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Global solutions for a nonlinear degenerate nonlocal problem

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Abstract

In this paper, we consider the existence and asymptotic behavior of solutions to the following new nonlocal problem

$$u_{tt} - M\Big(\int_\Omega |\nabla u|^2\,dx\Big)\triangle u + \delta u_t = |u|^{\rho-2}u \qquad \text{in } \Omega\times]0,\infty[,$$

where

$$M(s) = \begin{cases} a - bs & \text{for } \; s \in [0, \frac{a}{b}[, \\ 0, & \text{for } s \in [\frac{a}{b}, +\infty[. \end{cases}$$

We first state a local existence theorem. Next, if the initial energy is appropriately small, by using Tartar's method and the decay rate of the energy, we derive the global existence theorem. As a biproduct, we also obtain the exponential decay property of the global solution.

Keywords: global solutions, degenerate nonlocal problem, asymptotic behavior
2010 MSC: 35L80, 35 L70, 35B33, 35J75

1 Introduction

In this research we study the following nonlocal problem

$$\begin{aligned} u_{tt} - M\Big(\int_\Omega |\nabla u|^2\,dx\Big)\triangle u + \delta u_t &= |u|^{\rho-2}u \quad \text{in } \Omega\times]0,\infty[, \\ u = 0, &\quad \text{on } \Gamma\times]0,\infty[, \\ u(x,0) = u^0(x), u_t(x,0) = u^1(x), &\quad x \in \Omega \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary Γ,

$$M(s) = \begin{cases} a - bs & \text{for } \; s \in [0, \frac{a}{b}[, \\ 0, & \text{for } s \in [\frac{a}{b}, +\infty[, \end{cases} \tag{1.2}$$

$a, b > 0, \rho > 2$. When $M(s) = a + bs,\ s \geq 0, a > 0, b \geq 0, \delta = 0 = \mu$ and Ω is a finite open interval, equation (1.1) was introduced by Kirchhoff [10] in the study of nonlinear vibrations of the elastic string and is called the wave equation of

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Kirchhoff type after his name. See also Lions [11]. Moreover, it is said a degenerate equation when $M(s)$ has zeros and a nondegenerate one when $M(s) \geq m_0 > 0, \forall s \geq 0$. Many results on the solutions to problem (1.1) have been established by many authors through various approaches and assumptive conditions (see [1, 2, 4, 5, 7, 17, 14, 15, 22, 23, 24, 26] and references therein).

A notable characteristic of (1.1) is the presence of the nonlocal coefficient $M\Big(\displaystyle\int_\Omega |\nabla u|^2\,dx\Big)$ which depends on the average $\displaystyle\int_\Omega |\nabla u|^2\,dx$ of the term $|\nabla u|^2$ in Ω, and hence the equation is no longer a pointwise identity. Nonlocal problems have gained considerable attention in recent years due to their relevance in modeling physical and biological phenomena. See [6, 11, 13, 21].

In [27] Yin et al. investigated the existence and multiplicity of nontrivial solution for the new nonlocal problem

$$\begin{aligned} -\Big(a - b\int_\Omega |\nabla u|^2\,dx\Big)\triangle u &= |u|^{\rho-2}u \quad \text{ in } \Omega, \\ u &= 0, \quad \text{ on } \Gamma. \end{aligned} \tag{1.3}$$

Also see [9, 20, 28] for generalizations of (1.3)

Motivated for their works, it is interesting to investigate the global solvability of (1.1) with the nonlocal operator given in (1.3). More precisely, under appropriate assumptions imposed on the initial data and the source term, we shall establish global existence of solutions by using Tartar's method [25] combined with suitable a priori estimates including $|\triangle u(t)|$ and $|\nabla_t u(t)|$ in addition to the usual energy estimate.

This article is organized as follows. In Section 2, we prepare some lemmas needed for our arguments and state the local existence theorem. In Section 3, we derive the global solution and its exponential decay.

