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On certain coupled fixed point theorems via C-class functions in S_b-metric spaces with applications

Gujjula Upender Reddy[a], Parkala Naresh[a,b,*], Bagathi Srinuvasa Rao[c]

[a]*Department of Mathematics, Mahatma Gandhi University, Nalgonda, Telangana, India*

[b]*Department of Mathematics, Sreenidhi Institute of Science and Technology, Ghatkesar, Hyderabad-501301,Telangana, India*

[c]*Department of Mathematics, Dr.B.R.Ambedkar University, Srikakulam, Etcherla-532410, Andhra Pradesh, India*

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Abstract

In this study, the concept of C-class functions in the setup of S_b-metric spaces, and some common coupled fixed point theorems for these mappings in complete S_b-metric spaces that involve altering distance functions and ultra altering distance functions are established. A few instances are given to support our major findings. We also provided an application for integral equations as well as Homotopy.

Keywords: C-class functions, coupled fixed point, complete S_b-metric spaces and ω-compatible mappings
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1 Introduction

In 1922, the Banach contraction principle was first introduced by S. Banach[7]. It is the most important tool in nonlinear analysis, and certain findings linked to the generalisation of various metric type spaces come from using it(see [8, 13, 18, 21, 22, 23]).

Previously, Sedghi et al. [27], using the concepts of S and b-metric spaces, developed S_b-metric spaces and proved common fixed point outcomes in these spaces. Following this, various authors developed numerous results on S_b-metric spaces (see e.g.[17, 25, 29, 30])

The idea of a coupled fixed point was first developed by Guo and Lakshmikantham [10] in 1987. Later, employing a weak contractivity type assumption, Bhaskar and Lakshmikantham [9] developed a novel fixed point theorem for a mixed monotone mapping in a metric space driven by partial ordering. Jungck and Rhoades [14] introduced the idea of weak compatibility in 1998 and demonstrated that compatible mappings are weakly compatible but the reverse is not true. See the results in ([1, 2, 3, 15, 19, 20]) and related references for additional results on coupled fixed point outcomes.

A.H. Ansari et al. [4] presented the idea of C-class functions in 2016 and proved some unique fixed point theorems for certain contractive mappings with regard to the C-class functions, which started a lot of work in this field (See, [5, 6, 11, 12, 24, 26, 28]).

*Corresponding author
Email addresses: upendermathsmgu@gmail.com (Gujjula Upender Reddy), parakala2@gmail.com (Parkala Naresh), srinivasabagathi@gmail.com (Bagathi Srinuvasa Rao)

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The purpose of the current paper is to provide common fixed point theorems for mappings of C-class function type in the context of S_b-metric spaces. We can also provide examples that are appropriate and relevant for integral equations and Homoopy theory. First we recall some basic results.

2 Preliminaries

Definition 2.1. ([27]) Let $\mathcal{G}$ be a non-empty set and $\kappa \geq 1$ be any real number.Let a mapping $S_b : \mathcal{G}^3 \to [0, \infty)$ satisfying the following properties :

(S_b1) $0 < S_b(\alpha, \beta, \gamma)$ for all $\alpha, \beta, \gamma \in \mathcal{G}$ with $\alpha \neq \beta \neq \gamma$,
(S_b2) $S_b(\alpha, \beta, \gamma) = 0 \Leftrightarrow \alpha = \beta = \gamma$,
(S_b3) $S_b(\alpha, \beta, \gamma) \leq \kappa(S_b(\alpha, \alpha, \theta) + S_b(\beta, \beta, \theta) + S_b(\gamma, \gamma, \theta))$ for all $\alpha, \beta, \gamma, \theta \in \mathcal{G}$.

Then the function S_b is called a S_b-metric on $\mathcal{G}$ and the pair $(\mathcal{G}, S_b)$ is called a S_b-metric space.

Example 2.2. ([27]) Let $(\mathcal{G}, S)$ be an S-metric space and $S_*(\alpha, \beta, \gamma) = S(\alpha, \beta, \gamma)^p$, where $p > 1$ is a real.Then S_* is a S_b-metric with $\kappa = 2^{2(p-1)}$.

Definition 2.3. ([27]) If $(\mathcal{G}, S_b)$ be a S_b-metric space. $\{\chi_n\}$ in $\mathcal{G}$ said to be:

(1) S_b-Cauchy sequence if, for each $\epsilon > 0$, there exists $n_0 \in \mathcal{N}$ such that $S_b(\chi_n, \chi_n, \chi_m) < \epsilon$ for each $m, n \geq n_0$.
(2) S_b - convergent to a point $\chi \in \mathcal{G}$ if, for each $\epsilon > 0$, there exists a positive integer n_0 such that $S_b(\chi_n, \chi_n, \chi) < \epsilon$ or $S_b(\chi, \chi, \chi_n) < \epsilon$ for all $n \geq n_0$ and we denote by $\lim_{n\to\infty} \chi_n = \chi$.
(3) if every S_b-Cauchy sequence is S_b-convergent in $\mathcal{G}$. Then S_b-metric space $(\mathcal{G}, S_b)$ is called complete.

Lemma 2.4. ([27]) According to the definition of S_b-metric space, we have

$$S_b(\alpha, \alpha, \beta) \leq \kappa S_b(\beta, \beta, \alpha) \text{ and } S_b(\beta, \beta, \alpha) \leq \kappa S_b(\alpha, \alpha, \beta).$$

Lemma 2.5. ([27]) In a S_b-metric space, we have

$$S_b(\alpha, \alpha, \beta) \leq 2\kappa S_b(\alpha, \alpha, \gamma) + \kappa^2 S_b(\gamma, \gamma, \beta).$$

Lemma 2.6. ([27]) If $(\mathcal{G}, S_b)$ be a S_b-metric space with $\kappa \geq 1$ and $\{\alpha_n\}$ be S_b-convergent to α, then we have

$$(i) \quad \frac{1}{2\kappa} S_b(\beta, \beta, \alpha) \leq \lim_{n\to\infty} \inf\ S_b(\beta, \beta, \alpha_n) \leq \lim_{n\to\infty} \sup S_b(\beta, \beta, \alpha_n) \leq 2\kappa S_b(\beta, \beta, \alpha) \text{ and}$$

$$(ii) \quad \frac{1}{\kappa^2} S_b(\alpha, \alpha, \beta) \leq \lim_{n\to\infty} \inf\ S_b(\alpha_n, \alpha_n, \beta) \leq \lim_{n\to\infty} \sup\ S_b(\alpha_n, \alpha_n, \beta) \leq \kappa^2 S_b(\alpha, \alpha, \beta)$$

for all $\beta \in \mathcal{G}$. In particular, if $\alpha = \beta$, then we have $\lim_{n\to\infty} S_b(\alpha_n, \alpha_n, \beta) = 0$.

Definition 2.7. [4] A continuous mapping $\Delta : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ is called a C-class function if for all $s^*, t^* \in [0, \infty)$,

(a) $\Delta(s^*, t^*) \leq s^\star$;
(b) $\Delta(s^*, t^*) = s^*$ implies that either $s^* = 0$ or $t^* = 0$.

The family of all C-class functions is denoted by C.

Example 2.8. [4] Each of the functions $\Delta : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ defined below are elements of C.

(a) $\Delta(s^*, t^*) = s^\star - t^\star$.
(b) $\Delta(s^*, t^*) = ms^*$ where $m \in (0, 1)$.
(c) $\Delta(s^*, t^*) = \frac{s^*}{(1+t^\star)^r}$ where $r \in (0, \infty)$.
(d) $\Delta(s^*, t^*) = s^\star \eta(s^\star)$ where $\eta : [0, \infty) \to [0, \infty)$ is a continuous function.
(e) $\Delta(s^*, t^*) = s^* - \varphi(s^\star)$ for all $s^*, t^* \in [0, +\infty)$ where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\varphi(s^\star) = 0 \Leftrightarrow s^\star = 0$.
(f) $\Delta(s^*, t^*) = s\Omega(s^\star, t^\star)$ for all $s^*, t^* \in [0, +\infty)$ where $\Omega : [0, \infty)^2 \to [0, \infty)$ is a continuous function such that $\Omega(s^\star, t^\star) < 1$.

3 Main Results

In this section, we give some common coupled fixed point theorems for C-class functions in complete S_b-metric spaces which involve altering distance functions and ultra altering distance functions.

Definition 3.1. Let $(\mathcal{G}, S_b)$ be a S_b-metric space. A pair $(\wp, \varpi)$ is called

(a) a coupled fixed point of $\Omega : \mathcal{G}^2 \to \mathcal{G}$ if $\Omega(\wp, \varpi) = \wp$ and $\Omega(\varpi, \wp) = \varpi$;

(b) a coupled coincident point of $\Omega : \mathcal{G}^2 \to \mathcal{G}$ and $\Lambda : \mathcal{G} \to \mathcal{G}$ if $F(\wp, \varpi) = \Lambda\wp,\ \ \Omega(\varpi, \wp) = \Lambda\varpi$;

(c) common fixed point of $\Omega : \mathcal{G}^2 \to \mathcal{G}$ and $\Lambda : \mathcal{G} \to \mathcal{G}$ if $\Omega(\wp, \varpi) = \Lambda\wp = \wp,\ \ \Omega(\varpi, \wp) = \Lambda\varpi = \varpi$;

(d) the pair (Ω, Λ) is weakly compatible if $\Lambda(\Omega(\wp, \varpi)) = \Omega(\Lambda\wp, \Lambda\varpi)$ whenever $\Omega(\wp, \varpi) = \Lambda\wp,\ \ \Omega(\varpi, \wp) = \Lambda\varpi$.

A new category of contractive fixed point results was addressed by Khan et al. [16] and A. H Ansari et al.[5]. In their study they introduced the notion of an altering distance and ultra altering distance functions which are control functions that alters distance between two points in a metric space.

Let $\mathfrak{F}$ be the class of all altering distance function $\psi_\star : [0, \infty) \to [0, \infty)$ and $\mathfrak{G}$ be the class of all ultra altering distance functions $\phi_\star : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(ψ_0) $\psi_\star$ is nondecreasing and continuous;

(ψ_1) $\psi_\star(t) = 0$ if and only if $t = 0$.

(ϕ_0) $\phi_\star$ is continuous;

(ϕ_1) $\phi_\star(t) > 0$, for all $t > 0$ and $\phi_\star(0) \geq 0$.

Theorem 3.2. Let $(\mathcal{G}, S_b)$ be a complete S_b-metric space with coefficient $\kappa > 1$. Let $\Gamma : \mathcal{G}^2 \to \mathcal{G}$, $\Lambda : \mathcal{G} \to \mathcal{G}$ be two mappings and $\Theta \geq 0$ such that

$$\psi_\star\left(2\kappa^4 S_b(\Gamma(u,v), \Gamma(u,v), \Gamma(p,q))\right) \leq \Delta\left(\psi_\star\left(M_b^S(u,v,p,q)\right), \phi_\star\left(M_b^S(u,v,p,q)\right)\right) + \Theta\phi_\star\left(N_b^S(u,v,p,q)\right) \tag{3.1}$$

where, $$M_b^S(u,v,p,q) = \max\left\{\begin{array}{c} S_b(\Lambda u, \Lambda u, \Lambda p), S_b(\Lambda v, \Lambda v, \Lambda q), S_b(\Lambda u, \Lambda u, \Gamma(u,v)), \\ S_b(\Lambda v, \Lambda v, \Gamma(v,u)), S_b(\Lambda p, \Lambda p, \Gamma(p,q)), S_b(\Lambda q, \Lambda q, \Gamma(q,p)), \\ \frac{1}{2\kappa^4}\left[S_b(\Lambda u, \Lambda u, \Gamma(p,q)) + S_b(\Lambda p, \Lambda p, \Gamma(u,v))\right], \\ \frac{1}{2\kappa^4}\left[S_b(\Lambda v, \Lambda v, \Gamma(q,p)) + S_b(\Lambda q, \Lambda q, \Gamma(v,u))\right] \end{array}\right\}$$

and

$$N_b^S(u,v,p,q) = \min\left\{\begin{array}{c} S_b(\Lambda u, \Lambda u, \Gamma(u,v)), S_b(\Lambda p, \Lambda p, \Gamma(p,q)), S_b(\Lambda u, \Lambda u, \Gamma(p,q)), \\ S_b(\Lambda v, \Lambda v, \Gamma(q,p)), S_b(\Lambda p, \Lambda p, \Gamma(u,v)), S_b(\Lambda q, \Lambda q, \Gamma(v,u)) \end{array}\right\}$$

for all $u, v, p, q \in \mathcal{G}$ where $\Delta \in C$, $\psi_\star \in \mathfrak{F}$ and $\phi_\star \in \mathfrak{G}$

(i) $\Gamma(\mathcal{G}^2) \subseteq \Lambda(\mathcal{G})$ and $\Lambda(\mathcal{G})$ is a complete subspace of $\mathcal{G}$,

(ii) the pair (Γ, Λ) is ω-compatible.

Then there is a unique common coupled fixed point of Γ and Λ in $\mathcal{G}$.

