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New model of invertible elements on multiplicative sample of quasilinear spaces

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Abstract

Invertible members of some of the most famous multiplicative quasilinear spaces are exactly equal to invertible elements of the linear subspace of such spaces. But with the old definition of invertible elements, we do not get anything extra. So, in the present paper, we develop the introductory algebra and introduce an extended model of invertible elements. This new concept plays a fundamental role in the definition of the generalized spectrum model. These new objects are constructed in such a way that can provide acceptable results. For example, we will be able to generalize the well-known theorem "Spectral mapping property for polynomials" with these new notions.

Keywords: Quasilinear space, Banach algebras, Banach quasi-algebras, Spectral theory
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1 Introduction

Interval analysis originated in studying the intervals created by the rounding errors in calculations, and nowadays it has important applications throughout engineering, economics, finance and industry. The concept of quasilinear spaces was introduced to present an abstract approach to the study of space of intervals, subsets and multivalued mappings. Since its creation by Aseev in 1985 [2], issues of quasilinear spaces has been continuously advanced in various fronts.

Recently attempts have also been made to present the counterpart of classical Riesz lemma [7], an analogue of Hahn-Banach theorem [6] and equivalents of quotient spaces on linear spaces in quasilinear spaces [8]. Among the latest researches in this field, we can refer to article [5] which analyzed a new continuous-time epidemic model including nonlinear delay differential equations by using parameters and functions selected from a class of intervals which is algebraically based on quasilinear spaces.

By studying the behavior of elements of quasilinear spaces, we have found that many examples of these spaces have a notion of multiplication. To characterize these examples, quasi-algebras, normed quasi-algebras and Banach quasi-algebras have been introduced. But here the classical definition of invertible elements is not sufficient to generalize such theorems to the new space. For example consider the space of all nonempty, compact subsets of $\mathbb{R}$ with the inclusion relation "$\subseteq$", the real-scalar multiplication

$$\lambda A = \{\lambda a : a \in A\},$$

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the algebraic sum operation

$$A + B = \{a + b : a \in A, b \in B\},$$

and the product operation

$$AB = \{ab : a \in A, b \in B\}.$$

This space is denoted by $\Omega(\mathbb{R})$. For every $0 \neq x \in \mathbb{R}$, $\{x\}$ is invertible and these are the only invertible elements. So in $\Omega(\mathbb{R}) - \mathbb{R}$, we do not have any invertible elements. This problem becomes more critical when we know that the spectrum with the old definition which is mentioned in [1], for element $[a, b]$ will be the whole set of real numbers. Therefore, we needed a re-creation.

For this purpose the first part of our paper provides preliminary explanations about quasilinear spaces. In the next part we introduce quasi-algebras and state some theorems and results about them. It is also suggested to examine the issues related to the automatic continuity of some specific homomorphisms of these new spaces. In the sequel, we will show how generalization of invertible elements can be done. Of course, sometimes it is necessary to limit the space to develop the theorems. Interested researchers can try to expand the range of these cases. Also, scholars can try to answer the question at the final part of this article.

2 Quailinear spaces

The reader is suggested to take a look at what is given in [2] as introductory definitions, examples and theorems. What is required in this paper will be restated.

Definition 2.1. A set X is said to be a quasilinear space if a partial order relation "$\leq$", an algebraic sum operation "$+$", and an operation of multiplication by real numbers "$\cdot$" are defined on it in such a way that the following conditions hold for any elements $x, y, z, v \in X$ and any $\alpha, \beta \in \mathbb{R}$:

1. $x + y = y + x$;
2. $x + (y + z) = (x + y) + z$;
3. There exists an element $0 \in X$, called zero element of X such that $x + 0 = x$;
4. $\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$;
5. $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$;
6. $1 \cdot x = x$;
7. $0 \cdot x = 0$;
8. $(\alpha + \beta) \cdot x \leq \alpha \cdot x + \beta \cdot x$;
9. $x + z \leq y + v$ if $x \leq y$ and $z \leq v$;
10. $\alpha \cdot x \leq \alpha \cdot y$ if $x \leq y$.

We call x regular if there exists $x' \in X$ such that $x + x' = 0$. Otherwise, it is called singular. X_r and X_s denote the sets of all regular and singular elements in X, respectively [3].