2 Preliminaries

Throughout this paper the functions are all real valued and the notations are as usual, in particular we shall denote by $\|\|_p, (p \geq 1)$ the ususal L^p-norm. Positive constants will be denote by C and will change from line to line.

Lemma 2.1 (Sobolev- Poincaré[8]). Let q be a number with $2 \leq q < \infty$ $(n = 1, 2)$ or $2 \leq q \leq \frac{2N}{N-2}$ $n \geq 3)$, then there is a positive constant $C_* = C(\Omega, q)$ such that

$$\|u\|_q \leq C_*\|\nabla u\|_2, \quad \text{ for } u \in H_0^1(\Omega). \tag{2.1}$$

Lemma 2.2 (Gagliardo- Nirenberg[8]). Let $1 \leq r < q \leq \infty$ and $q \leq p$. Then the inequality

$$\|u\|_q \leq C\|u\|_{W^{m,q}}^{\theta}\|u\|_r^{1-\theta} \quad \text{ for } u \in W^{m,q}(\Omega)\bigcap L^r(\Omega). \tag{2.2}$$

holds with some constant $C > 0$ and

$$\theta = \left(\frac{1}{r} - \frac{1}{p}\right)\left(\frac{1}{r} + \frac{m}{N} - \frac{1}{q}\right)^{-1}$$

provided that $0 < \theta \leq 1$ $(0 < \theta < 1$if $p = \infty)$.

Lemma 2.3 (Nakao[16]). Let $\Phi(t)$ be a nonincreasing and nonnegative function on $[0, T]$, $T > 1$, such that

$$\Phi(t)^{1+r} \leq k_0(\Phi(t) - \Phi(t + 1)) \quad \text{ on } [0, T], \tag{2.3}$$

where k_0 is a positive constant and r a nonnegative constant. Then

(i) if $r > 0$, then $\Phi(t) \leq \left(\Phi(0)^{-r} + k_0^{-1} r[t-1]^+\right)^{-1/r}$, on [0, T],
where $[t - 1]^+ = \max\{t - 1, 0\}$,

(ii) if $r = 0$, then $\Phi(t) \leq \Phi(0)e^{-k_1[t-1]^+}$, on [0, T], where $k_1 = \log\left(\frac{k_0}{k_0-1}\right)$.

For sake of completeness, we recall the following local existence result, which may be proved by the Banach contraction mapping principle (See [3, 18]).

Theorem 2.4 (Local existence). Let δ be a nonnegative constant and let $M(s)$ be a nonnegative locally Lipschitz function for $s \geq 0$. We assume that $f(u)$ is a nonlinear C^1-function such that

$$|f(u)| \leq k_1|u|^{\alpha+1} \text{ and } |f'(u)| \leq k_2|u|^{\alpha} \tag{2.4}$$

with certain constants k_1, k_2, and

$$0 \leq \alpha \leq 2/(N-4) \text{ if } N \geq 5 \quad (0 \leq \alpha < +\infty \text{ if } N \leq 4). \tag{2.5}$$

If the initial data $\{u_0, u_1\}$ belong to $(H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$ and satisfy the nondegeneracy condition

$$M(\|\nabla u_0\|_2) > 0, \tag{2.6}$$

then there exists $T = T(\|\triangle u_0\|_2, \|\nabla u_1\|_2) > 0$ such the problem (1.1) admits a unique local solution u in the class

$$C^0\left([0,T); H_0^1(\Omega) \cap H^2(\Omega)\right) \cap C^1\left([0,T); H_0^1(\Omega)\right) \cap C^2\left([0,T); L^2(\Omega)\right).$$

Moreover, if $M(\|\nabla u(t)\|_2) > 0$ for $T > t \geq 0$, then at least one of the following statements is valid

(i) $T = +\infty$,

(ii) $\|\nabla u_t(t)\|_2^2 + \|\triangle u(t)\|_2^2 \to +\infty \quad \text{as } t \to T^-$,

(iii) $\|\nabla u(t)\|_2^2 \to 0 \quad \text{as } t \to T^-$.

