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A new univalent integral operator defined by the Opoola differential operator

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Abstract

In this investigation, using Opoola differential operator $(D^m(\mu,\beta,t)f(z))$, a new integral operator: $I^{m,\sigma}_{t,\beta,\mu}(f_1,...,f_n)(z) : A^n \to A$ is defined in the unit disk, $U = \{z \in C : |z| < 1\}$; and we investigated the Univalence conditions of this generalized operator. Finally, a number of corollaries and remarks which show the extension of our results are presented.

Keywords: Analytic functions, Univalent functions starlike functions,convex functions, close-to-convex functions, Integral operator
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1 Introduction and definitions

Let A denote the family of functions, $f(z)$ that are analytic in the unit disk ,$U = \{z \in C : |z| < 1\}$. Let S be the subclass of $f(z) \in A$ that are of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are univalent in U Let S^*, C and K denote respectively the subclasses of S known as class of starlike functions with respect to the origin,class of convex functions and class of Close-to-convex functions. Let A and S denote the classes of all functions as defined above. Next, we define some well known subclasses of A, denoted by $A_2, K, K_2, K_{2,\delta}$ and $S(p)$ respectively as follow:

$$A_2 \subset A = \left\{ f \in A : f(z) = z + \sum_{k=3}^{\infty} a_k z^k \quad (z \in U, a_2 = 0) \right\}$$

$$K \subset A = \left\{ f \in A : \left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < 1 \quad z \in U \right\}$$

$$K_2 \subset K = \{ f \in K : f''(0) = 0 \quad z \in U \}$$

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$$K_{2,\delta} \subset K_2 = \left\{ f \in K_2 : \left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < 1 - \delta, \quad 0 \leq \delta < 1 \quad z \in U \right\}$$

$$S(p) \subset A = \left\{ f \in A : \left| \left(\frac{z}{f(z)} \right)'' \right| \leq p, \quad 0 < p \leq 2, \quad p \in \Re \quad z \in U \right\}$$

In [22], Opoola introduced the following differential operator:

$$D^m(\mu, \beta, t) f(z) : A \to A$$

$$D^0(\mu, \beta, t) f(z) = f(z)$$

$$D^1(\mu, \beta, t) f(z) = z D_t f(z) = t z f'(z) - z(\beta - \mu)t + [1 + (\beta - \mu - 1)t] f(z)$$

$$D^m(\mu, \beta, t) f(z) = z D_t (D^{m-1}(\mu, \beta, t) f(z)), \quad m \in N \tag{1.2}$$

If $f(z)$ is given by (1.1),then from (1.2), we see that

$$D^m(\mu, \beta, t) f(z) = z + \sum_{k=2}^{\infty} [1 + (k + \beta - \mu - 1)\, t]^m \, a_k z^k \tag{1.3}$$

where, $0 \leq \mu \leq \beta$, $t \geq 0$ and $m \in N_0 = N \cup 0)$.

Remark 1.1. (i) when $\beta = \mu, t = \lambda$, $D^n(\mu, \mu, \lambda) f(z) = D^n_\lambda f(z)$ is the Al-Oboudi Differential operator in [1].

(ii) when $\beta = \mu$, $t = 1$,$D^n(\mu, \mu, 1) f(z) = D^n f(z)$ is the Salagean Differential operator introduced in [30].

Alexander in [2] introduced and investigated a univalent Integral operator and was continued by Libera in[18]. In [4], Bernardi generalized the work of Libera where he gave the conditions for which the operator is starlike , convex and close-to-convex in the unit disk,U.Being an interesting area of research,many results have been obtained by many scholars.We refer interested readers to the works in [3, 5, 7, 8, 13, 14, 16, 17, 19, 23, 24, 25, 27, 28, 31] for more information on different univalent integral operators.

Motivated by the works of Bulut [9, 10, 11, 12, 15], where a differential operator was used to define integral operators; we generalize Bulut's work and others by using Opoola differential operator and defined a generalized Integral operator,and investigated the conditions for which the operator is univalent in the unit disk, $U = \{z \in C : |z| < 1\}$. The operator and the results generalize some existing results.

Now we introduce a new general integral operator by means of Opoola differential operator.

Definition 1.2. Let $n \in N, m \in N_0, t, \beta \geq 0, 0 \leq \mu \leq \beta, \sigma \in C$ with $\Re(\sigma) > 0$ and $\alpha_j \in C, (j = 1, 2, ..., n)$, we define the integral operator $I^{m,\sigma}_{t,\beta,\mu}(f_1, ..., f_n)(z) : A^n :\to A$ by

$$I^{m,\sigma}_{t,\beta,\mu}(f_1, ..., f_n)(z) := \left\{ \sigma \int_0^z u^{(\sigma-1)} \prod_{j=1}^{n} \left(\frac{D^m(\mu, \beta, t) f_j(u)}{u} \right)^{\alpha_j} du \right\}^{\frac{1}{\sigma}}, \quad z \in U \tag{1.4}$$

where $f_1, ..., f_n \in A$ and $D^m(\mu, \beta, t) f_j(z)$ is the Opoola differential operator defined in (1.3).

Remark 1.3. (i) For $n \in N, \sigma = 1, \alpha_j \in C, j = 1, 2, ..., n$ in (1.4), we have the integral operator

$$I^{m}_{t,\beta,\mu}(f_1, ..., f_n)(z) := \left\{ \int_0^z \prod_{j=1}^{n} \left(\frac{D^m(\mu, \beta, t) f_j(u)}{u} \right)^{\alpha_j} du \right\}, \quad z \in U \tag{1.5}$$

where $f_1, ..., f_n \in A$ and $D^m(\mu, \beta, t) f_j(z)$ is the Opoola differential operator defined in (1.3).

(ii) For $n = 1, \sigma = 1, \alpha \in C$ in (1.4), we have the integral operator

$$I^{m}_{t,\beta,\mu} f(z) := \left\{ \int_0^z \left(\frac{D^m(\mu, \beta, t) f(u)}{u} \right)^{\alpha} du \right\}, \quad z \in U \tag{1.6}$$

where $f \in A$ and $D^m(\mu,\beta,t)f(z)$ is the Opoola differential operator defined in (1.3).