Proof . Let $\lambda_0, \zeta_0 \in \mathcal{G}$ be arbitrary, and from (i), we construct the sequences $\{\lambda_p\}, \{\zeta_p\}$ in $\mathcal{G}$ as

$$\Gamma(\lambda_p, \zeta_p) = \Lambda\lambda_{p+1} = \alpha_p, \quad \Gamma(\zeta_p, \lambda_p) = \Lambda\zeta_{p+1} = \beta_p, \text{ where } p = 0, 1, 2, \ldots.$$

Now we show that Γ and Λ have a common coupled fixed point in $\mathcal{G}$. Assume that $S_b(\alpha_p, \alpha_p, \alpha_{p+1}) > 0$ and $S_b(\beta_p, \beta_p, \beta_{p+1}) > 0$ $\forall p$. Otherwise, there exists some positive integer p such that $\alpha_p = \alpha_{p+1}$, $\beta_p = \beta_{p+1}$ and so (Λ_p, ζ_p) is a coupled coincidence point of Γ, Λ, and the proof is complete. By using (3.1), for each $p \in \mathbb{N}$, we have

$$\begin{aligned} \psi_\star\left(2\kappa^4 S_b(\alpha_p, \alpha_p, \alpha_{p+1})\right) &= \psi_\star\left(2\kappa^4 S_b(\Gamma(\lambda_p, \zeta_p), \Gamma(\lambda_p, \zeta_p), \Gamma(\lambda_{p+1}, \zeta_{p+1}))\right) \\ &\leq \Delta\left(\psi_\star\left(M_b^S(\lambda_p, \zeta_p, \lambda_{p+1}, \zeta_{p+1})\right), \phi_\star\left(M_b^S(\lambda_p, \zeta_p, \lambda_{p+1}, \zeta_{p+1})\right)\right) \\ &\quad +\Theta\phi_\star\left(N_b^S(\lambda_p, \zeta_p, \lambda_{p+1}, \zeta_{p+1})\right) \end{aligned} \tag{3.2}$$

where,

$$\begin{aligned}
& M_b^S(\lambda_p, \zeta_p, \lambda_{p+1}, \zeta_{p+1}) \\
= & \max\left\{ \begin{array}{c} S_b\left(\Lambda\lambda_p, \Lambda\lambda_p, \Lambda\lambda_{p+1}\right), S_b\left(\Lambda\zeta_p, \Lambda\zeta_p, \Lambda\zeta_{p+1}\right), S_b\left(\Lambda\lambda_p, \Lambda\lambda_p, \Gamma(\lambda_p, \zeta_p)\right), \\ S_b\left(\Lambda\zeta_p, \Lambda\zeta_p, \Gamma(\zeta_p, \lambda_p)\right), S_b\left(\Lambda\lambda_{p+1}, \Lambda\lambda_{p+1}, \Gamma(\lambda_{p+1}, \zeta_{p+1})\right), S_b\left(\Lambda\zeta_{p+1}, \Lambda\zeta_{p+1}, \Gamma(\zeta_{p+1}, \lambda_{p+1})\right), \\ \frac{1}{2\kappa^4}\left[S_b\left(\Lambda\lambda_p, \Lambda\lambda_p, \Gamma(\lambda_{p+1}, \zeta_{p+1})\right) + S_b\left(\Lambda\lambda_{p+1}, \Lambda\lambda_{p+1}, \Gamma(\lambda_p, \zeta_p)\right)\right], \\ \frac{1}{2\kappa^4}\left[S_b\left(\Lambda\zeta_p, \Lambda\zeta_p, \Gamma(\zeta_{p+1}, \lambda_{p+1})\right) + S_b\left(\Lambda\zeta_{p+1}, \Lambda\zeta_{p+1}, \Gamma(\zeta_p, \lambda_p)\right)\right] \end{array} \right\} \\
= & \max\left\{ \begin{array}{c} S_b\left(\alpha_{p-1}, \alpha_{p-1}, \alpha_p\right), S_b\left(\beta_{p-1}, \beta_{p-1}, \beta_p\right), S_b\left(\alpha_{p-1}, \alpha_{p-1}, \alpha_p\right), \\ S_b\left(\beta_{p-1}, \beta_{p-1}, \beta_p\right), S_b\left(\alpha_p, \alpha_p, \alpha_{p+1}\right), S_b\left(\beta_p, \beta_p, \beta_{p+1}\right), \\ \frac{1}{2\kappa^4}\left[S_b\left(\alpha_{p-1}, \alpha_{p-1}, \alpha_{p+1}\right) + S_b\left(\alpha_p, \alpha_p, \alpha_p\right)\right], \\ \frac{1}{2\kappa^4}\left[S_b\left(\beta_{p-1}, \beta_{p-1}, \beta_{p+1}\right) + S_b\left(\beta_p, \beta_p, \beta_p\right)\right] \end{array} \right\} \\
= & \max\left\{ \begin{array}{c} S_b\left(\alpha_{p-1}, \alpha_{p-1}, \alpha_p\right), S_b\left(\beta_{p-1}, \beta_{p-1}, \beta_p\right), \\ S_b\left(\alpha_p, \alpha_p, \alpha_{p+1}\right), S_b\left(\beta_p, \beta_p, \beta_{p+1}\right) \end{array} \right\}
\end{aligned}$$

and

$$\begin{aligned}
& N_b^S(\lambda_p, \zeta_p, \lambda_{p+1}, \zeta_{p+1}) \\
= & \min\left\{ \begin{array}{c} S_b\left(\Lambda\lambda_p, \Lambda\lambda_p, \Gamma(\lambda_p, \zeta_p)\right), S_b\left(\Lambda\lambda_{p+1}, \Lambda\lambda_{p+1}, \Gamma(\lambda_{p+1}, \zeta_{p+1})\right), S_b\left(\Lambda\lambda_p, \Lambda\lambda_p, \Gamma(\lambda_{p+1}, \zeta_{p+1})\right), \\ S_b\left(\Lambda\zeta_p, \Lambda\zeta_p, \Gamma(\zeta_{p+1}, \lambda_{p+1})\right), S_b\left(\Lambda\lambda_{p+1}, \Lambda\lambda_{p+1}, \Gamma(\lambda_p, \zeta_p)\right), S_b\left(\Lambda\zeta_{p+1}, \Lambda\zeta_{p+1}, \Gamma(\zeta_p, \lambda_p)\right) \end{array} \right\} \\
= & \min\left\{ \begin{array}{c} S_b\left(\alpha_{p-1}, \alpha_{p-1}, \alpha_p\right), S_b\left(\alpha_p, \alpha_p, \alpha_{p+1}\right), S_b\left(\alpha_{p-1}, \alpha_{p-1}, \alpha_{p+1}\right), \\ S_b\left(\beta_{p-1}, \beta_{p-1}, \beta_{p+1}\right), S_b\left(\alpha_p, \alpha_p, \alpha_p\right), S_b\left(\beta_p, \beta_p, \beta_p\right) \end{array} \right\} = 0.
\end{aligned}$$

From (3.2), we deduce

$$\begin{aligned}
\psi_\star\left(2\kappa^4 S_b(\alpha_p, \alpha_p, \alpha_{p+1})\right) \leq\ & \Delta\left(\psi_\star\left(\max\left\{ \begin{array}{c} S_b\left(\alpha_{p-1}, \alpha_{p-1}, \alpha_p\right), \\ S_b\left(\beta_{p-1}, \beta_{p-1}, \beta_p\right), \\ S_b\left(\alpha_p, \alpha_p, \alpha_{p+1}\right), \\ S_b\left(\beta_p, \beta_p, \beta_{p+1}\right) \end{array} \right\}\right), \phi_\star\left(\max\left\{ \begin{array}{c} S_b\left(\alpha_{p-1}, \alpha_{p-1}, \alpha_p\right), \\ S_b\left(\beta_{p-1}, \beta_{p-1}, \beta_p\right), \\ S_b\left(\alpha_p, \alpha_p, \alpha_{p+1}\right), \\ S_b\left(\beta_p, \beta_p, \beta_{p+1}\right) \end{array} \right\}\right)\right) \\
\leq\ & \psi_\star\left(\max\left\{ \begin{array}{c} S_b\left(\alpha_{p-1}, \alpha_{p-1}, \alpha_p\right), S_b\left(\beta_{p-1}, \beta_{p-1}, \beta_p\right), \\ S_b\left(\alpha_p, \alpha_p, \alpha_{p+1}\right), S_b\left(\beta_p, \beta_p, \beta_{p+1}\right) \end{array} \right\}\right).
\end{aligned}$$

By using (ψ_0), we have

$$S_b(\alpha_p, \alpha_p, \alpha_{p+1}) \leq \max\left\{ \begin{array}{c} \frac{1}{2\kappa^4} S_b\left(\alpha_{p-1}, \alpha_{p-1}, \alpha_p\right), \frac{1}{2\kappa^4} S_b\left(\beta_{p-1}, \beta_{p-1}, \beta_p\right), \\ \frac{1}{2\kappa^4} S_b\left(\alpha_p, \alpha_p, \alpha_{p+1}\right), \frac{1}{2\kappa^4} S_b\left(\beta_p, \beta_p, \beta_{p+1}\right) \end{array} \right\}.$$

If for some $p \in \mathbb{N}$, $\frac{1}{2\kappa^4} S_b\left(\alpha_{p-1}, \alpha_{p-1}, \alpha_p\right) < \frac{1}{2\kappa^4} S_b\left(\alpha_p, \alpha_p, \alpha_{p+1}\right)$ and $\frac{1}{2\kappa^4} S_b\left(\beta_{p-1}, \beta_{p-1}, \beta_p\right) < \frac{1}{2\kappa^4} S_b\left(\beta_p, \beta_p, \beta_{p+1}\right)$, then we have

$$S_b(\alpha_p, \alpha_p, \alpha_{p+1}) \leq \max\left\{ \frac{1}{2\kappa^4} S_b\left(\alpha_p, \alpha_p, \alpha_{p+1}\right), \frac{1}{2\kappa^4} S_b\left(\beta_p, \beta_p, \beta_{p+1}\right) \right\}. \tag{3.3}$$

By similar arguments we obtain

$$S_b(\beta_p, \beta_p, \beta_{p+1}) \leq \max\left\{ \frac{1}{2\kappa^4} S_b\left(\beta_p, \beta_p, \beta_{p+1}\right), \frac{1}{2\kappa^4} S_b\left(\alpha_p, \alpha_p, \alpha_{p+1}\right) \right\}. \tag{3.4}$$

Combining (3.3) and (3.4), we can get

$$\max\left\{ \begin{array}{c} S_b(\alpha_p, \alpha_p, \alpha_{p+1}), \\ S_b(\beta_p, \beta_p, \beta_{p+1}) \end{array} \right\} \leq \frac{1}{2\kappa^4} \max\left\{ \begin{array}{c} S_b\left(\alpha_p, \alpha_p, \alpha_{p+1}\right), \\ S_b\left(\beta_p, \beta_p, \beta_{p+1}\right) \end{array} \right\}.$$

This is contradiction. Hence

$$\begin{aligned}\max\left\{\ S_b(\alpha_p,\alpha_p,\alpha_{p+1}),S_b(\beta_p,\beta_p,\beta_{p+1})\ \right\} &\leq \frac{1}{2\kappa^4}\max\left\{\ S_b\left(\alpha_{p-1},\alpha_{p-1},\alpha_p\right),S_b\left(\beta_{p-1},\beta_{p-1},\beta_p\right)\ \right\}\\ &\leq \frac{1}{(2\kappa^4)^2}\max\left\{\begin{array}{c} S_b\left(\alpha_{p-2},\alpha_{p-2},\alpha_{p-1}\right),\\ S_b\left(\beta_{p-2},\beta_{p-2},\beta_{p-1}\right)\end{array}\right\}\\ &\ \ \vdots\\ &\leq \frac{1}{(2\kappa^4)^p}\max\left\{\ S_b\left(\alpha_0,\alpha_0,\alpha_1\right),S_b\left(\beta_0,\beta_0,\beta_1\right)\ \right\}\to 0 \text{ as } p\to\infty.\end{aligned}$$

Now, we prove that $\{\alpha_p\}$ and $\{\beta_p\}$ are Cauchy sequences in $(\mathcal{G},S_b)$. On contrary we suppose that $\{\alpha_p\}$ and $\{\beta_p\}$ are not Cauchy. Then there exist $\epsilon>0$ and monotonically increasing sequences of natural numbers $\{q_k\}$ and $\{p_k\}$ such that $p_k>q_k$.

$$S_b\left(\alpha_{q_k},\alpha_{q_k},\alpha_{p_k}\right)\geq\epsilon \qquad S_b\left(\beta_{q_k},\beta_{q_k},\beta_{p_k}\right)\geq\epsilon \tag{3.5}$$

and

$$S_b\left(\alpha_{q_k},\alpha_{q_k},\alpha_{p_k-1}\right)<\epsilon \qquad S_b\left(\beta_{q_k},\beta_{q_k},\beta_{p_k-1}\right)<\epsilon. \tag{3.6}$$