Lemma 2.2. [2] Suppose that any element x in the quasilinear space X is regular. Then the partial order in X is determined by equality, and consequently, X is a linear space.

Corollary 2.3. [2] In a real linear space, equality is the only way to define a partial order such that the conditions of a quasilinear space hold.

It will be assumed that $-x = (-1) \cdot x$ and $x - y$ means $x + (-y)$.

Lemma 2.4. [20] In a quasilinear space X, $x \in X_r$ if and only if $x - x = 0$.

Lemma 2.5. [3] In a quasilinear space every regular element is minimal, that means $x = 0$ if $x \leq 0$. Especially 0 is minimal.

Definition 2.6. A norm on a quasilinear space X is a real-valued function $\|\cdot\|$ on X such that

1. $\|x\|_X > 0$ if $x \neq 0$;

2. $\|x+y\|_X \leq \|x\|_X + \|y\|_X$;
3. $\|\alpha \cdot x\|_X = |\alpha|\|x\|_X$;
4. If $x \leq y$, then $\|x\|_X \leq \|y\|_X$;
5. If for any $\epsilon > 0$ there exists an element $x_\epsilon \in X$ such that $x \leq y + x_\epsilon$ and $\|x_\epsilon\|_X \leq \epsilon$, then $x \leq y$;

for all $x, y \in X$ and all $\alpha \in \mathbb{R}$.

Let X be a normed quasilinear space. the Hausdorff metric on X is defined by

$$h_X(x,y) = \inf\{r \geq 0 : \exists a_1^r, a_2^r \in X : x \leq y + a_1^r, y \leq x + a_2^r, \|a_i^r\|_X \leq r\}. \tag{2.1}$$

It is easy to see that $h_X(x,y) \leq \|x-y\|_X$ [2].

Theorem 2.7. Let X be a quasilinear space and $x, y \in X$. If $y \in X_r$ then

$$h_X(x,y) = \|x-y\|_X.$$

Proof . For any $\epsilon > 0$ there exists a_ϵ such that

$$x \leq y + a_\epsilon, \qquad \|a_\epsilon\|_X \leq h_X(x,y) + \epsilon.$$

Because $y \in X_r$, so

$$x - y \leq a_\epsilon.$$

Thus $\|x-y\|_X \leq \|a_\epsilon\|_X \leq h_X(x,y) + \epsilon$. Hence $h_X(x,y) = \|x-y\|_X$. □

Definition 2.8. A normed quasilinear space X is called an Ω-space if there exists an element $0 \neq B_X \in X$ such that:

$$\text{if } \|x\|_X \leq \|B_X\|_X, \text{ then } x \leq B_X.$$

For more details, the reader can refer to [2].

Theorem 2.9. Suppose that X be an Ω-space. Then any bounded subset of X_r has upper bound.

If $A \subseteq X_r$ is bounded, there exists $r > 0$ such that $h_X(x,y) < r$ for every $x, y \in A$. Let $a \in A$. fix $b \in A$. We have therefore shown that

$$\|a\|_X = \|a - b + b\|_X \leq \|a-b\|_X + \|b\|_X = h_X(a,b) + \|b\|_X \leq r + \|b\|_X.$$

So there exists $B_X \in X$ by $\|B_X\|_X = 1$ such that $a \leq (r + \|b\|_X)B_X$.

Definition 2.10. Let X be a quasilinear space and $Y \subseteq X$. Y is called a subspace of X whenever Y is a quasilinear space with the same partial ordering and the same operations on X. In a special case, if Y is a linear space with the same operations on X, it is a linear subspace of X.

3 Quasi-algebras

In preliminary analysis, during the study of linear spaces, the concept of an algebra was introduced. We have also repeated this process in quasilinear spaces. Of course, some work has been done in this field before us [18].

Definition 3.1. A set X is called a quasi-algebra where it has a quasilinear structure, with a multiplication or product operation

$$* : (x,y) \mapsto x * y, \qquad X \times X \longrightarrow X,$$

in such a way that the following conditions hold for any elements $x, y, z, v \in X$ and any $\alpha, \beta \in \mathbb{R}$:

1. $x * (y * z) = (x * y) * z$;

2. $\alpha \cdot (x * y) = (\alpha \cdot x) * y = x * (\alpha \cdot y)$;
3. $x * (y + z) \leq x * y + x * z$ and $(x + y) * z \leq x * z + y * z$;
4. $x * z \leq y * v$ if $x \leq y$ and $z \leq v$.