(iii) For $n \in N, m \in N_0, \sigma = 1, \alpha_j \in C, \beta = \mu, t = \lambda$ and $D^m(\mu,\mu,\lambda)f_j(z) = D_\lambda^m f_j(z), j = 1,2,...,n$, we have the integral operator

$$I(f_1,...,f_n)(z) := \int_0^z \left(\frac{D_\lambda^m f_1(u)}{u}\right)^{\alpha_1} ... \left(\frac{D_\lambda^m f_n(u)}{u}\right)^{\alpha_n} du, \quad z \in U \tag{1.7}$$

which was introduced by Bulut, $D_\lambda^m f_j(z)$ is the Al-Oboudi differential operator in [15].

(iv) For $n \in N, \sigma, \alpha_j \in C, D^0(\mu,\beta,t)f_j(z) = f_j(z) \in S, j = 1,2,...,n$, we have the integral operator

$$I_\sigma(f_1,...,f_n)(z) := \left\{\sigma \int_0^z \left(\frac{f_1(u)}{u}\right)^{\alpha_1} ... \left(\frac{f_n(u)}{u}\right)^{\alpha_n} du\right\}^{\frac{1}{\sigma}}, \quad z \in U \tag{1.8}$$

in [6].

(v) For $n \in N, \sigma = 1, \alpha_j \in C, D^0(\mu,\beta,t)f_j(z) = f_j(z) \in S, j = 1,2,...,n$, we have the integral operator

$$I(f_1,...,f_n)(z) := \left\{\int_0^z \left(\frac{f_1(u)}{u}\right)^{\alpha_1} ... \left(\frac{f_n(u)}{u}\right)^{\alpha_n} du\right\}, \quad z \in U \tag{1.9}$$

in [6].

(vi) For $m = 0, n = 1, \sigma = 1, \alpha_1 = 1, \alpha_2 = \alpha_3 = ... = \alpha_n = 0$, and $D^0(\mu,\beta,t)f_1(z) = f(z) \in A$, we have Alexander integral operator

$$I(f)(z) := \int_0^z \frac{f(u)}{u} du, \quad z \in U \tag{1.10}$$

in [2].

(vii) For $m = 0, n = 1, \sigma = 1, \alpha[0,1], \alpha_2 = \alpha_3 = \cdots = \alpha_n = 0$, and $D^0(\mu,\mu,t)f_1(z) = f(z) \in A$, we have the integral operator

$$I(f)(z) := \int_0^z \left(\frac{f(u)}{u}\right)^\alpha du \quad z \in U, \tag{1.11}$$

in [20].

2 Relevant Lemmas

In order to prove our main results, we need the following Lemmas.

Lemma 2.1 (General Schwarz Lemma). [21] Let the analytic function $f(z)$ be regular in the disk

$$U_R = \{z \in C : |z| < R\},$$

with $|f(z)| \leq M$ for fixed M. If $f(z)$ has one zero and multiplicity order bigger than m for $z = 0$, then

$$|f(z)| \leq \frac{M}{R^m} |z|^m \quad (z \in U_R).$$

The equality can hold only if $f(z) = e^{i\theta} \frac{M}{R^m} z^m$, where θ is a constant.

Lemma 2.2. [29] Let α be a complex number with $Re\alpha > 0$ and $f(z) \in A$. If $f(z)$ satisfies

$$\frac{1 - |z|^{2\Re\alpha}}{\Re\alpha} \left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \quad (z \in U),$$

then, for any complex number β with $\Re(\beta) \geq \Re(\alpha)$, the integral operator

$$F_\beta(z) := \left\{\beta \int_0^z t^{\beta-1} f'(t) dt\right\}^{\frac{1}{\beta}} = z + \cdots$$

is analytic and univalent in U.

Lemma 2.3. [26] Let $\alpha \in C, (\Re(\alpha) > 0)$ and $c \in C(|c| \leq 1 : c \neq -1)$. Suppose also that the function $f(z)$ given by (1.1) is analytic in U. If

$$\left| c\,|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zf''(z)}{\alpha f'(z)} \right| \leq 1 \quad (z \in U),$$

then the function $F_\alpha(z)$ defined by

$$F_\alpha(z) := \left\{ \alpha \int_0^z t^{\alpha-1} f'(t) dt \right\}^{\frac{1}{\alpha}} = z + \cdots$$

is analytic and univalent in U.

Lemma 2.4. [32] If a function $f(z) \in S(p)$, then $\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| \leq p\,|z|^2$.

3 Main Results

Theorem 3.1. Let $\alpha_1, ..., \alpha_n, \sigma \in C, m \in N_0, t \geq 0, \beta \geq 0, 0 \leq \mu \leq \beta$, and each of the functions $f_j \in A, j = 1, ..., n$, and

$$\left| \frac{z(D^m(\mu, \beta, t) f_j(z))'}{(D^m(\mu, \beta, t) f_j(z)} - 1 \right| \leq 1, \quad z \in U \tag{3.1}$$

and $\Re(\sigma) \geq \sum_{i=1}^n |\alpha_j| > 0$, then the integral operator $I^{m;\sigma}_{t,\beta,\mu}(f_1, ..., f_n)(z)$ defined by (1.4) is in the univalent class S.

Proof . Since $f_j \in A, j = 1, ..., n$ by (3.1), we have

$$\begin{aligned} \frac{(D^m(\mu, \beta, t) f_j(z))}{z} &= \frac{z + \sum_{k=2}^{\infty} [1 + (k + \beta - \mu - 1)t]^m a_{k,j} z^k}{z} \\ &= 1 + \sum_{k=2}^{\infty} [1 + (k + \beta - \mu - 1)]^m a_{k,j} z^{k-1} \end{aligned}$$

for all $z \in U$ and $m \in N_0$. Let us define a function $h(z)$ by

$$h(z) = \int_o^z \prod_{j=1}^n \left(\frac{(D^m(\mu, \beta, t) f_j(u))}{u} \right)^{\alpha_j} du. \tag{3.2}$$

So that by Fundamental Theorem of Calculus, we have

$$h'(z) = \prod_{j=1}^n \left(\frac{(D^m(\mu, \beta, t) f_j(z))}{z} \right)^{\alpha_j} \tag{3.3}$$

for all $z \in U$. This equality implies that

$$\begin{aligned} \ln h'(z) &= \alpha_1 \ln \left(\frac{(D^m(\mu, \beta, t) f_1(z))}{z} \right) + \cdots + \alpha_n \ln \left(\frac{(D^m(\mu, \beta, t) f_n(z))}{z} \right) \\ &= \alpha_1 \left[\ln (D^m(\mu, \beta, t) f_1(z)) - \ln z \right] + \cdots + \alpha_n \left[\ln (D^m(\mu, \beta, t) f_n(z)) - \ln z \right]. \end{aligned} \tag{3.4}$$