Now, we set

$$B_\rho = \sup_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\|u\|\rho}{\|\nabla u\|_2}, \quad \gamma_1 = \frac{b}{4a}, \quad \gamma_2 = \frac{B_\rho^\rho}{3\rho a}.$$

Define the function

$$h(\lambda) = \frac{1}{4}\lambda^2 - \gamma_1\lambda^4 - \frac{3}{2}\gamma_2\lambda^\rho,$$

then

$$h'(\lambda) = \lambda\left(\frac{1}{2} - 4\gamma_1\lambda^2 - \frac{3}{2}\rho\gamma_2\lambda^{\rho-2}\right).$$

So, choosing $\lambda \in \mathbb{R}$, such that

$$0 \leq \lambda^2 \leq \frac{1}{16\gamma_1} \text{ and } 0 \leq \lambda^{\rho-2} \leq \frac{1}{6\rho\gamma_2},$$

we get that this $\lambda's$ satisfy the inequality

$$\frac{1}{2} - 4\gamma_1\lambda^2 - \frac{3}{2}\rho\gamma_2\lambda^{\rho-2} \geq 0$$

and $h'(\lambda) \geq 0$ for $0 \leq \lambda \leq \lambda_1$, where

$$\lambda_1 = \min\{(16\gamma_1)^{-1/2}, (6\rho\gamma_2)^{-1/(\rho-2)}\}. \tag{2.7}$$

Thus, $h(0) = 0$ and $h(\lambda) \geq 0$, $\forall\lambda \in [0, \lambda_1]$. From this, we get

$$h_0(\lambda) = \frac{1}{2}\lambda^2 - \gamma_1\lambda^4 - \gamma_2\lambda^\rho \geq \frac{1}{4}\lambda^2 + \frac{1}{2}\gamma_2\lambda^\rho, \quad \forall\lambda \in [0, \lambda_1]. \tag{2.8}$$

The energy associated with the problem (1.1) is given by

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + J(u(t)), \quad \text{for } u \in H_0^1(\Omega),$$

where

$$J(u(t)) = \frac{a}{2}\|u(t)\|^2 - \frac{b}{4}\|u(t)\|^4 - \frac{1}{\rho}\|u(t)\|_\rho^\rho.$$

By a simple calculation, we see that the energy $E(t)$ satisfies

$$\frac{d}{dt}E(t) + \delta\|u_t(t)\|_2^2 = 0. \tag{2.9}$$

Therefore, $E(t)$ is a nonincreasing function on t, and

$$E(t) + \delta\int_0^t \|u_t(s)\|_2^2\,ds = E(0). \tag{2.10}$$

3 Global existence and exponential decay

In this section we state the main results of this paper. Firstly, we give the following two propositions.

Proposition 3.1. If the local solution $u(t)$ of (1.1) satisfies $0 < \|\nabla u(t)\|_2 < \lambda_1$ on $[0, T_0]$, then

$$(E(t) \geq)\ J(u(t)) \geq a\left(\frac{1}{4}\|u(t)\|^2 + \frac{\gamma_2}{2}\|u(t)\|^\rho\right) \tag{3.1}$$

$$\text{and} \qquad \|u(t)\| \leq \left[\frac{4}{a}E(t)\right]^{1/2}. \tag{3.2}$$

Proof . It is obvious, from (2.8) □

Proposition 3.2. Under the assumption of Proposition 3.1, the energy $E(t)$ satisfies

$$E(t) \leq E(0)e^{-kt}, \tag{3.3}$$

where $k = Log\left(\frac{k_0}{k_0-1}\right)$, k_0 is defined in (3.8).