From Lemma (2.5), (3.5) and (3.6), we have

$$\begin{aligned}\epsilon &\leq S_b\left(\alpha_{q_k},\alpha_{q_k},\alpha_{p_k}\right)\\ &\leq 2\kappa S_b\left(\alpha_{q_k},\alpha_{q_k},\alpha_{q_k+1}\right)+\kappa^2 S_b\left(\alpha_{q_k+1},\alpha_{q_k+1},\alpha_{p_k}\right).\end{aligned}$$

So

$$2\kappa^2\epsilon \leq 4\kappa^3 S_b\left(\alpha_{q_k},\alpha_{q_k},\alpha_{q_k+1}\right)+2\kappa^4 S_b\left(\alpha_{q_k+1},\alpha_{q_k+1},\alpha_{p_k}\right).$$

Letting $k\to\infty$ and applying $\psi_\star$ on both sides, we have that

$$\begin{aligned}\psi_\star(2\kappa^2\epsilon) &\leq \lim_{k\to\infty}\psi_\star\left(2\kappa^4 S_b\left(\alpha_{q_k+1},\alpha_{q_k+1},\alpha_{p_k}\right)\right)\\ &= \lim_{k\to\infty}\psi\left(2\kappa^4 S_b\left(\Gamma(\lambda_{q_{k+1}},\zeta_{q_{k+1}}),\Gamma(\lambda_{q_{k+1}},\zeta_{q_{k+1}}),\Gamma(\lambda_{p_k},\zeta_{p_k})\right)\right)\\ &\leq \lim_{k\to\infty}\Delta\left(\psi_\star\left(M_b^S(\lambda_{q_{k+1}},\zeta_{q_{k+1}},\lambda_{p_k},\zeta_{p_k})\right),\phi_\star\left(M_b^S(\lambda_{q_{k+1}},\zeta_{q_{k+1}},\lambda_{p_k},\zeta_{p_k})\right)\right)\\ &\quad+\lim_{k\to\infty}\Theta\phi_\star\left(N_b^S(\lambda_{q_{k+1}},\zeta_{q_{k+1}},\lambda_{p_k},\zeta_{p_k})\right)\end{aligned} \tag{3.7}$$

where,

$$\begin{aligned}&\lim_{k\to\infty}M_b^S(\lambda_{q_{k+1}},\zeta_{q_{k+1}},\lambda_{p_k},\zeta_{p_k})\\ &= \lim_{k\to\infty}\max\left\{\begin{array}{c} S_b\left(\Lambda\lambda_{q_{k+1}},\Lambda\lambda_{q_{k+1}},\Lambda\lambda_{p_k}\right),S_b\left(\Lambda\zeta_{q_{k+1}},\Lambda\zeta_{q_{k+1}},\Lambda\zeta_{p_k}\right),S_b\left(\Lambda\lambda_{q_{k+1}},\Lambda\lambda_{q_{k+1}},\Gamma(\lambda_{q_{k+1}},\zeta_{q_{k+1}})\right),\\ S_b\left(\Lambda\zeta_{q_{k+1}},\Lambda\zeta_{q_{k+1}},\Gamma(\zeta_{q_{k+1}},\lambda_{q_{k+1}})\right),S_b\left(\Lambda\lambda_{p_k},\Lambda\lambda_{p_k},\Gamma(\lambda_{p_k},\zeta_{p_k})\right),S_b\left(\Lambda\zeta_{p_k},\Lambda\zeta_{p_k},\Gamma(\zeta_{p_k},\lambda_{p_k})\right),\\ \frac{1}{2\kappa^4}\left[S_b\left(\Lambda\lambda_{q_{k+1}},\Lambda\lambda_{q_{k+1}},\Gamma(\lambda_{p_k},\zeta_{p_k})\right)+S_b\left(\Lambda\lambda_{p_k},\Lambda\lambda_{p_k},\Gamma(\lambda_{q_{k+1}},\zeta_{q_{k+1}})\right)\right],\\ \frac{1}{2\kappa^4}\left[S_b\left(\Lambda\zeta_{q_{k+1}},\Lambda\zeta_{q_{k+1}},\Gamma(\zeta_{p_k},\lambda_{p_k})\right)+S_b\left(\Lambda\zeta_{p_k},\Lambda\zeta_{p_k},\Gamma(\zeta_{q_{k+1}},\lambda_{q_{k+1}})\right)\right]\end{array}\right\}\\ &= \lim_{k\to\infty}\max\left\{\begin{array}{c} S_b\left(\alpha_{q_k},\alpha_{q_k},\alpha_{p_{k-1}}\right),S_b\left(\beta_{q_k},\beta_{q_k},\beta_{p_{k-1}}\right),S_b\left(\alpha_{q_k},\alpha_{q_k},\alpha_{q_{k+1}}\right),\\ S_b\left(\beta_{q_k},\beta_{q_k},\beta_{q_{k+1}}\right),S_b\left(\alpha_{p_{k-1}},\alpha_{p_{k-1}},\alpha_{p_k}\right),S_b\left(\beta_{p_{k-1}},\beta_{p_{k-1}},\beta_{p_k}\right),\\ \frac{1}{2\kappa^4}\left[S_b\left(\alpha_{q_k},\alpha_{q_k},\alpha_{p_k}\right)+S_b\left(\alpha_{p_{k-1}},\alpha_{p_{k-1}},\alpha_{q_{k+1}}\right)\right],\\ \frac{1}{2\kappa^4}\left[S_b\left(\beta_{q_k},\beta_{q_k},\beta_{p_k}\right)+S_b\left(\beta_{p_{k-1}},\beta_{p_{k-1}},\beta_{q_{k+1}}\right)\right]\end{array}\right\}\\ &= \lim_{k\to\infty}\max\left\{\begin{array}{c}\epsilon,\epsilon,0,0,0,0,\\ \frac{1}{2\kappa^4}\left[S_b\left(\alpha_{q_k},\alpha_{q_k},\alpha_{p_k}\right)+S_b\left(\alpha_{p_{k-1}},\alpha_{p_{k-1}},\alpha_{q_{k+1}}\right)\right],\\ \frac{1}{2\kappa^4}\left[S_b\left(\beta_{q_k},\beta_{q_k},\beta_{p_k}\right)+S_b\left(\beta_{p_{k-1}},\beta_{p_{k-1}},\beta_{q_{k+1}}\right)\right]\end{array}\right\}.\end{aligned}$$

But

$$\begin{aligned}&\lim_{k\to\infty}\frac{1}{2\kappa^4}\left[S_b\left(\alpha_{q_k},\alpha_{q_k},\alpha_{p_k}\right)+S_b\left(\alpha_{p_{k-1}},\alpha_{p_{k-1}},\alpha_{q_{k+1}}\right)\right]\\ &\leq \lim_{k\to\infty}\frac{1}{2\kappa^4}\left[\begin{array}{c}[2\kappa S_b\left(\alpha_{q_k},\alpha_{q_k},\alpha_{p_{k-1}}\right)+\kappa^2 S_b\left(\alpha_{p_{k-1}},\alpha_{p_{k-1}},\alpha_{p_k}\right)]+\\ [2\kappa S_b\left(\alpha_{p_{k-1}},\alpha_{p_{k-1}},\alpha_{q_k}\right)+\kappa^2 S_b\left(\alpha_{q_k},\alpha_{q_k},\alpha_{q_{k+1}}\right)]\end{array}\right]<\frac{2}{\kappa^3}\epsilon.\end{aligned}$$

Also,

$$\begin{aligned}&\lim_{k\to\infty}\frac{1}{2\kappa^4}\left[S_b\left(\beta_{q_k},\beta_{q_k},\beta_{p_k}\right)+S_b\left(\beta_{p_{k-1}},\beta_{p_{k-1}},\beta_{q_{k+1}}\right)\right]\\ \leq\ &\lim_{k\to\infty}\frac{1}{2\kappa^4}\left[\begin{array}{c}[2\kappa S_b\left(\beta_{q_k},\beta_{q_k},\beta_{p_{k-1}}\right)+\kappa^2 S_b\left(\beta_{p_{k-1}},\beta_{p_{k-1}},\beta_{p_k}\right)]+\\ [2\kappa S_b\left(\beta_{p_{k-1}},\beta_{p_{k-1}},\beta_{q_k}\right)+\kappa^2 S_b\left(\beta_{q_k},\beta_{q_k},\beta_{q_{k+1}}\right)]\end{array}\right]<\frac{2}{\kappa^3}\epsilon.\end{aligned}$$

Therefore,

$$\lim_{k\to\infty}M_b^S(\lambda_{q_{k+1}},\zeta_{q_{k+1}},\lambda_{p_k},\zeta_{p_k})\leq\lim_{k\to\infty}\max\left\{\ \epsilon,\epsilon,0,\tfrac{2}{\kappa^3}\epsilon\ \right\}=\epsilon,$$

and

$$\begin{aligned}\lim_{k\to\infty}N_b^S(\lambda_{q_{k+1}},\zeta_{q_{k+1}},\lambda_{p_k},\zeta_{p_k})&=\lim_{k\to\infty}\min\left\{\begin{array}{c}S_b\left(\Lambda\lambda_{q_{k+1}},\Lambda\lambda_{q_{k+1}},\Gamma(\lambda_{q_{k+1}},\zeta_{q_{k+1}})\right),S_b\left(\Lambda\lambda_{p_k},\Lambda\lambda_{p_k},\Gamma(\lambda_{p_k},\zeta_{p_k})\right),\\ S_b\left(\Lambda\lambda_{q_{k+1}},\Lambda\lambda_{q_{k+1}},\Gamma(\lambda_{p_k},\zeta_{p_k})\right),S_b\left(\Lambda\zeta_{q_{k+1}},\Lambda\zeta_{q_{k+1}},\Gamma(\zeta_{p_k},\lambda_{p_k})\right),\\ S_b\left(\Lambda\lambda_{p_k},\Lambda\lambda_{p_k},\Gamma(\lambda_{q_{k+1}},\zeta_{q_{k+1}})\right),S_b\left(\Lambda\zeta_{p_k},\Lambda\zeta_{p_k},\Gamma(\zeta_{q_{k+1}},\lambda_{q_{k+1}})\right)\end{array}\right\}\\ &=\lim_{k\to\infty}\min\left\{\begin{array}{c}S_b\left(\alpha_{q_k},\alpha_{q_k},\alpha_{q_{k+1}}\right),S_b\left(\alpha_{p_{k-1}},\alpha_{p_{k-1}},\alpha_{p_k}\right),S_b\left(\alpha_{q_k},\alpha_{q_k},\alpha_{p_k}\right),\\ S_b\left(\beta_{q_k},\beta_{q_k},\beta_{p_k}\right),S_b\left(\alpha_{p_{k-1}},\alpha_{p_{k-1}},\alpha_{q_{k+1}}\right),S_b\left(\beta_{p_{k-1}},\beta_{p_{k-1}},\beta_{q_{k+1}}\right)\end{array}\right\}=0.\end{aligned}$$

Therefore, from (3.7), we deduce

$$\psi_\star(2\kappa^2\epsilon)\quad\leq\quad\Delta\left(\psi_\star\left(\epsilon\right),\phi_\star\left(\epsilon\right)\right)\leq\psi_\star\left(\epsilon\right)$$

by the definition of (ψ_0), we have that $2\kappa^2\epsilon\leq\epsilon$ which is a contradiction. Hence $\{\alpha_p\}$ is an S_b-Cauchy sequence in complete S_b-metric spaces $(\mathcal{G},S_b)$. By similar arguments, we obtain $\{\beta_p\}$ is an S_b-Cauchy sequence in $\mathcal{G}$. Since $\Lambda(\mathcal{G}\})$ is a complete subspace of $(\mathcal{G},S_b)$, the sequences $\{\alpha_p\}$, $\{\beta_p\}$ are convergence to u,v respectively in $\Lambda(\mathcal{G})$. Thus, there exist $a,b\in\Lambda(\mathcal{G})$ such that

$$\lim_{p\to\infty}\alpha_p=u=\Lambda a\quad\ \textit{and}\quad\ \lim_{p\to\infty}\beta_p=v=\Lambda b \tag{3.8}$$

We claim that $\Gamma(a,b)=u$ and $\Gamma(b,a)=v$. Suppose $\Gamma(a,b)\neq u$ and $\Gamma(b,a)\neq v$. By Lemma (2.6), we have that

$$\tfrac{1}{2\kappa}S_b\left(\Gamma(a,b),\Gamma(a,b),u\right)\quad\leq\lim_{k\to\infty}\inf S_b\left(\Gamma(a,b),\Gamma(a,b),\alpha_p\right).$$