By differentiating the above equality logarithmical, we have

$$\frac{h''(z)}{h'(z)} = \alpha_1 \left[\frac{(D^m(\mu, \beta, t) f_1(z))'}{D^m(\mu, \beta, t) f_1(z)} - \frac{1}{z} \right] + \cdots + \alpha_n \left[\frac{(D^m(\mu, \beta, t) f_n(z))'}{D^m(\mu, \beta, t) f_n(z)} - \frac{1}{z} \right]. \tag{3.5}$$

Then

$$\frac{zh''(z)}{h'(z)} = \sum_{j=1}^n \alpha_j \left[\frac{z(D^m(\mu, \beta, t) f_j(z))'}{D^m(\mu, \beta, t) f_j(z)} - 1 \right]. \tag{3.6}$$

So by the condition of the Theorem, we find

$$\begin{aligned}\frac{1-|z|^{2\Re(\sigma)}}{\Re(\sigma)}\left|\frac{zh''(z)}{h'(z)}\right| \leq & \frac{1-|z|^{2\Re(\sigma)}}{\Re(\sigma)}\sum_{j=1}^{n}|\alpha_j|\left(\left|\frac{z(D^m(\mu,\beta,t)f_j(z))'}{D^m(\mu,\beta,t)f_j(z)}-1\right|\right)\\ \leq & \frac{1-|z|^{2\Re(\sigma)}}{\Re(\sigma)}\sum_{j=1}^{n}|\alpha_j| \leq \frac{1}{\Re(\sigma)}\sum_{j=1}^{n}|\alpha_j| \leq 1 \end{aligned} \tag{3.7}$$

since $\left|\frac{z(D^m(\mu,\beta,t)f_j(z))'}{D^m(\mu,\beta,t)f_j(z)}-1\right| \leq 1$, and $\Re(\sigma) \geq \sum_{j=1}^{n}|\alpha_j|$. Finally by applying Lemma 2.2, for the function $h(z)$, we proved that $I^{m,\sigma}_{t,\beta,\mu}(f_1,...,f_n)(z) \in S$. □

Remark 3.2. If we set $\beta = \mu, \sigma = 1$ in Theorem 3.1, then we have [15, Theorem 2.3].

Remark 3.3. If we set $m = 0, \sigma = 1$ in Theorem 3.1, then we have [6, Theorem 1].

Corollary 3.4. Let $\alpha_j > 0, \sigma \in C, m \in N_0, t \geq 0, \beta \geq 0, 0 \leq \mu \leq \beta$, and each of the functions $f_j \in A, j = 1,2,...,n$. If $f_j \in A, j = 1,2,...,n$, satisfies the inequality (3.1) and $\Re(\sigma) \geq \sum_{i=1}^{n}\alpha_j$, then the integral operator $I^{m,\sigma}_{t,\beta,\mu}(f_1,...,f_n)$ defined by (1.4) is in the univalent class S.

Remark 3.5. If we set $\beta = \mu, \sigma = 1$ in Corollary 3.4, we get [15, Corollary 2.5].

Theorem 3.6. Let $M_j \geq 1$ and suppose that each of the functions $f_j \in A, j = 1,2,...,n$ satisfies the inequality

$$\left|\frac{z^2(D^m(\mu,\beta,t)f_j(z))'}{(D^m(\mu,\beta,t)f_j(z))^2}-1\right| \leq 1, \quad z \in U, m \in N_0. \tag{3.8}$$

Also, let $\alpha_1,...,\alpha_n,\sigma \in C$ with $\Re(\sigma) \geq \sum_{i=1}^{n}|\alpha_j|(2M_j+1) > 0$. If $|D^m(\mu,\beta,t)f_j(z)| \leq M_J, z \in U, j = 1,2,...,n$, then the integral operator $I^{m,\sigma}_{t,\beta,\mu}(f_1,...,f_n)$ defined by (1.4) is in the univalent function class S.

Proof . We know from the proof of Theorem 3.1 that

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq \sum_{j=1}^{n}|\alpha_j|\left(\left|\frac{z(D^m(\mu,\beta,t)f_j(z))'}{D^m(\mu,\beta,t)f_j(z)}-1\right|\right). \tag{3.9}$$

So by the imposed condition, we find

$$\frac{1-|z|^{2\Re(\sigma)}}{\Re(\sigma)}\left|\frac{zh''(z)}{h'(z)}\right| \leq \frac{1-|z|^{2\Re(\rho)}}{\Re(\rho)}\sum_{j=1}^{n}|\alpha_j|\left(\left|\frac{z(D^m(\mu,\beta,t)f_j(z))'}{D^m(\mu,\beta,t)f_j(z)}\right|+1\right) \tag{3.10}$$

By Schwarz Lemma, we get $|D^m(\mu,\beta,t)g_i| \leq M_j|z|, R = 1$, therefore

$$\begin{aligned}\frac{1-|z|^{2\Re(\sigma)}}{\Re(\sigma)}\left|\frac{zh''(z)}{h'(z)}\right| \leq & \frac{1-|z|^{2\Re(\sigma)}}{\Re(\sigma)}\sum_{j=1}^{n}|\alpha_j|\left(\left|\frac{z^2(D^m(\mu,\beta,t)f_j(z))'}{(D^m(\mu,\beta,t)f_j(z))^2}\right|M_j+1\right)\\ \leq & \frac{1-|z|^{2\Re(\sigma)}}{\Re(\sigma)}\sum_{j=1}^{n}|\alpha_j|\left(\left|\frac{z^2(D^m(\mu,\beta,t)f_j(z))'}{(D^m(\mu,\beta,t)f_j(z))^2}-1\right|M_j+M_j+1\right)\\ \leq & \frac{1}{\Re(\sigma)}\sum_{j=1}^{n}|\alpha_j|(2M_J+1) \leq 1 \end{aligned} \tag{3.11}$$

since $\left|\frac{z^2(D^m(\mu,\beta,t)f_j(z))'}{(D^m(\mu,\beta,t)f_j(z))^2}-1\right| \leq 1$ and $\Re(\sigma) \geq \sum_{j=1}^{n}|\alpha_j|(2M_j+1)$. By applying Lemma 2.2, for function $h(z)$, we prove that $I^{m,\sigma}_{t,\beta,\mu}(f_1,...,f_n) \in S$. □

Corollary 3.7. Let $M_j \geq 1, \alpha_j > 0$ and suppose that each of the functions $f_j \in A, j = 1, 2, ..., n$, satisfies the inequality (3.8). Also, let $\alpha, \sigma \in C$ with $\Re(\sigma) \geq \sum_{i=1}^{n} |\alpha_j| (2M_j+1) > 0$. If $|D^m(\mu,\beta,t)f_j(z)| \leq M_J, z \in U, j = 1, ..., n$, then for any complex number σ with $\Re(\sigma) \geq \Re(\alpha)$, the integral operator $I^{m,\sigma}_{t,\beta,\mu}(f_1, ..., f_n)(z)$ defined by (1.4) is in the univalent function class S.

