\mathbf{m}}(\beta-1)_{\mathbf{m}}}{(\gamma-1)_{\mathbf{m}}(\frac{a}{2}+1)_{\mathbf{m}}}\right)u^{m_1+m_2-2}\times\\ &(m_1-m_2)(m_1-m_2+a)(m_1-m_2+\frac{a}{2})\times\prod_{j=1}^{a-1}(m_1-m_2+j)u^{m_1+m_2-2}C^{\frac{a}{2}}_{m_1-m_2}(\cosh t).\Big]\end{aligned} \tag{3.3}$$

From the fact that $C^{\frac{a}{2}}_{m_1-m_2}(x)$ is a solution of the following Gegenbauer differential equation

$$(m_1-m_2)(m_1-m_2+a)C^{\frac{a}{2}}_{m_1-m_2}(x)=(a+1)x\frac{dC^{\frac{a}{2}}_{m_1-m_2}(x)}{dx}-(1-x^2)\frac{d^2C^{\frac{a}{2}}_{m_1-m_2}(x)}{dx^2}.$$

and

$$\begin{cases}\dfrac{df(\cosh t)}{d(\cosh t)}=\dfrac{1}{\sinh t}\dfrac{df(\cosh t)}{dt}\\ \dfrac{d^2f(\cosh t)}{d(\cosh t)^2}=-\dfrac{\cosh t}{\sinh^3 t}\dfrac{df(\cosh t)}{dt}+\dfrac{1}{\sinh^2 t}\dfrac{d^2f(\cosh t)}{dt^2}\end{cases}$$

we have

$$(m_1-m_2)(m_1-m_2+a)C^{\frac{a}{2}}_{m_1-m_2}(\cosh t)=(a+1-\sinh t)\frac{\cosh t}{\sinh t}\frac{dC^{\frac{a}{2}}_{m_1-m_2}(\cosh t)}{dt}+\frac{d^2C^{\frac{a}{2}}_{m_1-m_2}(\cosh t)}{dt^2}. \tag{3.4}$$

Substituting (3.4) in (3.3). We obtain the desired result. □

We will now generalize the Theorem 3.1 in the case $(t_1,t_2)\in\mathbb{R}^2$ such that $t_1>t_2>0$. More precisely, let ${}_2H_1^{(a)}(\alpha,\beta;\gamma;t_1,t_2)$ denote the following function

$${}_2H_1^{(a)}(\alpha,\beta;\gamma;t_1,t_2)=(t_1-t_2)^{\frac{a}{2}}\ {}_2F_1^{(a)}(\alpha,\beta;\gamma;t_1,t_2).$$

Then, we have

Theorem 3.2. For all $(t_1,t_2)) = u(w_1,w_2) \in \mathbb{R}^2$ such that $t_1 > t_2 > 0$, it holds that

$$\begin{aligned}
{}_2F_1^{(a)}(\alpha,\beta;\gamma;t_1,t_2) =& \frac{(\gamma-1)_{\mathbf{1}}}{(\alpha-1)_{\mathbf{1}}(\beta-1)_{\mathbf{1}}t_1t_2}\Big[\frac{u^2}{4}\frac{d^2}{du^2}\ {}_2F_1^{(a)}(\alpha-1,\beta-1;\gamma-1;t_1,t_2)\\
&+\frac{(3+2a)u}{8}\frac{d}{du}\ {}_2F_1^{(a)}(\alpha-1,\beta-1;\gamma-1;t_1,t_2)\\
&-\frac{t_1t_2}{4(t_1-t_2)^{\frac{a}{2}}}\sum_{j=1}^{2}\Big(\frac{\partial^2}{\partial t_j\partial t_{3-j}}-\frac{t_j}{t_{3-j}}\frac{\partial^2}{\partial t_j^2}+\frac{(-1)^{j+1}}{t_j}\frac{\partial}{\partial t_{3-j}}\Big)\ {}_2H_1^{(a)}(\alpha-1,\beta-1;\gamma-1;t_1,t_2)\\
&+\frac{a(a+1)}{16}\ {}_2F_1^{(a)}(\alpha-1,\beta-1;\gamma-1;t_1,t_2)\Big].
\end{aligned}$$

Proof. By using the identity (3.1) ${}_2F_1^{(a)}(\alpha,\beta;\gamma;t_1,t_2)$ can be written as

$$ {}_2F_1^{(a)}(\alpha,\beta;\gamma;t_1,t_2) = \frac{(\gamma-1)_{\mathbf{1}}}{(\alpha-1)_{\mathbf{1}}(\beta-1)_{\mathbf{1}}t_1t_2}\sum_{\mathbf{m}\geq 0}\frac{(\alpha-1)_{\mathbf{m}}(\beta-1)_{\mathbf{m}}}{(\gamma-1)_{\mathbf{m}}}\frac{\pi_{\mathbf{m}}}{(\frac{a}{2}+1)_{\mathbf{m}}}m_2(\frac{a}{2}+m_1)\phi_{\mathbf{m}}^{(a)}(t_1,t_2). \quad (3.5)$$