Now from (3.1) and applying $\psi_\star$ on both sides, we have

$$\begin{aligned}\psi_\star\left(\kappa^3 S_b\left(\Gamma(a,b),\Gamma(a,b),u\right)\right)\quad&\leq\quad\lim_{p\to\infty}\inf\psi_\star\left(2\kappa^4 S_b\left(\Gamma(a,b),\Gamma(a,b),\alpha_p\right)\right)\\ &=\quad\lim_{p\to\infty}\inf\psi_\star\left(2\kappa^4 S_b\left(\Gamma(a,b),\Gamma(a,b),\Gamma(\lambda_p,\zeta_p)\right)\right)\\ &\leq\quad\lim_{p\to\infty}\inf\Delta\left(\psi_\star\left(M_b^S(a,b,\lambda_p,\zeta_p)\right),\phi_\star\left(M_b^S(a,b,\lambda_p,\zeta_p)\right)\right)\\ &\quad\quad+\lim_{p\to\infty}\inf\Theta\phi_\star\left(N_b^S(a,b,\lambda_p,\zeta_p)\right)\\ &\leq\quad\lim_{p\to\infty}\inf\psi_\star\left(M_b^S(a,b,\lambda_p,\zeta_p)\right)+\lim_{p\to\infty}\inf\Theta\phi_\star\left(N_b^S(a,b,\lambda_p,\zeta_p)\right)\end{aligned}\tag{3.9}$$

where,

$$\begin{aligned}&\lim_{p\to\infty}\inf M_b^S(a,b,\lambda_p,\zeta_p)\\ =\ &\lim_{p\to\infty}\inf\max\left\{\begin{array}{c}S_b\left(\Lambda a,\Lambda a,\Lambda\lambda_p\right),S_b\left(\Lambda b,\Lambda b,\Lambda\zeta_p\right),S_b\left(\Lambda a,\Lambda a,\Gamma(a,b)\right),\\ S_b\left(\Lambda b,\Lambda b,\Gamma(b,a)\right),S_b\left(\Lambda\lambda_p,\Lambda\lambda_p,\Gamma(\lambda_p,\zeta_p)\right),S_b\left(\Lambda\zeta_p,\Lambda\zeta_p,\Gamma(\zeta_p,\lambda_p)\right),\\ \frac{1}{2\kappa^4}\left[S_b\left(\Lambda a,\Lambda a,\Gamma(\lambda_p,\zeta_p)\right)+S_b\left(\Lambda\lambda_p,\Lambda\lambda_p,\Gamma(a,b)\right)\right],\\ \frac{1}{2\kappa^4}\left[S_b\left(\Lambda b,\Lambda b,\Gamma(\zeta_p,\lambda_p)\right)+S_b\left(\Lambda\zeta_p,\Lambda\zeta_p,\Gamma(b,a)\right)\right]\end{array}\right\}\end{aligned}$$

$$
\begin{aligned}
&\leq \lim_{p\to\infty}\sup\max\left\{\begin{array}{c} S_b(\Lambda a,\Lambda a,\alpha_{p-1}),S_b(\Lambda b,\Lambda b,\beta_{p-1}),S_b(\Lambda a,\Lambda a,\Gamma(a,b)),\\ S_b(\Lambda b,\Lambda b,\Gamma(b,a)),S_b(\alpha_{p-1},\alpha_{p-1},\alpha_p),S_b(\beta_{p-1},\beta_{p-1},\beta_p),\\ \frac{1}{2\kappa^4}\left[S_b(\Lambda a,\Lambda a,\alpha_p)+S_b(\alpha_{p-1},\alpha_{p-1},\Gamma(a,b))\right],\\ \frac{1}{2\kappa^4}\left[S_b(\Lambda b,\Lambda b,\beta_p)+S_b(\beta_{p-1},\beta_{p-1},\Gamma(b,a))\right]\end{array}\right\}\\
&\leq \max\left\{ S_b(u,u,\Gamma(a,b)),S_b(v,v,\Gamma(b,a)) \right\}\\
&\leq \max\left\{ \kappa S_b(\Gamma(a,b),\Gamma(a,b),u),\kappa S_b(\Gamma(b,a),\Gamma(b,a),v) \right\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{p\to\infty} N_b^S(a,b,\lambda_p,\zeta_p)\\
&= \lim_{p\to\infty}\min\left\{\begin{array}{c} S_b(\Lambda a,\Lambda a,\Gamma(a,b)),S_b(\Lambda\lambda_p,\Lambda\lambda_p,\Gamma(\lambda_p,\zeta_p)),S_b(\Lambda a,\Lambda a,\Gamma(\lambda_p,\zeta_p)),\\ S_b(\Lambda b,\Lambda b,\Gamma(\zeta_p,\lambda_p)),S_b(\Lambda\lambda_p,\Lambda\lambda_p,\Gamma(b,a)),S_b(\Lambda\zeta_p,\Lambda\zeta_p,\Gamma(b,a))\end{array}\right\}\\
&= \lim_{p\to\infty}\min\left\{\begin{array}{c} S_b(u,u,\Gamma(a,b)),S_b(\alpha_{p-1},\alpha_{p-1},\alpha_p),S_b(u,u,\alpha_p),\\ S_b(v,v,\beta_p),S_b(\alpha_{p-1},\alpha_{p-1},\Gamma(b,a)),S_b(\beta_{p-1},\beta_{p-1},\Gamma(b,a))\end{array}\right\}=0.
\end{aligned}
$$

Hence, from (3.9) we have that

$$
\psi_\star\left(\kappa^3 S_b(\Gamma(a,b),\Gamma(a,b),u)\right) \leq \psi_\star\left(\max\left\{ \kappa S_b(\Gamma(a,b),\Gamma(a,b),u),\kappa S_b(\Gamma(b,a),\Gamma(b,a),v) \right\}\right)+\lim_{k\to\infty}\inf\Theta\phi_\star(0).
$$

By the definition of (ψ_0), we get $S_b(\Gamma(a,b),\Gamma(a,b),u)\leq\max\left\{ \frac{1}{\kappa^2}S_b(\Gamma(a,b),\Gamma(a,b),u),\frac{1}{\kappa^2}S_b(\Gamma(b,a),\Gamma(b,a),v) \right\}$. Similarly, we can prove that $S_b(\Gamma(b,a),\Gamma(b,a),v)\leq\max\left\{ \frac{1}{\kappa^2}S_b(\Gamma(a,b),\Gamma(a,b),u),\frac{1}{\kappa^2}S_b(\Gamma(b,a),\Gamma(b,a),v) \right\}$.

Therefore,

$$
\max\left\{ S_b(\Gamma(a,b),\Gamma(a,b),u),S_b(\Gamma(b,a),\Gamma(b,a),v) \right\}\leq\max\left\{ \frac{1}{\kappa^2}S_b(\Gamma(a,b),\Gamma(a,b),u),\frac{1}{\kappa^2}S_b(\Gamma(b,a),\Gamma(b,a),v) \right\}
$$

which is a contradiction. So $\Gamma(a,b)=u$ and $\Gamma(b,a)=v$. It follows that $\Gamma(a,b)=u=\Lambda a$ and $\Gamma(b,a)=v=\Lambda b$. Since $\{\Gamma,\Lambda\}$ is a weakly compatible pair, we have $\Gamma(u,v)=\Lambda u$, $\Gamma(v,u)=\Lambda v$. Now we prove that $\Lambda u=u$ and $\Lambda v=v$. From Lemma (2.6), we have that

$$
\frac{1}{\kappa^2}S_b(\Lambda u,\Lambda u,u)\leq\lim_{p\to\infty}\inf\ S_b(\Lambda u,\Lambda u,\alpha_p).
$$

Now from (3.1) and applying $\psi_\star$ on both sides, we have

$$
\begin{aligned}
\psi_\star\left(2\kappa^2 S_b(\Lambda u,\Lambda u,u)\right) &\leq \lim_{p\to\infty}\inf\psi_\star\left(2\kappa^4 S_b(\Gamma(u,v),\Gamma(u,v),\Gamma(\lambda_p,\zeta_p))\right)\\
&\leq \lim_{p\to\infty}\inf\Delta\left(\psi_\star\left(M_b^S(u,v,\lambda_p,\zeta_p)\right),\phi_\star\left(M_b^S(u,v,\lambda_p,\zeta_p)\right)\right)\\
&\quad+\lim_{p\to\infty}\inf\Theta\phi_\star\left(N_b^S(u,v,\lambda_p,\zeta_p)\right)\\
&\leq \lim_{p\to\infty}\inf\psi_\star\left(M_b^S(u,v,\lambda_p,\zeta_p)\right)+\lim_{p\to\infty}\inf\Theta\phi_\star\left(N_b^S(u,v,\lambda_p,\zeta_p)\right)
\end{aligned}
\tag{3.10}
$$

where,

$$
\begin{aligned}
&\lim_{p\to\infty}\inf M_b^S(u,v,\lambda_p,\zeta_p)\\
&= \lim_{p\to\infty}\inf\max\left\{\begin{array}{c} S_b(\Lambda u,\Lambda u,\Lambda\lambda_p),S_b(\Lambda v,\Lambda v,\Lambda\zeta_p),S_b(\Lambda u,\Lambda u,\Gamma(u,v)),\\ S_b(\Lambda v,\Lambda v,\Gamma(v,u)),S_b(\Lambda\lambda_p,\Lambda\lambda_p,\Gamma(\lambda_p,\zeta_p)),S_b(\Lambda\zeta_p,\Lambda\zeta_p,\Gamma(\zeta_p,\lambda_p)),\\ \frac{1}{2\kappa^4}\left[S_b(\Lambda u,\Lambda u,\Gamma(\lambda_p,\zeta_p))+S_b(\Lambda\lambda_p,\Lambda\lambda_p,\Gamma(u,v))\right],\\ \frac{1}{2\kappa^4}\left[S_b(\Lambda v,\Lambda v,\Gamma(\zeta_p,\lambda_p))+S_b(\Lambda\zeta_p,\Lambda\zeta_p,\Gamma(v,u))\right]\end{array}\right\}\\
&\leq \lim_{p\to\infty}\sup\max\left\{\begin{array}{c} S_b(\Lambda u,\Lambda u,\alpha_{p-1}),S_b(\Lambda v,\Lambda v,\beta_{p-1}),S_b(\Lambda u,\Lambda u,\Gamma(u,v)),\\ S_b(\Lambda v,\Lambda v,\Gamma(v,u)),S_b(\alpha_{p-1},\alpha_{p-1},\alpha_p),S_b(\beta_{p-1},\beta_{p-1},\beta_p),\\ \frac{1}{2\kappa^4}\left[S_b(\Lambda u,\Lambda u,\alpha_p)+S_b(\alpha_{p-1},\alpha_{p-1},\Gamma(u,v))\right],\\ \frac{1}{2\kappa^4}\left[S_b(\Lambda v,\Lambda v,\beta_p)+S_b(\beta_{p-1},\beta_{p-1},\Gamma(v,u))\right]\end{array}\right\}\\
&\leq \max\left\{ S_b(\Lambda u,\Lambda u,u),S_b(\Lambda v,\Lambda v,v) \right\}
\end{aligned}
$$

and

$$\begin{aligned}
&\lim_{p\to\infty} N_b^S(u,v,\lambda_p,\zeta_p)\\
=&\lim_{p\to\infty}\min\left\{\begin{array}{c} S_b\left(\Lambda u,\Lambda v,\Gamma(u,v)\right), S_b\left(\Lambda\lambda_p,\Lambda\lambda_p,\Gamma(\lambda_p,\zeta_p)\right), S_b\left(\Lambda u,\Lambda u,\Gamma(\lambda_p,\zeta_p)\right),\\ S_b\left(\Lambda v,\Lambda v,\Gamma(\zeta_p,\lambda_p)\right), S_b\left(\Lambda\lambda_p,\Lambda\lambda_p,\Gamma(v,u)\right), S_b\left(\Lambda\zeta_p,\Lambda\zeta_p,\Gamma(v,u)\right)\end{array}\right\}\\
=&\lim_{p\to\infty}\min\left\{\begin{array}{c} S_b\left(u,u,\Gamma(u,v)\right), S_b\left(\alpha_{p-1},\alpha_{p-1},\alpha_p\right), S_b\left(u,u,\alpha_p\right),\\ S_b\left(v,v,\beta_p\right), S_b\left(\alpha_{p-1},\alpha_{p-1},\Gamma(v,u)\right), S_b\left(\beta_{p-1},\beta_{p-1},\Gamma(v,u)\right)\end{array}\right\}=0.
\end{aligned}$$

Hence from (3.10) we have that

$$\psi_\star\left(2\kappa^2 S_b\left(\Lambda u,\Lambda u,u\right)\right) \leq \psi_\star\left(\max\left\{\ S_b\left(\Lambda u,\Lambda u,u\right), S_b\left(\Lambda v,\Lambda v,v\right)\ \right\}\right)+\lim_{p\to\infty}\inf\Theta\phi_\star\left(0\right).$$