Corollary 3.8. Let $M \geq 1$ and suppose that each of the functions $f_j \in A, j = 1, 2, .., n$, satisfies the inequality (3.8). Also, let $\alpha_1, ..., \alpha_n, \sigma \in C$ with $\Re(\sigma) \geq (2M+1)\sum_{i=1}^{n} |\alpha_j| > 0$. If $|D^m(\mu,\beta,t)f_j(z)| \leq M, z \in U, j = 1, 2, ..., n$, then for any complex number σ with $\Re(\sigma) \geq \Re(\alpha)$, the integral operator $I^{m,\sigma}_{t,\beta,\mu}(f_1, ..., f_n)(z)$ defined by (1.4) is in the univalent function class S.

Corollary 3.9. Suppose that each of the functions $f_j \in A, j = 1, 2, ..., n$ satisfies the inequality (3.8). Also, let $\alpha_1, ..., \alpha_n, \sigma \in C$ with $\Re(\sigma) \geq 3\sum_{i=1}^{n} |\alpha_j| > 0$. If $|D^m(\mu,\beta,t)f_j(z)| \leq 1, z \in U, j = 1, 2, ..., n$, then for any complex number σ with $\Re(\sigma) \geq \Re(\alpha)$, the integral operator $I^{m,\sigma}_{t,\beta,\mu}(f_1, ..., f_n)(z)$ defined by (3.10) is in the univalent function class S.

Remark 3.10. In Corollary 3.9

(i) If we set $\sigma = 1, \beta = \mu$, then we obtain Theorem 2.6.

(ii) If we set $\sigma = 1, \alpha_j > 0, j = 1, 2, ..., n$, then we have [15, Corollary 2.8].

Theorem 3.11. Suppose that each of the functions $f_j \in A, j = 1, 2, ..., n$, satisfies the inequality

$$\left|\frac{z(D^m(\mu,\beta,t)f_j(z))'}{D^m(\mu,\beta,t)f_j(z)} - 1\right| \leq 1, \quad z \in U. \tag{3.12}$$

Also, let $\alpha_1, ..., \alpha_n, \sigma \in C$, with $\Re(\sigma) \geq \sum_{i=1}^{n} |\alpha_j| > 0$, and let $c \in C$ be such that $|c| \leq 1 - \frac{1}{\Re(\sigma)}\sum_{j=1}^{n} |\alpha_j|$, then the integral operator $I^{m,\sigma}_{t,\beta,\mu}(f_1, ..., f_n)(z)$ defined by (1.4) is in the univalent class S.

Proof . We know from the proof of Theorem 3.1 that

$$\frac{zh''(z)}{h'(z)} = \sum_{j=1}^{n} \alpha_j \left[\frac{z(D^m(\mu,\beta,t)f_j(z))'}{D^m(\mu,\beta,t)f_j(z)} - 1\right]. \tag{3.13}$$

We have that

$$c\,|z|^{2\sigma} + (1 - |z|^{2\sigma})\frac{zh''(z)}{\sigma h'(z)} = c\,|z|^{2\sigma} + (1 - |z|^{2\sigma})\frac{1}{\sigma}\sum_{j=1}^{n} \alpha_j \left[\frac{z(D^m(\mu,\beta,t)f_j(z))'}{D^m(\mu,\beta,t)f_j(z)} - 1\right]. \tag{3.14}$$

Then

$$\begin{aligned}\left|c\,|z|^{2\sigma} + (1 - |z|^{2\sigma})\frac{zh''(z)}{\sigma h'(z)}\right| &= \left|c\,|z|^{2\sigma} + (1 - |z|^{2\sigma})\frac{1}{\sigma}\sum_{j=1}^{n} \alpha_j \left[\frac{z(D^m(\mu,\beta,t)f_j(z))'}{D^m(\mu,\beta,t)f_j(z)} - 1\right]\right| \\ &\leq |c| + \left|\frac{1 - |z|^{2\sigma}}{\sigma}\right| \sum_{j=1}^{n} |\alpha_j| \left(\left|\frac{z(D^m(\mu,\beta,t)f_j(z))'}{D^m(\mu,\beta,t)f_j(z)} - 1\right|\right) \\ &\leq |c| + \frac{1}{|\sigma|}\sum_{j=1}^{n} |\alpha_j| \leq |c| + \frac{1}{\Re(\sigma)}\sum_{j=1}^{n} |\alpha_j| \leq 1 \end{aligned} \tag{3.15}$$

since $\left|\frac{z(D^m(\mu,\beta,t)f_j(z))'}{D^m(\mu,\beta,t)f_j(z)} - 1\right| \leq 1$, $\Re(\sigma) \geq \sum_{j=1}^{n} |\alpha_j|$ and $|c| \leq 1 - \frac{1}{\Re(\sigma)}\sum_{j=1}^{n} |\alpha_j|$. Finally, applying Lemma 2.3 for the function $h(z)$, we prove that $I^{m,\sigma}_{t,\beta,\mu}(f_1, ..., f_n)(z) \in S$. □

Corollary 3.12. Suppose that each of the functions $f_j \in A, j \in \{1, ..., n\}$, satisfies the inequality (3.12). Also,let $\alpha_j > 0, \sigma \in C$, with $\Re(\sigma) \geq \sum_{i=1}^{n} \alpha_j$, and let $c \in C$ be such that $|c| \leq 1 - \frac{1}{\Re(\sigma)}\sum_{j=1}^{n} \alpha_j$, then the integral operator $I^{m,\sigma}_{t,\beta,\mu}(f_1, ..., f_n)(z)$ defined by (1.4) is in the univalent class S.