Proof . For a moment, we suppose that $T > 1$. Integrating (2.9) from t to $t+1$ we find

$$\delta\int_t^{t+1} \|u(s)\|_2^2\,ds = E(t) - E(t+1) \equiv \delta F^2(t)$$

Using the mean value theorem for integrals, there exist two points $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|u(t_i)\|_2 \leq 2F(t), \qquad i = 1, 2.$$

Next, multiplying (1.1) by u and integrating over Ω , we obtain

$$a\|u(t)\|^2 - b\|u(t)\|^4 - \|u(t)\|_\rho^\rho = \|u_t(t)\|_2^2 - (u_t(t), u(t)) - \frac{d}{dt}(u_t(t), u(t)). \tag{3.4}$$

On the other hand, it follows from the Sobolev-Poincaré inequality and (3.2) that

$$\|u(t)\|_\rho^\rho \leq B_\rho^\rho\|u(t)\|^\rho \leq B_\rho^\rho\|u(t)\|^{\rho-2}\|u(t)\|^2 \leq B_\rho^\rho\left[\frac{4}{a}E(0)\right]^{(\rho-2)/2}\|u(t)\|^2$$

and

$$b\|u(t)\|^4 \leq b\left[\frac{4}{a}E(0)\right]\|u(t)\|^2.$$

Thus, we get

$$\begin{aligned} b\|u(t)\|^4 + \|u(t)\|_\rho^\rho \leq & \frac{1}{a}\left[B_\rho^\rho\left(\frac{4}{a}E(0)\right)^{(\rho-2)/2} + \frac{4b}{a}E(0)\right](a\|u(t)\|^2) \\ & \equiv (1-\eta_0)(a\|u(t)\|^2), \qquad 0 < \eta_0 < 1. \end{aligned} \tag{3.5}$$

Then

$$\eta_0 a\|u(t)\|^2 \leq a\|u(t)\|^2 - b\|u(t)\|^4 - \|u(t)\|_\rho^\rho \equiv I(t). \tag{3.6}$$

From (2.10) and (3.4), integrating the resultant inequality over $[t_1, t_2]$, we have

$$\begin{aligned}\eta_0 a \int_{t_1}^{t_2} \|u(s)\|^2\, ds \leq \int_{t_1}^{t_2} I(s)\, ds &\leq \int_{t_1}^{t_2} \|u_t(s)\|^2\, ds + \int_{t_1}^{t_2} |(u_t(s), u(s))|\, ds - (u_t(t), u(t))|_{t_1}^{t_2} \\ &\leq F^2(t) + \left[\left(\int_t^{t+1} \|u_t(s)\|^2\, ds\right)^{1/2} \sum_{i=1}^{2} \|u(t)\|\right] \sup_{s\in[t,t+1]} \|u(s)\| \\ &\leq F^2(t) + 5B_2 F(t) \left(\frac{4}{a}E(t)\right)^{1/2}.\end{aligned} \tag{3.7}$$

On the other hand, integrating (2.10) over $[t, t_2]$, noting that $E(t_2) \leq 2\int_{t_1}^{t_2} E(s)\, ds$ due to $t_2 - t_1 \geq \frac{1}{2}$, using (3.7) and the Young inequality, we have

$$\begin{aligned}E(t) =& E(t_2) + \delta \int_t^{t_2} \|u_t(s)\|_2^2\, ds \\ \leq& 2\int_{t_1}^{t_2} E(s)\, ds + \delta \int_t^{t+1} \|u_t(s)\|_2^2\, ds \\ \leq& (1+\delta) \int_t^{t+1} \|u_t(s)\|_2^2\, ds + a \int_{t_1}^{t_2} \|u(s)\|_2^2\, ds \\ \leq& (1+\delta+\frac{1}{\eta_0})F^2(t) + \frac{1}{2}\left(\frac{5}{\eta_0}B_2\right)\frac{4}{a}F^2(t) + \frac{1}{2}E(t)\end{aligned}$$

Thus

$$E(t) \leq k_0(E(t) - E(t+1))$$

where

$$k_0 = 2\left[1+\delta+\frac{1}{\eta_0}+\frac{2}{a}(\frac{5}{\eta_0}B_2)^2\right]+1. \tag{3.8}$$