By the definition of (ψ_0), we get $S_b\left(\Lambda u,\Lambda u,u\right)\leq\max\left\{\ \frac{1}{2\kappa^2}S_b\left(\Lambda u,\Lambda u,u\right), \frac{1}{2\kappa^2}S_b\left(\Lambda v,\Lambda v,v\right)\ \right\}$. Similarly, we can prove that $S_b\left(\Lambda v,\Lambda v,v\right)\leq\max\left\{\ \frac{1}{2\kappa^2}S_b\left(\Lambda u,\Lambda u,u\right), \frac{1}{2\kappa^2}S_b\left(\Lambda v,\Lambda v,v\right)\ \right\}$. Therefore,

$$\max\left\{\ S_b\left(\Lambda u,\Lambda u,u\right), S_b\left(\Lambda v,\Lambda v,v\right)\ \right\}\leq\max\left\{\ \frac{1}{2\kappa^2}S_b\left(\Lambda u,\Lambda u,u\right), \frac{1}{2\kappa^2}S_b\left(\Lambda v,\Lambda v,v\right)\ \right\}$$

which is a contradiction. So $\Lambda u=u$ and $\Lambda v=v$. It follows that $\Gamma(u,v)=\Lambda u=u$ and $\Gamma(v,u)=\Lambda v=v$. Thus, (u,v) is a coupled fixed point of Γ and Λ. In the following we will show the uniqueness of common coupled fixed point in $\mathcal{G}$. For this purpose, assume that there is another common coupled fixed point (u',v') of Γ,Λ. Then From (3.1), we have

$$\begin{aligned}
\psi_\star\left(2\kappa^4 S_b(u,u,u')\right) &= \psi_\star\left(2\kappa^4 S_b(\Gamma(u,v),\Gamma(u,v),\Gamma(u',v'))\right)\\
&\leq\Delta\left(\psi_\star\left(M_b^S(u,v,u',v')\right),\phi_\star\left(M_b^S(u,v,u',v')\right)\right)+\Theta\phi_\star\left(N_b^S(u,v,u',v')\right)\\
&\leq\psi_\star\left(M_b^S(u,v,u',v')\right)+\Theta\phi_\star\left(N_b^S(u,v,u',v')\right)
\end{aligned}\tag{3.11}$$

where,

$$\begin{aligned}
M_b^S(u,v,u',v') &= \max\left\{\begin{array}{c} S_b\left(\Lambda u,\Lambda u,\Lambda u'\right), S_b\left(\Lambda v,\Lambda v,\Lambda v'\right), S_b\left(\Lambda u,\Lambda u,\Gamma(u,v)\right),\\ S_b\left(\Lambda v,\Lambda v,\Gamma(v,u)\right), S_b\left(\Lambda u',\Lambda u',\Gamma(u',v')\right), S_b\left(\Lambda v',\Lambda v',\Gamma(v',u')\right),\\ \frac{1}{2\kappa^4}\left[S_b\left(\Lambda u,\Lambda u,\Gamma(u',v')\right)+S_b\left(\Lambda u',\Lambda u',\Gamma(u,v)\right)\right],\\ \frac{1}{2\kappa^4}\left[S_b\left(\Lambda v,\Lambda v,\Gamma(v',u')\right)+S_b\left(\Lambda v',\Lambda v',\Gamma(v,u)\right)\right]\end{array}\right\}\\
&= \max\left\{\ S_b\left(u,u,u'\right), S_b\left(v,v,v'\right)\ \right\}
\end{aligned}$$

and

$$N_b^S(u,v,u',v')=\min\left\{\begin{array}{c} S_b\left(\Lambda u,\Lambda u,\Gamma(u,v)\right), S_b\left(\Lambda u',\Lambda u',\Gamma(u',v')\right), S_b\left(\Lambda u,\Lambda u,\Gamma(u',v')\right),\\ S_b\left(\Lambda v,\Lambda v,\Gamma(v',u')\right), S_b\left(\Lambda u',\Lambda u',\Gamma(u,v)\right), S_b\left(\Lambda v',\Lambda v',\Gamma(v,u)\right)\end{array}\right\}=0.$$

From (3.11), we have

$$\psi_\star\left(2\kappa^4 S_b(u,u,u')\right) \leq\psi_\star\left(\max\left\{\ S_b\left(u,u,u'\right), S_b\left(v,v,v'\right)\ \right\}\right)+\Theta\phi_\star\left(0\right).$$

By the definition of $\psi_\star$, we deduce that

$$S_b(u,u,u')\leq\max\left\{\ \frac{1}{2\kappa^4}S_b\left(u,u,u'\right), \frac{1}{2\kappa^4}S_b\left(v,v,v'\right)\ \right\}.$$

Similarly, we get that

$$S_b(v,v,v')\leq\max\left\{\ \frac{1}{2\kappa^4}S_b\left(u,u,u'\right), \frac{1}{2\kappa^4}S_b\left(v,v,v'\right)\ \right\}.$$

Therefore, we have

$$\max\left\{\ S_b\left(u,u,u'\right), S_b\left(v,v,v'\right)\ \right\}\leq\max\left\{\ \frac{1}{2\kappa^4}S_b\left(u,u,u'\right), \frac{1}{2\kappa^4}S_b\left(v,v,v'\right)\ \right\}$$

which is a contradiction unless $S_b\left(u,u,u'\right)=0$ and $S_b\left(v,v,v'\right)=0$ that is $u=u'$ and $v=v'$. Hence Γ and Λ have a unique common coupled fixed point. □

Corollary 3.3. Let $(\mathcal{G}, S_b)$ be a complete S_b-metric space with coefficient $\kappa \geq 1$. Let $\Gamma : \mathcal{G}^2 \to \mathcal{G}$ be a mapping and $\Theta \geq 0$ such that

$$\begin{aligned} & \psi_\star\left(2\kappa^4 S_b(\Gamma(u,v),\Gamma(u,v),\Gamma(p,q))\right) \\ \leq \;& \Delta\left(\eta\left(\psi_\star\left(M_b^S(u,v,p,q)\right)\right)\psi_\star\left(M_b^S(u,v,p,q)\right),\eta\left(\phi_\star\left(M_b^S(u,v,p,q)\right)\right)\phi_\star\left(M_b^S(u,v,p,q)\right)\right) \quad +\Theta\phi_\star\left(N_b^S(u,v,p,q)\right)\end{aligned}$$

where,

$$M_b^S(u,v,p,q) = \max\left\{\begin{array}{c} S_b\left(u,u,p\right), S_b\left(v,v,q\right), S_b\left(u,u,\Gamma(u,v)\right), \\ S_b\left(v,v,\Gamma(v,u)\right), S_b\left(p,p,\Gamma(p,q)\right), S_b\left(q,q,\Gamma(q,p)\right), \\ \frac{1}{2\kappa^4}\left[S_b\left(u,u,\Gamma(p,q)\right)+S_b\left(p,p,\Gamma(u,v)\right)\right], \\ \frac{1}{2\kappa^4}\left[S_b\left(v,v,\Gamma(q,p)\right)+S_b\left(q,q,\Gamma(v,u)\right)\right] \end{array}\right\}$$

and

$$N_b^S(u,v,p,q) = \min\left\{\begin{array}{c} S_b\left(u,u,\Gamma(u,v)\right), S_b\left(p,p,\Gamma(p,q)\right), S_b\left(u,u,\Gamma(p,q)\right), \\ S_b\left(v,v,\Gamma(q,p)\right), S_b\left(p,p,\Gamma(u,v)\right), S_b\left(q,q,\Gamma(v,u)\right) \end{array}\right\}$$

for all $u,v,p,q \in \mathcal{G}$ and $\Delta \in C$, $\psi_\star \in \mathfrak{F}$, $\phi_\star \in \mathfrak{G}$ and $\eta : [0,1) \to [0,\infty)$ is a continuous mapping. Then there is a unique coupled fixed point of Γ in $\mathcal{G}$.

Proof . The proof Follows along similar lines of Theorem 3.2 if we take identity function $I_{\mathcal{G}}$ in place of $\Lambda : \mathcal{G} \to \mathcal{G}$ in Theorem 3.2. □

Corollary 3.4. Let $(\mathcal{G}, S_b)$ be a complete S_b-metric space with coefficient $\kappa \geq 1$. Let $\Gamma : \mathcal{G}^2 \to \mathcal{G}$ be a mapping and $\Theta \geq 0$ such that

$$2\kappa^4 S_b(\Gamma(u,v),\Gamma(u,v),\Gamma(p,q)) \leq \eta\left(M_b^S(u,v,p,q)\right) M_b^S(u,v,p,q) + \Theta N_b^S(u,v,p,q)$$

for all $u,v,p,q \in \mathcal{G}$ and $\eta : [0,1) \to [0,\infty)$ is a continuous mapping. Then there is a unique coupled fixed point of Γ in $\mathcal{G}$.

Proof . The proof follows from Theorems 3.2 by taking $\psi_\star(t) = t = \phi_\star(t)$ and $\Delta(s,t) = s\eta(s)$. □

Corollary 3.5. Let $(\mathcal{G}, S_b)$ be a complete S_b-metric space with coefficient $\kappa \geq 1$. Let $\Gamma : \mathcal{G}^2 \to \mathcal{G}$ be a mapping satiesfying

$$S_b(\Gamma(u,v),\Gamma(u,v),\Gamma(p,q)) \leq \tau M_b^S(u,v,p,q)$$

for all $u,v,p,q \in \mathcal{G}$ and $\tau \in [0, \frac{1}{2\kappa^4})$. Then there is a unique coupled fixed point of Γ in $\mathcal{G}$.

Proof . Let us take $\Delta(s,t) \leq s$ and $\psi_\star(t) = t$, $\phi_\star(t) = 0$, from Theorem (3.2), we see that Γ has a unique coupled fixed point. □

Example 3.6. Let $S_b : \mathcal{G}^3 \to \mathbb{R}^+$ be a mapping defined as

$$S_b(\alpha,\beta,\gamma) = (|\beta+\gamma-2\alpha| + |\beta-\gamma|)^2$$

where $\mathcal{G} = [0,\infty)$. So clearly $(\mathcal{G}, S_b)$ is a complete S_b-metric space with $\kappa = 2$. Define $\Gamma : \mathcal{G}^2 \to \mathcal{G}$ and $\Lambda :\to \mathcal{G}$ by $\Gamma(a,b) = \frac{a+b}{128\sqrt{2}}$ and $\Lambda a = \frac{a}{4}$. Let $\psi_\star : [0,\infty) \to [0,\infty)$ and $\phi_\star : [0,\infty) \to [0,\infty)$ be defined as $\psi_\star(t) = t$ and $\phi_\star(t) = \frac{t}{2}$. Also let $\Delta : [0,+\infty)^2 \to R$ by $\Delta(s,t) = s - t$.Then obviously, $\Gamma(\mathcal{G}^2) \subseteq \Lambda(\mathcal{G})$ and the pair (Γ, Λ) is ω-compatible and clearly for all $\imath, \jmath, \eth, \mho \in \mathcal{G}$, we have

$$\begin{aligned} \psi_\star\left(2\kappa^4\left(S_b(\Gamma(\imath,\jmath),\Gamma(\imath,\jmath),\Gamma(\eth,\mho))\right)\right) &= 2\kappa^4\left(S_b(\Gamma(\imath,\jmath),\Gamma(\imath,\jmath),\Gamma(\eth,\mho))\right) \\ &= 2\kappa^4\left(|\Gamma(\imath,\jmath)+\Gamma(\eth,\mho)-2\Gamma(\imath,\jmath)| + |\Gamma(\imath,\jmath)-\Gamma(\eth,\mho)|\right)^2 \\ &= 2\kappa^4\left(2|\frac{\imath+\jmath}{128\sqrt{2}} - \frac{\eth+\mho}{128\sqrt{2}}|\right)^2 = \frac{2\kappa^4}{8\kappa^8}\left(2|\frac{\imath-\eth}{4} - \frac{\jmath-\mho}{4}|\right)^2 \\ &\leq \frac{1}{2}\left[\frac{1}{2\kappa^4}\left(S_b(\Lambda\imath,\Lambda\imath,\Lambda\eth) + S_b(\Lambda\jmath,\Lambda\jmath,\Lambda\mho)\right)\right] \\ &\leq \frac{1}{2}M_b^S(\imath,\jmath,\eth,\mho) \\ &\leq \psi_\star\left(M_b^S(\imath,\jmath,\eth,\mho)\right) - \phi_\star\left(M_b^S(\imath,\jmath,\eth,\mho)\right) \\ &\leq \Delta\left(\psi_\star\left(M_b^S(\imath,\jmath,\eth,\mho)\right), \phi_\star\left(M_b^S(\imath,\jmath,\eth,\mho)\right)\right) + \Theta\phi_\star\left(N_b^S(\imath,\jmath,\eth,\mho)\right). \end{aligned}$$

Thus, all assumptions of Theorem 3.2 are satisfied and $(0,0)$ is the unique common coupled fixed point of Γ and Λ.

4 Application to Integral Equations

In this section, we apply our Corollary 3.3 to the existence theorem for solution of the following nonlinear integral equations:

$$\alpha(t) = \int_0^T K(t,\alpha(s),\beta(s))ds, \text{ and } \beta(t) = \int_0^T K(t,\beta(s),\alpha(s))ds, \quad t \in I = [0,T] \tag{4.1}$$

where T is a real number such that $T > 0$ and $K : I \times \mathbb{R}^2 \to \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$. Let $\mathcal{G} = C(I, \mathbb{R})$ be the space of all $\mathbb{R}$-valued continuous functions on I. An element $(x,y) \in \mathcal{G} \times \mathcal{G}$ is called a coupled solution of the integral Eq. (4.1) if $x(t) \le y(t)$ and

$$x(t) = \int_0^T K(t,x(s),y(s))ds \text{ and } y(t) = \int_0^T K(t,y(s),x(s),)ds, \text{ where } t \in I = [0,T]$$

Now, we consider the following assumptions:

($\star_1$) $K : I \times \mathbb{R}^2 \to \mathbb{R}$ is continuous;

($\star_2$) for all $t \in I$ and for all $x, y, u, v \in \mathbb{R}$, we have

$$0 \le |K(t,u(s),v(s)) - K(t,x(s),y(s))| \le \frac{1}{8\kappa^2 T}|u(s) - x(s) + y(s) - v(s)|.$$

Next, we give the existence theorem for solution of the integral Eq. (4.1)

Theorem 4.1. Suppose that $\star_1$ and $\star_2$ hold. Then,there is a solution of integral equation 4.1.