Theorem 3.13. Let $M_j \geq 1$ and suppose that each of the functions $f_j \in A, j = 1, 2, ..., n$, satisfies the inequality

$$\left| \frac{z^2(D^m(\mu,\beta,t)f_j(z))'}{(D^m(\mu,\beta,t)f_j(z))^2} - 1 \right| \leq 1, \quad z \in U, m \in N_0. \tag{3.16}$$

Also, let $\alpha_1, ..., \alpha_n, \sigma \in C$ with $\Re(\sigma) \geq \sum_{i=1}^{n} |\alpha_j| (2M_j + 1) > 0$, and let $c \in C$ be such that $|c| \leq 1 - \frac{1}{\Re(\sigma)} \sum_{j=1}^{n} |\alpha_j| (2M_j + 1)$. If $|D^m(\mu,\beta,t)f_j(z)| \leq M_J$ $(z \in U, j = 1, 2, ..., n)$, then, for any complex number σ with $\Re(\sigma) \geq \Re(\alpha)$, the integral operator $I^{m,\sigma}_{t,\beta,\mu}(f_1, ..., f_n)(z)$ defined by (1.4) is in the univalent function class S.

Proof . We know from the proof of Theorem 3.1 that

$$\frac{zh''(z)}{h'(z)} = \sum_{j=1}^{n} \alpha_j \left[\frac{z(D^m(\mu,\beta,t)f_j(z))'}{D^m(\mu,\beta,t)f_j(z)} - 1 \right]. \tag{3.17}$$

Therefore,

$$c\,|z|^{2\sigma} + (1 - |z|^{2\sigma}) \frac{zh''(z)}{\sigma h'(z)} = c\,|z|^{2\sigma} + (1 - |z|^{2\sigma}) \frac{1}{\sigma} \sum_{j=1}^{n} \alpha_j \left[\frac{z(D^m(\mu,\beta,t)f_j(z))'}{D^m(\mu,\beta,t)f_j(z)} - 1 \right]. \tag{3.18}$$

Then

$$\begin{aligned} \left| c\,|z|^{2\sigma} + (1 - |z|^{2\sigma}) \frac{zh''(z)}{\sigma h'(z)} \right| &= \left| c\,|z|^{2\sigma} + (1 - |z|^{2\sigma}) \frac{1}{\sigma} \sum_{j=1}^{n} \alpha_j \left[\frac{z(D^m(\mu,\beta,t)f_j(z))'}{D^m(\mu,\beta,t)f_j(z)} - 1 \right] \right| \\ &\leq |c| + \left| \frac{1 - |z|^{2\sigma}}{\sigma} \right| \sum_{j=1}^{n} |\alpha_j| \left| \frac{z(D^m(\mu,\beta,t)f_j(z))'}{D^m(\mu,\beta,t)f_j(z)} - 1 \right| \\ &\leq |c| + \left| \frac{1 - |z|^{2\sigma}}{\sigma} \right| \sum_{j=1}^{n} |\alpha_j| \left(\left| \frac{z(D^m(\mu,\beta,t)f_j(z))'}{D^m(\mu,\beta,t)f_j(z)} \right| + 1 \right). \end{aligned} \tag{3.19}$$

By Schwarz Lemma, we get $|D^m(\mu,\beta,t)g_i| \leq M_j\,|z|\,, R = 1$, therefore

$$\begin{aligned} \left| c\,|z|^{2\sigma} + (1 - |z|^{2\sigma}) \frac{zh''(z)}{\sigma h'(z)} \right| &\leq |c| + \frac{1}{|\sigma|} \sum_{j=1}^{n} |\alpha_j| \left(\left| \frac{z^2(D^m(\mu,\beta,t)f_j(z))'}{(D^m(\mu,\beta,t)f_j(z))^2} \right| M_j + 1 \right) \\ &\leq |c| + \frac{1}{|\sigma|} \sum_{j=1}^{n} |\alpha_j| \left(\left| \frac{z^2(D^m(\mu,\beta,t)f_j(z))'}{(D^m(\mu,\beta,t)f_j(z))^2} - 1 \right| M_j + M_j + 1 \right) \\ &\leq |c| + \frac{1}{|\sigma|} \sum_{j=1}^{n} |\alpha_j| (2M_j + 1) \\ &\leq |c| + \frac{1}{\Re(\sigma)} \sum_{j=1}^{n} |\alpha_j| (2M_j + 1) \leq 1, \end{aligned} \tag{3.20}$$

since $\left| \frac{z^2(D^m(\mu,\beta,t)f_j(z))'}{(D^m(\mu,\beta,t)f_j(z))^2} - 1 \right| \leq 1$ and $|c| \leq 1 - \frac{1}{\Re(\sigma)} \sum_{j=1}^{n} |\alpha_j| (2M_j + 1)$. Finally, applying Lemma 2.3 for function $h(z)$, we proved that $I^{m,\sigma}_{t,\beta,\mu}(f_1, ..., f_n)(z) \in S$. □

Corollary 3.14. Let $M_j \geq 1, \alpha_j > 0$ and suppose that each of the functions $f_j \in A, j = 1, 2, , ..., n$ satisfies the inequality (3.16). Also,let $\sigma \in C$ with $\Re(\sigma) \geq \sum_{i=1}^{n} \alpha_j(2M_j + 1) > 0$, and let $c \in C$ be such that $|c| \leq 1 - \frac{1}{\Re(\sigma)} \sum_{j=1}^{n} \alpha_j(2M_j + 1)$. If $|D^m(\mu,\beta,t)f_j(z)| \leq M_J, z \in U, j = 1, ..., n$, then the integral operator $I^{m,\sigma}_{t,\beta,\mu}(f_1, ..., f_n)(z)$ defined by (1.4) is in the univalent function class S.

Corollary 3.15. Let $M \geq 1$, and suppose that each of the functions $f_j \in A, j = 1, ..., n$ satisfies the inequality (3.16) Also, let $\alpha_1, ..., \alpha_n, \sigma \in C$ with $\Re(\sigma) \geq (2M + 1) \sum_{i=1}^{n} |\alpha_j| > 0$, and let $c \in C$ be such that $|c| \leq 1 - \frac{(2M_j+1)}{\Re(\sigma)} \sum_{j=1}^{n} \alpha_j$. If $|D^m(\mu,\beta,t)f_j(z)| \leq M, z \in U, j = 1, 2, ..., n$, then the integral operator $I^{m,\sigma}_{t,\beta,\mu}(f_1, ..., f_n)(z)$ defined by (1.4) is in the univalent function class S.