Therefore, noting (2.10) and applying Lemma 2.3, we obtain (3.3) □

Theorem 3.3. Let $N = 3$ and $\rho > 4$. Assume further that $\{u_0, u_1\}$ belongs to $(H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$ with

$$\|u_0\| < \min\{\left(\frac{a}{b}\right)^{1/2}, \lambda_1\}, \qquad [4E(0)]^{1/2} < \lambda_1 \tag{3.9}$$

$$\text{and} \quad \left[(2bE^{1/2}(0)^{\rho-4} + d_1(\tau_0 a)^{(\rho-4)/2}\right]^2 \left(\frac{\|\nabla u_1\|_2^2}{a - b\|\nabla u_1\|_2^2} + \|\triangle u_0\|_2^2\right)^{\rho-4} < (\tau_0 a)^{\rho-4}, \tag{3.10}$$

then problem (1.1) admits a unique global solution

$$u \in C([0,+\infty[; H_0^1(\Omega)\cap H^2(\Omega)) \cap C^1([0,+\infty[; H_0^1(\Omega)) \cap C^2([0,+\infty[; L^2(\Omega)),$$

and the energy satisfies

$$E(t) \leq Ce^{-kt} \qquad \text{for} \quad t \geq 0, \tag{3.11}$$

with some constant $k > 0$.

Proof . Let $u(t)$ be a unique solution of the problem (1.1) in the sense Theorem 2.4 on $[0, T_0[$, with T_0 the maximal time where the solution exists. Now, we introduce the function

$$M(t) = a - b\|\nabla u(t)\|_2^2, \qquad \text{for} \quad t \in [0, T_0[.$$

So, under the assumption (3.9), there exists a positive constant τ_0 such that

$$M(0) = a - b\|\nabla u_0\|_2^2 = \tau_0.$$

Then there exists T_0' such that $0 < T_0' \le T_0$ and

$$M(t) > \tau_0 \qquad \text{for} \;\; 0 \le t \le T_0'.$$

We can define

$$T_1 = \sup\{t \in [0, +\infty[: M(t) > \tau_0 \qquad 0 \le s < t\}.$$

We see that $T_1 > 0$ and $M(t) > \tau_0 \qquad \text{for} \quad 0 \le s < T_1$. Let us set

$$H(t) = \frac{\|\nabla u_t(t)\|_2^2}{M(t)} + \|\triangle u_t(t)\|_2^2 \qquad \text{and} \;\; f(u(t)) = |u(t)|^{\rho-2} u(t).$$

So, for $0 \le s < T_1$, taking the derivative of $H(t)$ we have

$$\frac{d}{dt} H(t) + 2\delta \frac{\|\nabla u_t(t)\|_2^2}{M(t)} \left(1 + \frac{M'(t)}{M(t)} \right) = \frac{2}{M(t)} (\nabla f(u(t)), \nabla u_t(t)).$$

Here, we observe that

$$\begin{aligned} \left| \frac{M'(t)}{M(t)} \right| &\le \left| \frac{b(\nabla u(t), \nabla u_t(t))}{M(t)} \right| \\ &\le \frac{b\|\nabla u(t)\|_2}{M^{1/2}(t)} H^{1/2}(t) \le \left(\frac{4b^2}{\tau_0 a} E(0) \right)^{1/2} H^{1/2}(t), \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} \left| \frac{2}{M(t)} (\nabla f(u(t)), \nabla u_t(t)) \right| &\le \frac{2(\rho-1)}{M(t)} \| |u(t)|^{\rho-2} u(t) \|_2 \|\nabla u_t(t)\|_2 \\ &\le \frac{2(\rho-1)}{M(t)} \|u(t)\|_{3(\rho-2)}^{\rho-2} \|\nabla u(t)\|_6 \|\nabla u_t(t)\|_2 \\ &\le \frac{2(\rho-1)}{M^{1/2}(t)} C_*^{\rho-1} \|\nabla u(t)\|_2^{(\rho-2)/2+1} \|\triangle u(t)\|_2^{(\rho-2)/2} \frac{\|\nabla u_t(t)\|_2}{M^{1/2}(t)} \\ &\le \left(\frac{\rho-1}{\delta^{1/2} \tau_0^{1/2}} C_*^{\rho-1} \|\nabla u(t)\|_2^{\rho/2} \right)^2 H^{(\rho-2)/2}(t) + \delta \frac{\|\nabla u_t(t)\|_2^2}{M(t)}. \end{aligned} \tag{3.13}$$