Proof . Let $\mathcal{G} = C(I,\mathbb{R})$ be the space of all $\mathbb{R}$-valued continuous functions on I. We endowed $\mathcal{G}$ with the S_b-metric $S_b : \mathcal{G}^3 \to R^+$ defined by $S_b(x,y,z) = \sup_{t\in[0,T]} \left(|y(t)+z(t)-2x(t)| + |y(t)-z(t)|\right)^2$ for $x,y,z \in \mathcal{G}$. Then it is clear that $(\mathcal{G}, S_b)$ is a complete S_b-metric space with $\kappa = 2^{2(2-1)} = 4$, Define $\psi_\star, \phi_\star : [0,\infty) \to [0,\infty)$ by $\psi_\star(t) = t$, $\phi_\star(t) = \frac{2t}{3}$. Let $\eta : [0,1) \to [0,\infty)$ and $\Delta : [0,+\infty)^2 \to R$ by $\eta(t) = \frac{1}{2}$, $\Delta(s,t) = ms$ where $m \in (0,1)$. Define $\Gamma : \mathcal{G}^2 \to \mathcal{G}$ by

$$\Gamma(\alpha,\beta)(t) = \int_0^T K(t,\alpha(s),\beta(s))ds, \quad t \in I = [0,T] \text{ and } \alpha,\beta \in \mathcal{G}.$$

Now, let $\rho, \varrho, \mu, \nu \in \mathcal{G}$ and using ($\star_2$), for all $t \in I = [0,T]$, we have

$$\begin{aligned}
|\Gamma(\rho,\varrho)(t) - \Gamma(\mu,\nu)(t)|^2 &= |\int_0^T K(s,\rho(s),\varrho(s))ds - \int_0^T K(s,\mu(s),\nu(s))ds|^2 \\
&= \left(\int_0^T |K(s,\rho(s),\varrho(s)) - K(s,\mu(s),\nu(s))|ds\right)^2 \\
&\le \left(\int_0^T \frac{1}{8\kappa^2 T}|\rho(s) - \mu(s) + \nu(s) - \varrho(s)|ds\right)^2
\end{aligned}$$

$$\leq \frac{1}{64\kappa^4 T^2}\left(\int_0^T\left(\sup_{z\in[0,T]}|\rho(z)-\mu(z)|+\sup_{z\in[0,T]}|\nu(z)-\varrho(z)|\right)ds\right)^2$$
$$\leq \frac{1}{32\kappa^4}\left[\sup_{z\in[0,T]}(|\rho(z)-\mu(z)|)^2+\sup_{z\in[0,T]}(|\nu(z)-\varrho(z)|)^2\right]$$

which implies that

$$2\kappa^4\sup_{t\in[0,T]}(2|\Gamma(\rho,\varrho)(t)-\Gamma(\mu,\nu)(t)|)^2 \leq \frac{1}{16}\left[\sup_{z\in[0,T]}(2|\rho(z)-\mu(z)|)^2+\sup_{z\in[0,T]}(2|\nu(z)-\varrho(z)|)^2\right].$$

Therefore, we get that

$$\begin{aligned}2\kappa^4 S_b(\Gamma(\rho,\varrho)(t),\Gamma(\rho,\varrho)(t),\Gamma(\mu,\nu)(t)) &\leq \frac{1}{16}\left(S_b(\rho,\rho,\mu)+S_b(\varrho,\varrho,\nu)\right)\\ &\leq \frac{1}{8}\max\{S_b(\rho,\rho,\mu),S_b(\varrho,\varrho,\nu)\}\\ &\leq \frac{1}{2}\left(\frac{1}{2}M_b^S(\rho,\varrho,\mu,\nu)\right).\end{aligned}$$

Thus

$$\begin{aligned}&\psi_\star\left(2\kappa^4 S_b(\Gamma(\rho,\varrho)(t),\Gamma(\rho,\varrho)(t),\Gamma(\mu,\nu)(t))\right)\\ &\leq\Delta\left(\eta\left(\psi_\star\left(M_b^S(\rho,\varrho,\mu,\nu)\right)\right)\psi_\star\left(M_b^S(\rho,\varrho,\mu,\nu)\right),\eta\left(\phi_\star\left(M_b^S(u\rho,\varrho,\mu,\nu)\right)\right)\phi_\star\left(M_b^S(\rho,\varrho,\mu,\nu)\right)\right)+\Theta\phi_\star\left(N_b^S(\rho,\varrho,\mu,\nu)\right).\end{aligned}$$

It follows from Corollary 3.3, that the equation (4.1) has a unique solution in $C(I,\mathbb{R})$. □

5 Application to Homotopy Theory

Now we present the main result regrading application to Homotopy theory.

Theorem 5.1. Let $(\mathcal{G},S_b)$ be a complete S_b-metric space, $\Im$ and $\overline{\Im}$ be two open and closed subsets of $\mathcal{G}$ such that $\Im\subseteq\overline{\Im}$. Suppose $H_b:\overline{\Im}^2\times[0,1]\to\mathcal{G}$ be an operator such that the following conditions are satisfied,

(τ_0) $p\neq H_b(p,q,\chi)$ and $q\neq H_b(q,p,\chi)$ for each $p,q\in\partial\Im$ and $\chi\in[0,1]$ (here $\partial\Im$ denotes the boundary of $\Im$ in $\mathcal{G}$),

(τ_1) $\psi_\star\left(2\kappa^4 S_b(H_b(u,v,\chi),H_b(u,v,\chi),H_b(p,q,\chi))\right)\leq\Delta\left(\psi_\star\left(M_b^S(u,v,p,q)\right),\phi_\star(N_b^S(u,v,p,q))\right)\ \forall\ u,v,p,q\in\overline{\Im}$, $\chi\in[0,1]$ and $\Delta\in C$, $\psi_\star\in\mathfrak{F}$, $\phi_\star\in\mathfrak{G}$ where

$$M_b^S(u,v,p,q)=\max\left\{\begin{array}{c}S_b(u,u,p),S_b(v,v,q),S_b(u,u,H_b(u,v,\chi)),\\ S_b(v,v,H_b(v,u,\chi)),S_b(p,p,H_b(p,q,\chi)),S_b(q,q,H_b(q,p,\chi)),\\ \frac{1}{2\kappa^4}\left[S_b(u,u,H_b(p,q,\chi))+S_b(p,p,H_b(u,v,\chi))\right],\\ \frac{1}{2\kappa^4}\left[S_b(v,v,H_b(q,p,\chi))+S_b(q,q,H_b(v,u,\chi))\right]\end{array}\right\}$$

and
$$N_b^S(u,v,p,q)=\min\left\{\begin{array}{c}S_b(u,u,H_b(u,v,\chi)),S_b(p,p,H_b(p,q,\chi)),\\ S_b(u,u,H_b(p,q,\chi)),S_b(v,v,H_b(q,p,\chi)),\\ S_b(p,p,H_b(u,v,\chi)),S_b(q,q,H_b(v,u,\chi))\end{array}\right\}$$

(τ_2) there exists $M\geq 0$ such that $S_b(H_b(p,q,\chi),H_b(p,q,\chi),H_b(p,q,\mu))\leq M|\chi-\mu|$, for all $p,q\in\overline{\Im}$ and $\chi,\mu\in[0,1]$.

Then $H_b(.,0)$ has a coupled fixed point $\Leftrightarrow$ $H_b(.,1)$ has a coupled fixed point.

Proof . Consider the set

$$U=\{\chi\in[0,1]:p=H_b(p,q,\chi)\&q=H_b(q,p,\chi)\text{ for some }p,q\in\Im\}.$$

Since $H_b(.,0)$ has a coupled fixed point in $\Im$, we have $(0,0)\in U^2$. So U is a non-empty set. We will show that U is both open and closed in $[0,1]$ and so by the connectedness we have that $U=[0,1]$. As a result, $H_b(.,1)$ has a coupled fixed point in $\Im$. First we show that U is closed in $[0,1]$. To see this let $\{\chi_n\}_{n=1}^{\infty} \subseteq \Im$ with $\chi_n \to \chi \in [0,1]$ as $n \to \infty$. We must show that $\chi \in \Im$. Since $\chi_n \in U$ for $n = 1,2,3,\cdots$, $\exists\ p_n, q_n \in \Im$ with $p_n = H_b(p_n, q_n, \chi_n)$ and $q_n = H_b(q_n, p_n, \chi_n)$. Consider

$$\begin{aligned} S_b(p_n,p_n,p_{n+1}) &= S_b(H_b(p_n,q_n,\chi_n),H_b(p_n,q_n,\chi_n),H_b(p_{n+1},q_{n+1},\chi_{n+1})) \\ &\leq 2\kappa S_b(H_b(p_n,q_n,\chi_n),H_b(p_n,q_n,\chi_n),H_b(p_{n+1},q_{n+1},\chi_n)) \\ &\quad +\kappa^2 S_b(H_b(p_{n+1},q_{n+1},\chi_n),H_b(p_{n+1},q_{n+1},\chi_n),H_b(p_{n+1},q_{n+1},\chi_{n+1})) \\ &\leq 2\kappa S_b(H_b(p_n,q_n,\chi_n),H_b(p_n,q_n,\chi_n),H_b(p_{n+1},q_{n+1},\chi_n)) + \kappa^2 M|\chi_n-\chi_{n+1}|. \end{aligned}$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} S_b(p_n,p_n,p_{n+1}) \leq \lim_{n\to\infty} 2\kappa S_b(H_b(p_n,q_n,\chi_n),H_b(p_n,q_n,\chi_n),H_b(p_{n+1},q_{n+1},\chi_n)) + 0.$$

Since $\psi_\star$ is a non-decreasing continuous function, we get

$$\begin{aligned} \lim_{n\to\infty} \psi_\star(\kappa^3 S_b(p_n,p_n,p_{n+1})) &\leq \lim_{n\to\infty} \psi_\star(2\kappa^4 S_b(H_b(p_n,q_n,\chi_n),H_b(p_n,q_n,\chi_n),H_b(p_{n+1},q_{n+1},\chi_n))) \\ &\leq \lim_{n\to\infty} \left[\Delta\left(\psi_\star(M_b^S(p_n,q_n,p_{n+1},q_{n+1})),\phi_\star(N_b^S(p_n,q_n,p_{n+1},q_{n+1}))\right)\right] \end{aligned} \tag{5.1}$$

where,

$$\begin{aligned} & M_b^S(p_n,q_n,p_{n+1},q_{n+1}) \\ = \ & \max\left\{\begin{array}{c} S_b(p_n,p_n,p_{n+1}), S_b(q_n,q_n,q_{n+1}), S_b(p_n,p_n,H_b(p_n,q_n,\chi_n)), \\ S_b(q_n,q_n,H_b(q_n,p_n,\chi_n)), S_b(p_{n+1},p_{n+1},H_b(p_{n+1},q_{n+1},\chi_{n+1})), \\ S_b(q_{n+1},q_{n+1},H_b(q_{n+1},p_{n+1},\chi_{n+1})), \\ \frac{1}{2\kappa^4}\left[S_b(p_n,p_n,H_b(p_{n+1},q_{n+1},\chi_{n+1})) + S_b(p_{n+1},p_{n+1},H_b(p_n,q_n,\chi_n))\right], \\ \frac{1}{2\kappa^4}\left[S_b(q_n,q_n,H_b(q_{n+1},p_{n+1},\chi_{n+1})) + S_b(q_{n+1},q_{n+1},H_b(q_n,p_n,\chi_n))\right] \end{array}\right\} \\ \leq \ & \max\left\{\begin{array}{c} S_b(p_n,p_n,p_{n+1}), S_b(q_n,q_n,q_{n+1}), S_b(p_n,p_n,p_n), \\ S_b(q_n,q_n,q_n), S_b(p_{n+1},p_{n+1},p_{n+1}), S_b(q_{n+1},q_{n+1},q_{n+1}), \\ \frac{1+\kappa}{2\kappa^4} S_b(p_n,p_n,p_{n+1}), \\ \frac{1+\kappa}{2\kappa^4} S_b(q_n,q_n,q_{n+1}) \end{array}\right\} \\ = \ & \max\left\{ S_b(p_n,p_n,p_{n+1}), S_b(q_n,q_n,q_{n+1}) \right\} \end{aligned}$$

and

$$\begin{aligned} N_b^S(p_n,q_n,p_{n+1},q_{n+1}) &= \min\left\{\begin{array}{c} S_b(p_n,p_n,H_b(p_n,q_n,\chi_n)), S_b(p_{n+1},p_{n+1},H_b(p_{n+1},q_{n+1},\chi_{n+1})), \\ S_b(p_n,p_n,H_b(p_{n+1},q_{n+1},\chi_{n+1})), S_b(q_n,q_n,H_b(q_{n+1},p_{n+1},\chi_{n+1})), \\ S_b(p_{n+1},p_{n+1},H_b(p_n,q_n,\chi_n)), S_b(q_{n+1},q_{n+1},H_b(q_n,p_n,\chi_n)) \end{array}\right\} \\ &= \min\left\{\begin{array}{c} S_b(p_n,p_np_n), S_b(p_{n+1},p_{n+1},p_{n+1}), \\ S_b(p_n,p_n,p_{n+1}), S_b(q_n,q_n,q_{n+1}), \\ S_b(p_{n+1},p_{n+1},p_n), S_b(q_{n+1},q_{n+1},q_n) \end{array}\right\} = 0. \end{aligned}$$