Corollary 3.16. Suppose that each of the functions $f_j \in A, j = 1, 2, ..., n$ satisfies the inequality (3.16). Also, let $\alpha_1, ..., \alpha_n, \sigma \in C$ with $\Re(\sigma) \geq 3\sum_{i=1}^{n} |\alpha_j| > 0$, and let $c \in C$ be such that $|c| \leq 1 - \frac{3}{\Re(\sigma)} \sum_{j=1}^{n} |\alpha_j|$. If $|D^m(\mu, \beta, t)f_j(z)| \leq 1, z \in U, j = 1, 2, ..., n$, then the integral operator $I^{m,\sigma}_{t,\beta,\mu}(f_1, ..., f_n)(z)$ defined by (1.4) is in the univalent function class S.

Theorem 3.17. Let c be a complex number, $|D^m(\mu, \beta, t)f_j(z)| \leq M_j, M_j \geq 1$, for all $j = 1, 2, \ldots, n$, $D^m(\mu, \beta, t)f_j(z) \in S(p_j)$, for $j = 1, 2, ..., n$, with $\alpha_1, ..., \alpha_n, \sigma \in C$ such that

$$\Re(\sigma) \geq \sum_{j=1}^{\infty} |\alpha_j| \frac{[(2M_j - 1)p_j + 2] M_j}{2M_j - 1}.$$

Also, let $|c| \leq 1 - \frac{1}{\Re(\sigma)} \sum_{j=1}^{n} |\alpha_j| \frac{[(2M_j-1)p_j+2]M_j}{2M_j-1}$, then the integral operator $I^{m,\sigma}_{t,\beta,\mu}(f_1, ..., f_n)(z)$ defined by (1.4) is in the univalent function class S.

Proof . Since $D^m(\mu, \beta, t)f_j(z) \in S(p_j)$, so by Lemma 2.4, we have

$$\left| \frac{z^2(D^m(\mu, \beta, t)f_j(z))'}{(D^m(\mu, \beta, t)f_j(z))^2} - 1 \right| \leq p_j |z|^2, \quad z \in U \tag{3.21}$$

From $|D^m(\mu, \beta, t)f_j(z)| \leq M_j$, for any $j = 1, ..., n$ and by using Lemma 2.1, we get $|D^m(\mu, \beta, t)f_j(z)| \leq M_j |z|, R = 1$. From the proof of Theorem 3.1, we have that

$$\frac{zh''(z)}{h'(z)} = \sum_{j=1}^{n} \alpha_j \left[\frac{z(D^m(\mu, \beta, t)f_j(z))'}{D^m(\mu, \beta, t)f_j(z)} - 1 \right] \tag{3.22}$$

and so

$$\left| \frac{zh''(z)}{h'(z)} \right| \leq \sum_{j=1}^{n} |\alpha_j| \left(\left| \frac{z(D^m(\mu, \beta, t)f_j(z))'}{D^m(\mu, \beta, t)f_j(z)} \right| + 1 \right). \tag{3.23}$$

By Schwarz Lemma, we get $|D^m(\mu, \beta, t)g_i| \leq M_j |z|, R = 1$, therefore

$$\left| \frac{zh''(z)}{h'(z)} \right| \leq \sum_{j=1}^{n} |\alpha_j| \left(\left| \frac{z^2(D^m(\mu, \beta, t)f_j(z))'}{(D^m(\mu, \beta, t)f_j(z))^2} \right| M_j + 1 \right) \tag{3.24}$$

and

$$\left| \frac{zh''(z)}{h'(z)} \right| \leq \sum_{j=1}^{n} |\alpha_j| \left(\left| \frac{z^2(D^m(\mu, \beta, t)f_j(z))'}{(D^m(\mu, \beta, t)f_j(z))^2} - 1 \right| M_j + M_j + 1 \right). \tag{3.25}$$

Thus,

$$\begin{aligned} \left| \frac{zh''(z)}{h'(z)} \right| &\leq \sum_{j=1}^{n} |\alpha_j| \left(p_j |z|^2 M_j + M_j + 1 \right) \leq \sum_{j=1}^{n} |\alpha_j| (p_j M_j + 2M_j) \\ &\leq \sum_{j=1}^{n} |\alpha_j| (p_j M_j + 2M_j + 4M_j^2 + 8M_j^3 + ...) \end{aligned} \tag{3.26}$$

because $2M_j, 4M_j^2, 8M_j^3, ... > 1$. Which implies that

$$\begin{aligned} \left| \frac{zh''(z)}{h'(z)} \right| &\leq \sum_{j=1}^{n} |\alpha_j| \left(p_j M_j + \frac{2M_j}{2M_j - 1} \right) \\ &\leq \sum_{j=1}^{n} |\alpha_j| \frac{[(2M_j - 1)p_j + 2] M_j}{2M_j - 1}. \end{aligned} \tag{3.27}$$

Now, from

$$\left|c\,|z|^{2\sigma}+(1-|z|^{2\sigma})\frac{zh''(z)}{\sigma h'(z)}\right|\leq|c|+\frac{1}{|\sigma|}\left|\frac{zh''(z)}{h'(z)}\right|\leq|c|+\frac{1}{\Re(\sigma)}\left|\frac{zh''(z)}{h'(z)}\right| \tag{3.28}$$

This implies

$$\left|c\,|z|^{2\sigma}+(1-|z|^{2\sigma})\frac{zh''(z)}{\sigma h'(z)}\right|\leq|c|+\frac{1}{\Re(\sigma)}\sum_{j=1}^{n}|\alpha_j|\,\frac{\left[(2M_j-1)p_j+2\right]M_j}{2M_j-1}\leq 1 \tag{3.29}$$

Since $|c|\leq 1-\frac{1}{\Re(\sigma)}\sum_{j=1}^{n}|\alpha_j|\,\frac{[(2M_j-1)p_j+2]M_j}{2M_j-1}$, by Lemma 2.3, the integral operator $I^{m,\sigma}_{t,\beta,\mu}(f_1,...,f_n)(z)$ defined by (1.4) is in the univalent function class S. □

Corollary 3.18. Let c be a complex number, $|D^m(\mu,\beta,t)f_j(z)|\leq M, M_j=M\geq 1$, for $j=1,2,...,n$, $D^m(\mu,\beta,t)f_j(z)\in S(p_j)$, for $j=1,2,...,n$, with

$$\Re(\sigma)\geq\sum_{j=1}^{\infty}|\alpha_j|\,\frac{\left[(2M-1)p_j+2\right]M}{2M-1}$$

where σ,α_j are complex numbers. If $|c|\leq 1-\frac{1}{\Re(\sigma)}\sum_{j=1}^{n}|\alpha_j|\,\frac{[(2M-1)p_j+2]M}{2M-1}$, then $I^{m,\sigma}_{t,\beta,\mu}(f_1,...,f_n)(z)\in S$.