From the Corollary 2.3 , the Lemma 2.4 and the fact that

$$m_2(m_1+\frac{a}{2}) = \frac{(m_1+m_2)(m_1+m_2-1)}{4}-\frac{(m_1-m_2+\frac{a}{2})^2}{4}+\frac{a+1}{4}(m_1+m_2)+\frac{a^2}{16},$$

we get

$$\begin{aligned}
& m_2(m_1+\frac{a}{2})(t_1-t_2)^{\frac{a}{2}}\phi_{\mathbf{m}}^{(a)}(t_1,t_2) \qquad (3.6)\\
=\ & \Big[\frac{(m_1+m_2)(m_1+m_2-1)}{4}+\frac{3+2a}{8}(m_1+m_2)+\frac{a(a+1)}{16}\Big](t_1-t_2)^{\frac{a}{2}}\phi_{\mathbf{m}}^{(a)}(t_1,t_2)\\
& -\frac{t_1t_2}{4}\sum_{j=1}^{2}\Big(\frac{\partial^2}{\partial t_j\partial t_{3-j}}-\frac{t_j}{t_{3-j}}\frac{\partial^2}{\partial t_j^2}+\frac{(-1)^{j+1}}{t_j}\frac{\partial}{\partial t_{3-j}}\Big)(t_1-t_2)^{\frac{a}{2}}\phi_{\mathbf{m}}^{(a)}(t_1,t_2).
\end{aligned}$$

Substituting (3.6) in (3.5) and by a straightforward computation, we obtain the desired result.

4 Infinite sums involving classical hypergeometric functions

Various infinite series with some classical hypergeometric functions of a single variable can be obtained using the expansion of Faraut-Korányi hypergeometric functions of two variables. More precisely, we have the following theorem.

Theorem 4.1. For all $(t_1,t_2)) \in \mathbb{R}^2$ such that $t_1 > t_2 > 0$, it holds that

$$\begin{aligned}
& {}_2F_1^{(a)}(\alpha,\beta;\gamma;t_1,t_2) = \frac{2}{a}\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k(a)_k(k+\frac{a}{2})}{(\gamma)_k(\frac{a}{2})_k k!}\phi_{k,0}^{(a)}(t_1,t_2) \qquad (4.1)\\
& {}_4F_3(\alpha+k,\alpha-\frac{a}{2},\beta+k,\beta-\frac{a}{2};\gamma+k,\gamma-\frac{a}{2},\frac{a}{2}+k;t_1t_2),\\
& {}_2F_1^{(a)}(\alpha,\frac{a}{2};\gamma;t_1,t_2) = \sum_{k=0}^{\infty}\frac{(\alpha)_k(\frac{a}{2})_k}{(\gamma)_k k!}t_1^k.\ {}_2F_1(\alpha+k,\frac{a}{2};\gamma+k;t_2), \qquad (4.2)
\end{aligned}$$

$$ {}_2F_1^{(a)}(\alpha,\frac{a}{2};\gamma;t_1,t_2) = (1-t_2)^{\gamma-(\frac{a}{2}+\alpha)}\sum_{k=0}^{\infty}\frac{(\alpha)_k(\frac{a}{2})_k}{(\gamma)_k k!}t_1^k.\ {}_2F_1(\gamma-\alpha,\gamma+k-\frac{a}{2};\gamma+k;t_2), \quad (4.3)$$

$$ {}_2F_1^{(a)}(\alpha,\frac{a}{2};\gamma;t_1,t_2) = \sum_{k=0}^{\infty}\frac{(\alpha)_k(\frac{a}{2})_k}{(\gamma)_k k!}\frac{t_1^k}{(1-t_2)^{k+\alpha}}.\ {}_2F_1(\alpha+k,\gamma+k-\frac{a}{2};\gamma+k;\frac{t_2}{t_2-1}), \quad (4.4)$$

$$ {}_2F_1^{(a)}(\alpha,\frac{a}{2};\gamma;t_1,t_2) = (1-t_2)^{-\frac{a}{2}}\sum_{k=0}^{\infty}\frac{(\alpha)_k(\frac{a}{2})_k}{(\gamma)_k k!}t_1^k.\ {}_2F_1(\gamma-\alpha,\frac{a}{2};\gamma+k;\frac{t_2}{t_2-1}). \quad (4.5)$$

Proof . Proof. By applying the fact that $(a)_{m_2+k} = (a+k)_{m_2}(a)_k$, $\frac{(a)_k}{(a+1)_k} = \frac{a}{a+k}$ and $\phi^{(a)}_{m_1,m_2}(t_1,t_2)$, we get

$$ {}_2F_1^{(a)}(\alpha,\beta;\gamma;t_1,t_2) = \frac{2}{a}\sum_{m_2=0}^{\infty}\sum_{m_1=m_2}^{\infty}\frac{(\alpha)_{m_1}(\beta)_{m_1}(\alpha-\frac{a}{2})_{m_2}(\beta-\frac{a}{2})_{m_2}(a)_{m_1-m_2}(m_1-m_2+\frac{a}{2})}{(\gamma)_{m_1}(\frac{a}{2}+1)_{m_1}(\gamma-\frac{a}{2})_{m_2}(1)_{m_2}(1)_{m_1-m_2}}\phi^{(a)}_{m_1,m_2}(t_1,t_2). $$