From (5.1), we deduced that

$$\begin{aligned} \lim_{n\to\infty} \psi(\kappa^3 S_b(p_n,p_n,p_{n+1})) &\leq \lim_{n\to\infty} \left[\Delta\left(\psi_\star\left(\max\left\{\begin{array}{c} S_b(p_n,p_n,p_{n+1}), \\ S_b(q_n,q_n,q_{n+1}) \end{array}\right\}\right), \phi_\star(0)\right)\right] \\ &= \lim_{n\to\infty} \psi_\star\left(\max\left\{\begin{array}{c} S_b(p_n,p_n,p_{n+1}), \\ S_b(q_n,q_n,q_{n+1}) \end{array}\right\}\right). \end{aligned}$$

By the properties of $\psi_\star$, it follows that

$$\lim_{n\to\infty} S_b(p_n,p_n,p_{n+1}) \leq \lim_{n\to\infty} \max\left\{\begin{array}{c} \frac{1}{\kappa^3} S_b(p_n,p_n,p_{n+1}), \\ \frac{1}{\kappa^3} S_b(q_n,q_n,q_{n+1}) \end{array}\right\}. \tag{5.2}$$

Similarly, we can prove

$$\lim_{n\to\infty} S_b(q_n, q_n, q_{n+1}) \leq \lim_{n\to\infty} \max \left\{ \begin{array}{c} \frac{1}{\kappa^3} S_b\left(p_n, p_n, p_{n+1}\right), \\ \frac{1}{\kappa^3} S_b\left(q_n, q_n, q_{n+1}\right) \end{array} \right\}. \tag{5.3}$$

Combining (5.2) and (5.3), we deduce that

$$\lim_{n\to\infty} \max \left\{ \begin{array}{c} S_b\left(p_n, p_n, p_{n+1}\right), \\ S_b\left(q_n, q_n, q_{n+1}\right) \end{array} \right\} \leq \lim_{n\to\infty} \max \left\{ \begin{array}{c} \frac{1}{\kappa^3} S_b\left(p_n, p_n, p_{n+1}\right), \\ \frac{1}{\kappa^3} S_b\left(q_n, q_n, q_{n+1}\right) \end{array} \right\}.$$

$$\lim_{n\to\infty} (1 - \frac{1}{\kappa^3}) \max \left\{ \begin{array}{c} S_b\left(p_n, p_n, p_{n+1}\right), \\ S_b\left(q_n, q_n, q_{n+1}\right) \end{array} \right\} \leq 0.$$

So

$$\lim_{n\to\infty} S_b(p_n, p_n, p_{n+1}) = 0 \quad \text{and} \quad \lim_{n\to\infty} S_b(q_n, q_n, q_{n+1}) = 0. \tag{5.4}$$

Now we prove that $\{p_n\}$ and $\{q_n\}$ are S_b - Cauchy sequence in $(\mathcal{G}, S_p)$. On contrary suppose that $\{p_n\}$ and $\{q_n\}$ are not S_b-Cauchy. There exists an $\epsilon > 0$ and monotone increasing sequences of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k$,

$$S_b(p_{m_k}, p_{m_k}, p_{n_k}) \geq \epsilon, \qquad S_b(q_{m_k}, q_{m_k}, q_{n_k}) \geq \epsilon \tag{5.5}$$

and

$$S_b(p_{m_k}, p_{m_k}, p_{n_k-1}) < \epsilon, \qquad S_b(q_{m_k}, q_{m_k}, q_{n_k-1}) < \epsilon. \tag{5.6}$$

From (5.5), (5.6) and using the Lemma (2.6), we obtain

$$\begin{aligned} \epsilon &\leq S_b(p_{m_k}, p_{m_k}, p_{n_k}) \\ &\leq 2\kappa S_b(p_{m_k}, p_{m_k}, p_{m_k+1}) + \kappa^2 S_b(p_{m_k+1}, p_{m_k+1}, p_{n_k}). \end{aligned}$$

Letting $k \to \infty$, and using the Lemma (2.6) we have

$$\begin{aligned} \epsilon &\leq \lim_{k\to\infty} \kappa^2 S_b(p_{m_k+1}, p_{m_k+1}, p_{n_k}) \\ &\leq \lim_{k\to\infty} \kappa^2 S_b\left(H_b(p_{m_{k+1}}, q_{m_{k+1}}, \chi_{m_{k+1}}), H_b(p_{m_{k+1}}, q_{m_{k+1}}, \chi_{m_{k+1}}), H_b(p_{n_k}, q_{n_k}, \chi_{n_k})\right) \\ &\leq \lim_{k\to\infty} 2\kappa^3 S_b\left(H_b(p_{m_{k+1}}, q_{m_{k+1}}, \chi_{m_{k+1}}), H_b(p_{m_{k+1}}, q_{m_{k+1}}, \chi_{m_{k+1}}), H_b(p_{n_k}, q_{n_k}, \chi_{m_{k+1}})\right) \\ &\quad + \lim_{k\to\infty} \kappa^4 S_b\left(H_b(p_{n_k}, q_{n_k}, \chi_{m_{k+1}}), H_b(p_{n_k}, q_{n_k}, \chi_{m_{k+1}}), H_b(p_{n_k}, q_{n_k}, \chi_{n_k})\right) \\ &\leq \lim_{k\to\infty} 2\kappa^3 S_b\left(H_b(p_{m_{k+1}}, q_{m_{k+1}}, \chi_{m_{k+1}}), H_b(p_{m_{k+1}}, q_{m_{k+1}}, \chi_{m_{k+1}}), H_b(p_{n_k}, q_{n_k}, \chi_{m_{k+1}})\right) \\ &\quad + \lim_{k\to\infty} M\kappa^4 |\chi_{m_{k+1}} - \chi_{n_k}| \end{aligned}$$

By the property of $\psi_\star$, we have

$$\begin{aligned} \psi_\star(\kappa\epsilon) &\leq \lim_{k\to\infty} \psi_\star\left(2\kappa^4 S_b\left(H_b(p_{m_{k+1}}, q_{m_{k+1}}, \chi_{m_{k+1}}), H_b(p_{m_{k+1}}, q_{m_{k+1}}, \chi_{m_{k+1}}), H_b(p_{n_k}, q_{n_k}, \chi_{m_{k+1}})\right)\right) \\ &\leq \lim_{k\to\infty} \left[\Delta\left(\psi_\star(M_b^S(p_{m_{k+1}}, q_{m_{k+1}}, p_{n_k}, q_{n_k})), \phi_\star(N_b^S(p_{m_{k+1}}, q_{m_{k+1}}, p_{n_k}, q_{n_k}))\right)\right] \end{aligned} \tag{5.7}$$

where,

$$\begin{aligned} &\lim_{k\to\infty} M_b^S(p_{m_{k+1}}, q_{m_{k+1}}, p_{n_k}, q_{n_k}) \\ &= \lim_{k\to\infty} \max \left\{ \begin{array}{c} S_b\left(p_{m_{k+1}}, p_{m_{k+1}}, p_{n_k}\right), S_b\left(q_{m_{k+1}}, q_{m_{k+1}}, q_{n_k}\right), \\ S_b\left(p_{m_{k+1}}, p_{m_{k+1}}, H_b(p_{m_{k+1}}, q_{m_{k+1}}, \chi_{m_{k+1}})\right), S_b\left(q_{m_{k+1}}, q_{m_{k+1}}, H_b(q_{m_{k+1}}, p_{m_{k+1}}, \chi_{m_{k+1}})\right), \\ S_b\left(p_{n_k}, p_{n_k}, H_b(p_{n_k}, q_{n_k}, \chi_{n_k})\right), S_b\left(q_{n_k}, q_{n_k}, H_b(q_{n_k}, p_{n_k}, \chi_{n_k})\right), \\ \frac{1}{2\kappa^4}\left[S_b\left(p_{m_{k+1}}, p_{m_{k+1}}, H_b(p_{n_k}, q_{n_k}, \chi_{n_k})\right) + S_b\left(p_{n_k}, p_{n_k}, H_b(p_{m_{k+1}}, q_{n_{k+1}}, \chi_{m_{k+1}})\right)\right], \\ \frac{1}{2\kappa^4}\left[S_b\left(q_{m_{k+1}}, q_{m_{k+1}}, H_b(q_{n_k}, p_{n_k}, \chi_{n_k})\right) + S_b\left(q_{n_k}, q_{n_k}, H_b(q_{m_{k+1}}, p_{m_{k+1}}, \chi_{m_{k+1}})\right)\right] \end{array} \right\} \\ &\leq \max\left\{ \epsilon, \epsilon, \frac{[1+k]\epsilon}{2\kappa^4}, \frac{[1+k]\epsilon}{2\kappa^4} \right\} = \epsilon \end{aligned}$$

and

$$\begin{aligned}
&\lim_{n\to\infty} N_b^S(p_{m_{k+1}}, q_{m_{k+1}}, p_{n_k}, q_{n_k}) \\
&= \lim_{n\to\infty} \min\left\{ \begin{array}{c} S_b\left(p_{m_{k+1}}, p_{m_{k+1}}, H_b(p_{m_{k+1}}, q_{m_{k+1}}, \chi_{m_{k+1}})\right), S_b\left(p_{n_k}, p_{n_k}, H_b(p_{n_k}, q_{n_k}, \chi_{n_k})\right), \\ S_b\left(p_{m_{k+1}}, p_{m_{k+1}}, H_b(p_{n_k}, q_{n_k}, \chi_{n_k})\right), S_b\left(q_{m_{k+1}}, q_{m_{k+1}}, H_b(q_{n_k}, p_{n_k}, \chi_{n_k})\right), \\ S_b\left(p_{n_k}, p_{n_k}, H_b(p_{m_{k+1}}, q_{m_{k+1}}, \chi_{m_{k+1}})\right), S_b\left(q_{n_k}, q_{n_k}, H_b(q_{m_{k+1}}, p_{m_{k+1}}, \chi_{m_{k+1}})\right) \end{array} \right\} \\
&= \min\left\{\ 0, 0, \epsilon, \epsilon, \kappa\epsilon, \kappa\epsilon\ \right\} = 0.
\end{aligned}$$

From (5.7), we deduce that

$$\psi_\star(\kappa\epsilon) \quad \leq \quad \Delta\left(\psi_\star(\epsilon), \phi_\star(0)\right) \leq \psi_\star(\epsilon).$$

Hence from the definition of $\psi_\star$, we have $\kappa\epsilon \leq \epsilon$, which is a contradiction. Hence $\{p_n\}$ is a S_b - Cauchy sequence in $(\mathcal{G}, S_b)$. Similarly, we can prove $\{q_n\}$ is a S_b-Cauchy sequence in $(\mathcal{G}, S_b)$ and by completeness of $(\mathcal{G}, S_b)$, there exists $\alpha, \beta \in \Im$ with

$$\lim_{n\to\infty} p_n = \alpha = \lim_{n\to\infty} p_{n+1} \quad and \quad \lim_{n\to\infty} q_n = \beta = \lim_{n\to\infty} q_{n+1}. \tag{5.8}$$