Corollary 3.19. Let c be a complex number, $|D^m(\mu,\beta,t)f_j(z)|\leq M, M_j=M\geq 1$, for $j=1,2,...,n$, $D^m(\mu,\beta,t)f_j(z)\in S(p_j)$, for $j=1,2,...,n$, $|\alpha_J|=|\alpha|$ with

$$\Re(\sigma)\geq|\alpha|\sum_{j=1}^{\infty}\frac{\left[(2M-1)p_j+2\right]M}{2M-1}$$

where σ,α are complex numbers. If $|c|\leq 1-\frac{|\alpha|}{\Re(\sigma)}\sum_{j=1}^{n}\frac{[(2M-1)p_j+2]M}{2M-1}, M\geq 1$, then by Lemma 2.3 the integral operator $I^{m,\sigma}_{t,\beta,\mu}(f_1,...,f_n)(z)\in S$.

Theorem 3.20. Let c be a complex number,$|D^m(\mu,\beta,t)f_j(z)|\leq M_j, M_j\geq 1$, for all $(j=1,...,n)$, $D^m(\mu,\beta,t)f_j(z)\in K_{2,\delta_j}$ for $(j=1,...,n)$, such that

$$\Re(\sigma)\geq\sum_{j=1}^{\infty}|\alpha_j|\left[(1-\delta_j)+(n(n+1))/2\right]M_j$$

where σ,α_j are complex numbers. If $|c|\leq 1-\frac{1}{\Re(\sigma)}\sum_{j=1}^{n}|\alpha_j|\left[(1-\delta_j)+(n(n+1))/2\right]M_j, M_j\geq 1$ then, the integral operator $I^{m,\sigma}_{t,\beta,\mu}(f_1,...,f_n)(z)$ defined by (1.4) is univalent in U.

Proof . Using the proof of Theorem 3.1, we have

$$\frac{zh''(z)}{h'(z)}=\sum_{j=1}^{n}\alpha_j\left[\frac{z(D^m(\mu,\beta,t)f_j(z))'}{D^m(\mu,\beta,t)f_j(z)}-1\right] \tag{3.30}$$

$$\left|\frac{zh''(z)}{h'(z)}\right|\leq\sum_{j=1}^{n}|\alpha_j|\left(\left|\frac{z(D^m(\mu,\beta,t)f_j(z))'}{D^m(\mu,\beta,t)f_j(z)}\right|+1\right) \tag{3.31}$$

By Schwarz Lemma, we get $|D^m(\mu,\beta,t)g_i|\leq M_j\,|z|\,, R=1$, therefore

$$\left|\frac{zh''(z)}{h'(z)}\right|\leq\sum_{j=1}^{n}|\alpha_j|\left(\left|\frac{z^2(D^m(\mu,\beta,t)f_j(z))'}{(D^m(\mu,\beta,t)f_j(z))^2}\right|M_j+1\right) \tag{3.32}$$

and

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq \sum_{j=1}^{n} |\alpha_j| \left(\left| \frac{z^2(D^m(\mu,\beta,t)f_j(z))'}{(D^m(\mu,\beta,t)f_j(z))^2} - 1 \right| M_j + M_j + 1 \right). \tag{3.33}$$

Thus,

$$\begin{aligned}\left|\frac{zh''(z)}{h'(z)}\right| &\leq \sum_{j=1}^{n} |\alpha_j| \left((1-\delta_j)M_j + M_j + 1\right) \leq \sum_{j=1}^{n} |\alpha_j| \left((1-\delta_j)M_j + 2M_j\right) \\ &\leq \sum_{j=1}^{n} |\alpha_j| \left((1-\delta_j)M_j + M_j + 2M_j + 3M_j + ... + nM_j\right) \\ &\leq \sum_{j=1}^{n} |\alpha_j| \left[(1-\delta_j)M_j + (n(n+1)/2)\, M_j\right]. \end{aligned} \tag{3.34}$$

Now, we evaluate the expression

$$\left| c\,|z|^{2\sigma} + (1-|z|^{2\sigma})\frac{zh''(z)}{\sigma h'(z)} \right| \leq |c| + \frac{1}{|\sigma|}\left|\frac{zh''(z)}{h'(z)}\right| \leq |c| + \frac{1}{\Re(\sigma)}\left|\frac{zh''(z)}{h'(z)}\right|. \tag{3.35}$$

Then

$$\left| c\,|z|^{2\sigma} + (1-|z|^{2\sigma})\frac{zh''(z)}{\sigma h'(z)} \right| \leq |c| + \frac{1}{|\sigma|}\left|\frac{zh''(z)}{h'(z)}\right| \leq |c| + \frac{1}{\Re(\sigma)}\sum_{j=1}^{n} |\alpha_j| \left[(1-\delta_j)M_j + (n(n+1)/2)\, M_j\right]$$

This implies that

$$\left| c\,|z|^{2\sigma} + (1-|z|^{2\sigma})\frac{zh''(z)}{\sigma h'(z)} \right| \leq 1.$$

Since $|c| \leq 1 - \frac{1}{\Re(\sigma)}\sum_{j=1}^{n} |\alpha_j| \left[(1-\delta_j) + (n(n+1)/2)\right] M_j$ by Lemma 2.3, the integral operator, $I^{m,\sigma}_{t,\beta,\mu}(f_1, ..., f_n)(z) \in S$. □

Corollary 3.21. Let c be a complex number, $|D^m(\mu,\beta,t)f_j(z)| \leq M, M \geq 1$ for all $(j = 1, 2, ..., n)$, $D^m(\mu,\beta,t)f_j(z) \in K_{2,\delta_j}$, for $j = 1, 2, ..., n$, such that

$$\Re(\sigma) \geq \sum_{j=1}^{\infty} |\alpha_j| \left[(1-\delta_j) + (n(n+1))/2\right] M$$

where σ, α_j are complex numbers. If $|c| \leq 1 - \frac{1}{\Re(\sigma)}\sum_{j=1}^{n} |\alpha_j| \left[(1-\delta_j) + (n(n+1))/2\right] M, M \geq 1$ then, the integral operator $I^{m,\sigma}_{t,\beta,\mu}(f_1, ..., f_n)(z)$ defined by (1.4) is univalent in U.