Finally, (4.1) follows by using the change of variable to $m_1 = m_2 + k$. The proof of (4.2) lies essentially on the observation that $(\frac{a}{2})_{\mathbf{m}} = 0$ if $m_2 > 0$. Then

$$ {}_2F_1^{(a)}(\alpha,\frac{a}{2};\gamma;t_1,t_2) = \frac{2}{a}\sum_{m_1=0}^{\infty}\frac{(\alpha)_{m_1}(\frac{a}{2})_{m_1}(a)_{m_1}(m_1+\frac{a}{2})}{(\gamma)_{m_1}(\frac{a}{2}+1)_{m_1}m_1!}\phi^{(a)}_{m_1,0}(t_1,t_2). $$

By using (1.7) and by a straightforward computation, we obtain (4.2). (4.3) follows by applying the Euler's transformation formula ${}_2F_1(a,b;c;x) = (1-x)^{c-a-b}\,{}_2F_1(c-a,c-b;c;x)$ in (4.2). By using the Pfaff's transformation formulas ${}_2F_1(a,b;c;x) = (1-x)^{-a}\,{}_2F_1(a,c-b;c;\frac{x}{x-1})$ and ${}_2F_1(a,b;c;x) = (1-x)^{-b}\,{}_2F_1(b,c-a;c;\frac{x}{x-1})$ in (4.2), we get (4.4) and (4.5) respectively. □

Remark 4.2. By analytic continuation, (4.2) hold for all $(t_1,t_2) \in \mathbb{R}^2$.

5 Application

5.1 Evaluation of some hypergeometric series

The formula (4.2) can be used to evaluate some Faraut-Korányi hypergeometric functions which are interpreted as an infinite sums involving classical hypergeometric functions which can be satisfactorily evaluated. For example, one can apply the Gauss's theorem [[2], Section 1.3, page 2]

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} $$

provided $\Re(c-a-b) > 0$, the Kummer's theorem [[2], Section 2.3, page 9]

$$ {}_2F_1(a,b;1+a-b;-1) = \frac{\Gamma(1+a-b)\Gamma(1+\frac{a}{2})}{\Gamma(1+\frac{a}{2}-b)\Gamma(1+a)} $$

and the Gauss's second theorem [[2], Eq.2, page 11]

$$ {}_2F_1(a,b;\frac{a+b-1}{2};\frac{1}{2}) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})} $$

to evaluate the following Faraut-Korányi hypergeometric functions:

$$ {}_2F_1^{(a)}(\alpha,\frac{a}{2};\frac{\alpha+\frac{3a}{2}+1}{2};\frac{1}{2},1) = \frac{\Gamma(\frac{\frac{a}{2}+1-\alpha}{2})\Gamma(\frac{\alpha+\frac{3a}{2}+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{\frac{3a}{2}+1-\alpha}{2})\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{a+2}{4})}, \quad \Re(1+\frac{a}{2}-\alpha) > 0, \tag{5.1} $$

$$ {}_2F_1^{(a)}(2\gamma-\frac{3a}{2}-1,\frac{a}{2};\gamma;\frac{1}{2},1) = \frac{\Gamma(a+1-\gamma)\Gamma(\gamma)\Gamma(\frac{1}{2})}{\Gamma(\frac{3a}{2}+1-\gamma)\Gamma(\gamma-\frac{3a}{4})\Gamma(\frac{a+2}{4})}, \quad \Re(a+1-\gamma) > 0, \tag{5.2} $$

$$ {}_2F_1^{(a)}(\alpha,\frac{a}{2};\gamma;1,1) = \frac{\Gamma(\gamma)\Gamma(\gamma-a-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-a)}, \quad \Re(\gamma-\alpha-a) > 0, \tag{5.3} $$

$$ {}_2F_1^{(a)}(\alpha,\frac{a}{2};1+\alpha;-1,1) = \frac{\Gamma(1-\frac{a}{2})\Gamma(1+\frac{\alpha}{2})}{\Gamma(1+\frac{\alpha}{2}-\frac{a}{2})}, \quad \Re(1+\frac{a}{2}) > 0. \tag{5.4} $$

6 Conclusion

The proofs of Theorem 3.1 and Theorem 3.2 rely on interesting integral representation for $\phi^a_{\mathbf{m}}$ on domains of rank 2 (Formula(1.7)). It would be nice to have an analogous representation for general rank r in order to generalize our results. Future work, we will generalize our results of this paper to domains of rank 3.

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