From Lemma (2.6), we have

$$\begin{aligned}
\psi_\star\left(\kappa^3 S_b(H_b(\alpha,\beta,\chi), H_b(\alpha,\beta,\chi), \alpha)\right) &\leq \lim_{n\to\infty}\inf\ \psi_\star\left(2\kappa^4 S_b(H_b(\alpha,\beta,\chi), H_b(\alpha,\beta,\chi), H_b(p_n,q_n,\chi))\right) \\
&\leq \lim_{n\to\infty}\inf[\Delta\left(\psi_\star(M_b^S(\alpha,\beta,p_n,q_n)), \phi_\star(N_b^S(\alpha,\beta,p_n,q_n))\right)]
\end{aligned} \tag{5.9}$$

where

$$\begin{aligned}
&\lim_{n\to\infty}\inf M_b^S(\alpha,\beta,p_n,q_n) \\
&= \lim_{n\to\infty}\inf\max\left\{ \begin{array}{c} S_b(\alpha,\alpha,p_n), S_b(\beta,\beta,q_n), S_b(\alpha,\alpha,H_b(\alpha,\beta,\chi)), \\ S_b(\beta,\beta,H_b(\beta,\alpha,\chi)), S_b(p_n,p_n,H_b(p_n,q_n,\chi_n)), S_b(q_n,q_n,H_b(q_n,p_n,\chi_n)), \\ \frac{1}{2\kappa^4}\left[S_b(\alpha,\alpha,H_b(p_n,q_n,\chi_n)) + S_b(p_n,p_n,H_b(\alpha,\beta,\chi))\right], \\ \frac{1}{2\kappa^4}\left[S_b(\beta,\beta,H_b(q_n,p_n,\chi_n)) + S_b(q_n,q_n,H_b(\beta,\alpha,\chi))\right] \end{array} \right\} \\
&\leq \sup\max\left\{ \begin{array}{c} \kappa S_b(H_b(\alpha,\beta,\chi), H_b(\alpha,\beta,\chi), \alpha), \\ \kappa S_b(H_b(\beta,\alpha,\chi), H_b(\beta,\alpha,\chi), \beta) \end{array} \right\} \\
&= \max\left\{ \begin{array}{c} \kappa S_b(H_b(\alpha,\beta,\chi), H_b(\alpha,\beta,\chi), \alpha), \\ \kappa S_b(H_b(\beta,\alpha,\chi), H_b(\beta,\alpha,\chi), \beta) \end{array} \right\}
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty}\inf N_b^S(\alpha,\beta,p_n,q_n) &= \lim_{n\to\infty}\inf\min\left\{ \begin{array}{c} S_b(\alpha,\alpha,H_b(\alpha,\beta,\chi)), S_b(p_n,p_n,H_b(p_n,q_n,\chi_n)), \\ S_b(\alpha,\alpha,H_b(p_n,q_n,\chi_n)), S_b(\beta,\beta,H_b(q_n,p_n,\chi_n)), \\ S_b(p_n,p_n,H_b(\alpha,\beta,\chi)), S_b(q_n,q_n,H_b(\beta,\alpha,\chi)) \end{array} \right\} \\
&= \min\left\{\ S_b(\alpha,\alpha,H_b(\alpha,\beta,\chi)), 0, S_b(\beta,\beta,H_b(\beta,\alpha,\chi))\ \right\} = 0.
\end{aligned}$$

From (5.9), we deduce that

$$\begin{aligned}
\psi_\star\left(\kappa^3 S_b(H_b(\alpha,\beta,\chi), H_b(\alpha,\beta,\chi), \alpha)\right) &\leq \Delta\left(\psi_\star\left(\max\left\{ \begin{array}{c} \kappa S_b(H_b(\alpha,\beta,\chi), H_b(\alpha,\beta,\chi), \alpha), \\ \kappa S_b(H_b(\beta,\alpha,\chi), H_b(\beta,\alpha,\chi), \beta) \end{array} \right\}\right), \phi_\star(0)\right) \\
&\leq \psi_\star\left(\max\left\{ \begin{array}{c} \kappa S_b(H_b(\alpha,\beta,\chi), H_b(\alpha,\beta,\chi), \alpha), \\ \kappa S_b(H_b(\beta,\alpha,\chi), H_b(\beta,\alpha,\chi), \beta) \end{array} \right\}\right).
\end{aligned}$$

By the property of $\psi_\star$, we have

$$S_b(H_b(\alpha,\beta,\chi), H_b(\alpha,\beta,\chi), \alpha) \quad \leq \quad \max\left\{ \begin{array}{c} \frac{1}{\kappa^2} S_b(H_b(\alpha,\beta,\chi), H_b(\alpha,\beta,\chi), \alpha), \\ \frac{1}{\kappa^2} S_b(H_b(\beta,\alpha,\chi), H_b(\beta,\alpha,\chi), \beta) \end{array} \right\}.$$

Similarly, we can prove that

$$S_b\left(H_b(\beta,\alpha,\chi),H_b(\beta,\alpha,\chi),\beta\right) \leq \max\left\{\begin{array}{c}\frac{1}{\kappa^2}S_b\left(H_b(\alpha,\beta,\chi),H_b(\alpha,\beta,\chi),\alpha\right),\\ \frac{1}{\kappa^2}S_b\left(H_b(\beta,\alpha,\chi),H_b(\beta,\alpha,\chi),\beta\right)\end{array}\right\}.$$

We conclude that

$$\max\left\{\begin{array}{c}S_b\left(H_b(\alpha,\beta,\chi),H_b(\alpha,\beta,\chi),\alpha\right),\\ S_b\left(H_b(\beta,\alpha,\chi),H_b(\beta,\alpha,\chi),\beta\right)\end{array}\right\} \leq \frac{1}{\kappa^2}\max\left\{\begin{array}{c}S_b\left(H_b(\alpha,\beta,\chi),H_b(\alpha,\beta,\chi),\alpha\right),\\ S_b\left(H_b(\beta,\alpha,\chi),H_b(\beta,\alpha,\chi),\beta\right)\end{array}\right\}.$$

It follows that $\alpha = H_b(\alpha,\beta,\chi)$ and $\beta = H_b(\beta,\alpha,\chi)$. Thus $\chi \in U$. Hence U is closed in [0, 1]. Let $\chi_0 \in U$. Then there exists $p_0, q_0 \in \Im$ with $p_0 = H_b(p_0,q_0,\chi_0)$ and $q_0 = H_b(q_0,p_0,\chi_0)$. Since U is open, then there exists $\delta > 0$ such that $B_{S_b}(p_0,\delta) \subseteq U$ and $B_{S_b}(q_0,\delta) \subseteq U$. Choose $\chi \in (\chi_0-\epsilon,\chi_0+\epsilon)$ such that $|\chi-\chi_0| \leq \frac{1}{M^n} < \epsilon$. Then, for $p \in \overline{B_b(p_0,\delta)} = \{p \in \mathcal{G}/S_b(p,p,p_0) \leq \delta+\kappa^2 S_b(p_0,p_0,p_0)\}$ and $q \in \overline{B_b(q_0,\delta)} = \{q \in \mathcal{G}/S_b(q,q,q_0) \leq \delta+\kappa^2 S_b(q_0,q_0,q_0)\}$ we have

$$\begin{aligned}
&S_b(H_b(p,q,\chi),H_b(p,q,\chi),p_0)\\
&\quad= S_b(H_b(p,q,\chi),H_b(p,q,\chi),H_b(p_0,q_0,\chi_0))\\
&\quad\leq 2\kappa S_b(H_b(p,q,\chi),H_b(p,q,\chi),H_b(p,q,\chi_0)) + \kappa^2 S_b(H_b(p,q,\chi_0),H_b(p,q,\chi_0),H_b(p_0,q_o,\chi_0))\\
&\quad\leq 2\kappa M|\chi-\chi_0| + \kappa^2 S_b(H_b(p,q,\chi_0),H_b(p,q,\chi_0),H_b(p_0,q_0,\chi_0))\\
&\quad\leq \frac{2\kappa}{M^{n-1}} + \kappa^2 S_b(H_b(p,q,\chi_0),H_b(p,q,\chi_0),H_b(p_0,q_0,\chi_0)).
\end{aligned}$$

Letting $n \to \infty$, we obtain

$$S_b(H_b(p,q,\chi),H_b(p,q,\chi),p_0) \leq \kappa^2 S_b(H_b(p,q,\chi_0),H_b(p,q,\chi_0),H_b(p_0,q_0,\chi_0)).$$

Since $\psi_\star$ is a non-decreasing continuous function, we have

$$\begin{aligned}
\psi_\star(S_b(H_b(p,q,\chi),H_b(p,q,\chi),p_0)) &\leq \psi_\star(2\kappa^2 S_b(H_b(p,q,\chi),H_b(p,q,\chi),p_0))\\
&\leq \psi_\star(2\kappa^4 S_b(H_b(p,q,\chi_0),H_b(p,q,\chi_0),H_b(p_0,q_0,\chi_0)))\\
&\leq \Delta\left(\psi_\star(M_b^S(p,q,p_0,q_0)),\phi_\star(N_b^S(p,q,p_0,q_0))\right) \qquad (5.10)
\end{aligned}$$

where

$$\begin{aligned}
M_b^S(p,q,p_0,q_0) &= \max\left\{\begin{array}{c}S_b\left(p,p,p_0\right),S_b\left(q,q,q_0\right),S_b\left(p,p,H_b(p,q,\chi)\right),\\ S_b\left(q,q,H_b(q,p,\chi)\right),S_b\left(p_0,p_0,H_b(p_0,q_0,\chi_0)\right),S_b\left(q_0,q_0,H_b(q_0,p_0,\chi_0)\right),\\ \frac{1}{2\kappa^4}\left[S_b\left(p,p,H_b(p_0,q_0,\chi_0)\right)+S_b\left(p_0,p_0,H_b(p,q,\chi)\right)\right],\\ \frac{1}{2\kappa^4}\left[S_b\left(q,q,H_b(q_0,p_0,\chi_0)\right)+S_b\left(q_0,q_0,H_b(q,q,\chi)\right)\right]\end{array}\right\}\\
&= \max\left\{\begin{array}{c}S_b\left(p,p,p_0\right),\text{‘}\\ S_b\left(q,q,q_0\right)\end{array}\right\}
\end{aligned}$$

and

$$\begin{aligned}
N_b^S(p,q,p_0,q_0) &= \min\left\{\begin{array}{c}S_b\left(p,p,H_b(p,q,\chi)\right),S_b\left(p_0,p_0,H_b(p_0,q_0,\chi_0)\right),\\ S_b\left(p,p,H_b(p_0,q_0,\chi_0)\right),S_b\left(q,q,H_b(q_0,p_0,\chi_0)\right),\\ S_b\left(p_0,p_0,H_b(p,q,\chi)\right),S_b\left(q_0,q_0,H_b(q,p,\chi)\right)\end{array}\right\}\\
&= \min\left\{\begin{array}{c}0,0,S_b\left(p,p,p_0\right),S_b\left(q,q,q_0\right),\\ \kappa S_b\left(p,p,p_0\right),\kappa S_b\left(q,q,q_0\right)\end{array}\right\} = 0.
\end{aligned}$$

From (5.10), we deduce that

$$\begin{aligned}
\psi_\star(S_b(H_b(p,q,\chi),H_b(p,q,\chi),p_0)) &\leq \Delta\left(\psi_\star\left(\max\left\{\begin{array}{c}S_b\left(p,p,p_0\right),\\ S_b\left(q,q,q_0\right)\end{array}\right\}\right),\phi_\star(0)\right)\\
&\leq \psi_\star\left(\max\left\{\begin{array}{c}S_b\left(p,p,p_0\right),\\ S_b\left(q,q,q_0\right)\end{array}\right\}\right).
\end{aligned}$$

Since $\psi_\star$ is a non-decreasing, we have

$$S_b(H_b(p,q,\chi),H_b(p,q,\chi),p_0) \quad \leq \quad \max\left\{\begin{array}{c} S_b\,(p,p,p_0), \\ S_b\,(q,q,q_0) \end{array}\right\} \leq \max\left\{\begin{array}{c} \delta+\kappa^2 S_b(p_0,p_0,p_0), \\ \delta+\kappa^2 S_b(q_0,q_0,q_0) \end{array}\right\}.$$

Similarly, we can prove that

$$S_b(H_b(q,p,\chi),H_b(q,p,\chi),q_0) \quad \leq \quad \max\left\{\begin{array}{c} \delta+\kappa^2 S_b(p_0,p_0,p_0), \\ \delta+\kappa^2 S_b(q_0,q_0,q_0) \end{array}\right\}.$$

Therefore, we conclude that

$$\max\left\{\begin{array}{c} S_b(H_b(p,q,\chi),H_b(p,q,\chi),p_0), \\ S_b(H_b(q,p,\chi),H_b(q,p,\chi),q_0) \end{array}\right\} \quad \leq \quad \max\left\{\begin{array}{c} \delta+\kappa^2 S_b(p_0,p_0,p_0), \\ \delta+\kappa^2 S_b(q_0,q_0,q_0) \end{array}\right\}.$$

Thus for each fixed $\chi \in (\chi_0-\epsilon, \chi_0+\epsilon)$, $H_b(.,\chi) : \overline{B_b(p_0,\delta)} \to \overline{B_b(p_0,\delta)}$ and $H_b(.,\chi) : \overline{B_b(q_0,\delta)} \to \overline{B_b(q_0,\delta)}$. Since also condition (τ_1) holds and $\psi_\star$ is continuous and non - decreasing and $\phi_\star$ is continuous with $\phi(\delta) > 0$ for $\delta > 0$, then all conditions of Theorem (5.1) are satisfied,Thus we deduce that $H_b(.,\chi)$ has a coupled fixed point in $\overline{\Im}$. But this coupled fixed point must be in $\Im$ since (τ_0) holds. Thus $\chi \in U$ for any $\chi \in (\chi_0-\epsilon,\chi_0+\epsilon)$. Hence $(\chi_0-\epsilon,\chi_0+\epsilon) \subseteq U$ and therefore U is open in [0, 1]. For the reverse implication, we use the same strategy. □

6 Conclusions

In this paper, we studied the existence and uniqueness of common coupled fixed point for two mappings via C-class functions in a complete S_b-metric space with an example. Also, we have provided an application to integral equations as well as to Homotopy by using altering distance functions and ultra altering distance functions in complete S_b-metric space.These generalizations could be useful in future research and applications.

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