Corollary 3.22. Let c be a complex number, $|D^m(\mu,\beta,t)f_j(z)| \leq M, M \geq 1, \alpha_1 = \cdots = \alpha_n = \alpha$ for all $(j = 1, 2, ..., n)$, $D^m(\mu,\beta,t)f_j(z) \in K_{2,\delta}$ for $j = 1, ..., n$, such that

$$\Re(\sigma) \geq |\alpha| \sum_{j=1}^{\infty} \left[(1-\delta_j) + (n(n+1))/2\right] M$$

where σ, α_j are complex numbers. If $|c| \leq 1 - \frac{|\alpha|}{\Re(\sigma)}\sum_{j=1}^{n} \left[(1-\delta_j) + (n(n+1))/2\right] M, M \geq 1$ then, the integral operator $I^{m,\sigma}_{t,\beta,\mu}(f_1, ..., f_n)(z)$ defined by (1.4) is univalent in class S.

Conclusion

In conclusion, a new generalized integral operator is defined in the unit disk U and conditions for univalency of the integral operators in U are investigated.Results obtained generalize some existing results.

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References

[1] F.M. Al-Oboudi, *On univalent functions defined by a Sălăgean differential operator*, Int. J. Math. Math. Sci. **2004** (2004), 1429–1436.

[2] J.W. Alexander, *Functions which map the interior of the unit circle upon simple regions*, Ann. Math. **17** (1915), no. 1, 12–22.

[3] C. Barbatu and D. Breaz, *Univalence criteria for some general integral operators*, An. St. Univ. Ovidius Const. **29** (2021), no. 1, 37–52.

[4] S.D. Bernardi, *Convex and starlike univalent functions*, Trans. Amer. Math. Soc. **135** (1969), 429–446.

[5] D. Breaz, N. Breaz, and H.M. Srivastava, *An extension of the univalent condition for a family of integral operators*, Appl. Math. Lett. **22** (2009), no. 1, 41–44.

[6] D. Breaz and N. Breaz, *Two integral operators*, Studia Univ. Babes-Bolyai Math. **47** (2002), 13–20.

[7] D. Breaz and N. Breaz, *Univalence of an integral operator*, Mathematica **70** (2005), 35–38.

[8] D. Breaz and N . Breaz, *An integral Univalent operator*, Acta Math. Univ. Comenianae. New Ser. **76** (2007), no. 2, 137–142.

[9] S. Bulut, *Univalence preserving integral operators defined by generalized Al-Oboudi differential Operator*, An St. Univ. Ovidius Constanta **17** (2009), no. 1, 37–50.

[10] S. Bulut, *Univalence condition for a new generalization of the family of integral operators*, Acta Univ. Apulensis Math. Inf. **18** (2009), 71–78.

[11] S. Bulut, *A new univalent integral operator defined by Al-Oboudi differential operator*, Gen. Math. **18** (2010), no. 2, 85–93.

[12] S. Bulut, *An integral univalent operator defined by generalized Al-Oboudi differential operator on the classes* $T_j, T_{j,\mu}, S_j(p)$, Novi Sad J. Math. **40** (2010), no. 1, 43–53.

[13] C. Barbatu and D. Breaz, *Some Univalence conditions of a certain general integral operator*, Eur. J. Pure Appl. Math. **13** (2020), no. 5, 1285–1299.

[14] S. Bulut and D. Breaz, *Univalency and convexity conditions for a general integral operator*, Chin. J. Math. **2014** (2014), 4 pages.

[15] S. Bulut, *Sufficient conditions for univalence of an integral operator defined by Al-Oboudi differential operator*, J. Inequal. Appl. **2008** (2008), 5 pages.

[16] E. Deniz, D. Raducanu, and H. Orhan, *On the univalence of an integral operator defined by Hadamard product*, Appl. Math. Lett. **24** (2012), 179–184.

[17] I. Faisal and M. Darus, *A study of Ahlfors' univalence criteria for a space of analytic functions: Criteria II*, Math. Comput. Model. **55** (2012), 1466–1470.

[18] R.J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc. **16** (1965), 755–758.

[19] S.S. Miller, P.T. Mocanu, and M.O. *Reade, Bezilevic functions and generalized convexity*, Rev. Roumaine. Math. Pure Appl. **19** (1974), 213–224.

[20] S.S. Miller, P.T. Mocanu and M.O. Reade, *Starlike integral operators*, Pacific J. Math. **79** (1978), no. 1, 157–168.

[21] Z. Nehari, *Conformal Mapping*, Dover, New York, NY, USA, 1975.

[22] T.O. Opoola, *On a subclass of univalent functions defined by a generalized differential operator*, Int. J. Math. Anal. **11** (2017), no.18, 869–876.

[23] G.I. Oros, G. Oros, and D. Breaz, *Sufficient conditions for univalence of an integral operator*, J. Inequal. Appl. **2008** (2008), 7 pages.

[24] A. Oprea, D. Breaz, and H.M. Srivastava, *Univalence conditions for a new family of integral operators*, Filomat **30** (2016), no. 5, 1243–1251.

[25] A. Oprea and D. Breaz, *Univalence conditions for a general integral operator*, An. St. Univ. Ovidius Constanta **23** (2015), no. 1, 213–224.

[26] V. Pescar, *A new generalization of Ahlfors's and Becker's criterion of univalence*, Bull. Malays. Math. Soc. (Ser. 2) **19** (1996), 53–54.

[27] V. Pescar, *New criteria for Univalence of certain integral operators*, Demonst. Math. **33** (2000), 51–54.

[28] V. Pescar, *On the Univalence of some integral operators*, J. Indian Acad. Math. **27** (2005), 239–243.

[29] N. Pascu, *An improvement on Becker's univalence criterion*, Proc. Commem. Session Stoilow, Brasov, 1987, pp. 43–48.

[30] G.S. Sălăgean, *Subclasses of Univalent Functions*, Complex Anal. Fifth Roman. Seminar, part I (Brucharest, 1983), Lecture Notes in Mathematics, vol. 1013, Springer, Berlin, 1983, pp 362–372.

[31] N. Seenivasagan and D. Breaz, *Certain sufficient conditions for univalence*, Gen. Math. **15** (2007), no. 4, 7–15.

[32] V. Singh, *On class of univalent functions*, Int. J. Maths. Math Soc. **23** (2000).