Thus, it follows from (3.12) and (3.13) that

$$\begin{aligned} &\frac{d}{dt} H(t) + \delta \frac{\|\nabla u_t(t)\|_2^2}{M(t)} \left[1 - \left(\frac{4b^2 E(0)}{\tau_0 a} \right)^{1/2} H^{1/2}(t) \right] \\ &\le \left(\frac{\rho-1}{\delta^{1/2} \tau_0^{1/2}} C_*^{\rho-1} \right)^2 \|\nabla u(t)\|_2^{\rho} H^{(\rho-2)/2}(t). \end{aligned} \tag{3.14}$$

If

$$1 - \left(\frac{4b^2 E(0)}{\tau_0 a} \right)^{1/2} H^{1/2}(t) > 0, \quad \forall t \in [0, T_1[\tag{3.15}$$

does not hold, it implies a contradiction. In fact, from (3.10) and the continuity of $H(t)$, we can see that there exists $t^* > 0$ such that

$$1 - \left(\frac{4b^2 E(0)}{\tau_0 a} \right)^{1/2} H^{1/2}(t^*) = 0, \tag{3.16}$$

$$1 - \left(\frac{4b^2 E(0)}{\tau_0 a} \right)^{1/2} H^{1/2}(t) > 0, \quad \forall t \in [0, t^*[. \tag{3.17}$$

Now, integrating (3.14) from 0 to t^* and using (3.17) we have

$$H(t^*) \le H(0) + \underbrace{\left(\frac{\rho-1}{\delta^{1/2}\tau_0^{1/2}} C_*^{\rho-1}\right)^2}_{\theta_0} \int_0^{t^*} \|\nabla u(s)\|_2^\rho H^{(\rho-2)/2}(s)\, ds$$

and , hence by Lemma 3.4 in [19]

$$H^{1/2}(t^*) \le \left[H(0)^{-\frac{1}{\rho-4}} - \left(\frac{\rho-4}{2}\right)\theta_0 \int_0^{t^*} \|\nabla u(s)\|_2^\rho\, ds\right]^{-\frac{1}{\rho-4}}. \tag{3.18}$$

On the other hand, we see from (3.2) and (3.3) that

$$\int_0^{t^*} \|\nabla u(s)\|_2^\rho\, ds \le \frac{2^\rho}{a^{\rho/2}} \int_0^{t^*} E^{\rho/2}(s)\, ds \le \left[\frac{4E(0)}{a}\right]^{\rho/2} \left(1 + \frac{e^k}{k}\right). \tag{3.19}$$

By (3.18) and (3.19), we infer that

$$H^{1/2}(t^*) \le \left[H(0)^{-\frac{\rho-4}{2}} - d_1 E^{\rho/2}(0)\right]^{-\frac{1}{\rho-4}} \tag{3.20}$$

with $d_1 = (\rho-4)\left[\frac{(\rho-1)C_*^{\rho-1}}{\delta^{1/2}\tau_0^{1/2}}\right]^2 \left(\frac{4}{a}\right)^{\rho/2}\left(1+\frac{e^k}{k}\right)$ Hence, using (3.10) we get

$$1 - \left(\frac{4b^2 E(0)}{\tau_0 a}\right)^{1/2} H^{1/2}(t^*) > 0, \tag{3.21}$$

which contradicts (3.16). Thus (3.15) is true, and then

$$H(t) < \left[\frac{4b^2}{\tau_0 a} E(0)\right]^{-1}, \quad \forall t \in [0, T_1[. \tag{3.22}$$

Now, if $T_1 < T_0$, then

$$M(T_1) = 0.$$

Thus, it follows by (3.22) that $\|\nabla u_t(T_1)\|_2 = 0$. Induce a variable $v(t) = u(T_1 - t)$. Hence $v(t)$ satisfies

$$\begin{aligned}
&v_{tt} - M(\|\nabla v(t)\|_2^2)\triangle v = \delta v_t + f(v) \quad \text{in } \Omega\times]0, T_1],\\
&v = 0 \qquad\qquad\qquad\qquad\qquad \text{on } \Gamma\times]0, T_1],\\
&v(0) = 0 = v_t(0) \qquad\qquad\quad \text{in } \Omega.
\end{aligned}$$

Multiplying this equation by v_t as in integrating it over Ω we get

$$\begin{aligned}
\frac{d}{dt}E(v(t)) = \delta\|v_t(t)\|_2^2 \le &C(\|v_t(t)\|_2^2 + J(v(t)) \\
&\le CE(v(t)), \quad \forall t \in [0, T_1].
\end{aligned} \tag{3.23}$$

Notice that $E(v(0)) = 0$. Then, it follows from (3.23), applying Gronwall's inequality, that

$$\frac{a}{4}\|v(t)\|_2^2 \le E(v(t)) = 0.$$

Consequently $\|\nabla u(T_1 - t)\|_2 = 0$, for $0 \le t \le T_1$. So $u(0) = 0$. Thus $a - b\|\nabla u(0)\|_2^2 = \tau_0$ implies $a = \tau_0$, which is a contradiction. Therefore $M(t) > 0$, $\forall t \ge 0$ and (3.22) holds for all $t \ge 0$. Moreover, from proposition 3.2 we obtain the decay estimate (3.11). $\square$

Remark 3.4. It seems to be interesting to study a global solution for Kirchhoff equation with nonlinear source and boundary damping term or with nonlinear boundary damping and source term, i.e.

$$\begin{gathered} u_{tt} - M\Big(\int_\Omega |\nabla u|^2\,dx\Big)\triangle u = |u|^{\rho-2}u \quad \text{ in } \Omega\times]0,\infty[,\\ u=0, \quad \text{ on } \Gamma_0\times]0,\infty[,\\ \frac{\partial}{\partial\nu}u = g(u_t), \quad \text{ on } \Gamma_1\times]0,\infty[,\\ u(x,0)=u^0(x), u_t(x,0)=u^1(x), \quad x\in\Omega, \end{gathered}$$

and

$$\begin{gathered} u_{tt} - M\Big(\int_\Omega |\nabla u|^2\,dx\Big)\triangle u = 0 \quad \text{ in } \Omega\times]0,\infty[,\\ u=0, \quad \text{ on } \Gamma_0\times]0,\infty[,\\ \frac{\partial}{\partial\nu}u = g(u_t)+|u|^{\rho-2}u, \quad \text{ on } \Gamma_1\times]0,\infty[,\\ u(x,0)=u^0(x), u_t(x,0)=u^1(x), \quad x\in\Omega, \end{gathered}$$

with $M(s)$ given in (1.2) or $M(s)=a-bs^\gamma,\ \forall\ s\in[0,\sqrt[\gamma]{\frac{a}{b}}[, \gamma\geq 1$. Also these equations could be studied in variable exponent spaces or fractional Sobolev spaces. We plan to address these questions in a future research.

4 Conclusions

In recent years, there has been published much work concerning considering the nonlinear wave equation with the presence of a Kirchhoff term. However, to the best of our knowledge, our results are the first time to deal with the type of problem (1.1). In this work, we used energy decay estimates, via Nakao's Lemma, combined with Tartar's method to establish the global existence of solutions and the exponential decay of the energy, under some small data conditions. We like to point out that when $a=0=\delta$ the equation (1.1) becomes the quasilinear non well-posed problem which can be seen as a boundary value problem for the potential equation as in [12]. This question has some interest in the study of the optimal control for singular distributed system and is still an open problem.

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