It is easy to see that $x * 0 = 0 * x = 0$. We say a quasi-algebra is proper if X_r is closed under its product operation. For convenience, henceforth we write αx and xy instead of $\alpha \cdot x$ and $x * y$ Respectively. Of course, it can be recognized whether it is an algebra multiplication or a scalar multiplication operation.

A quasi-algebra X has an identity 1 or 1_X, if this element satisfies $1x = x1 = x$ for every $x \in X$. In this case it is called a unital quasi-algebra.

Definition 3.2. Let X be a quasi-algebra. A real function $\|\cdot\|_X : X \to \mathbb{R}$ is called a norm if $(X; \|\cdot\|_X)$ is a normed quasilinear space and $\|xy\|_X \leq \|x\|_X \|y\|_X$.

A normed quasi-algebra is a pair $(X; \|\cdot\|_X)$, where X is a non-zero quasi-algebra and $\|\cdot\|_X$ is a given quasi-algebra norm on it. A unital normed quasi-algebra, is a normed quasi-algebra with an identity element 1 such that $\|1\|_X = 1$.

Theorem 3.3. The operations of algebraic sum, product and multiplication by real numbers are continuous with respect to the Hausdorff metric. Moreover, the norm is a continuous function with respect to the Hausdorff metric.

Proof . We prove that the operation of algebraic product is continuous. Proof of other parts is in [2]. Suppose that $x_n \to x$ and $y_n \to y$. Then for any $\epsilon > 0$ there exists an index N such that the following conditions hold for $n \geq N$:

$$x \leq x_n + a^{\epsilon}_{1,n}, x_n \leq x + a^{\epsilon}_{2,n}, \|a^{\epsilon}_{i,n}\|_X \leq \min\{\frac{\epsilon}{6\|y\|}, \frac{\sqrt{\epsilon}}{\sqrt{6}}\},$$

$$y \leq y_n + b^{\epsilon}_{1,n}, y_n \leq y + b^{\epsilon}_{2,n}, \|b^{\epsilon}_{i,n}\|_X \leq \min\{\frac{\epsilon}{6\|x\|}, \frac{\sqrt{\epsilon}}{\sqrt{6}}\}.$$

Consequently,

$$xy \leq x_n y_n + x_n b^{\epsilon}_{1,n} + a^{\epsilon}_{1,n} y_n + a^{\epsilon}_{1,n} b^{\epsilon}_{1,n}$$

and

$$x_n y_n \leq xy + x b^{\epsilon}_{2,n} + a^{\epsilon}_{2,n} y + a^{\epsilon}_{2,n} b^{\epsilon}_{2,n}.$$

Since

$$\begin{aligned} &\|x_n b^{\epsilon}_{1,n} + a^{\epsilon}_{1,n} y_n + a^{\epsilon}_{1,n} b^{\epsilon}_{1,n}\|_X \\ &\leq \|x_n b^{\epsilon}_{1,n}\|_X + \|a^{\epsilon}_{1,n} y_n\|_X + \|a^{\epsilon}_{1,n} b^{\epsilon}_{1,n}\|_X \\ &\leq \|x_n\|_X \|b^{\epsilon}_{1,n}\|_X + \|a^{\epsilon}_{1,n}\|_X \|y_n\|_X + \|a^{\epsilon}_{1,n}\|_X \|b^{\epsilon}_{1,n}\|_X \\ &\leq (\|x\|_X + \|a^{\epsilon}_{2,n}\|_X)\|b^{\epsilon}_{1,n}\|_X + \|a^{\epsilon}_{1,n}\|_X (\|y\|_X + \|b^{\epsilon}_{2,n}\|_X) + \|a^{\epsilon}_{1,n}\|_X \|b^{\epsilon}_{1,n}\|_X \\ &\leq \epsilon, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &\|x b^{\epsilon}_{2,n} + a^{\epsilon}_{2,n} y + a^{\epsilon}_{2,n} b^{\epsilon}_{2,n}\|_X \\ &\leq \|x b^{\epsilon}_{2,n}\|_X + \|a^{\epsilon}_{2,n} y\|_X + \|a^{\epsilon}_{2,n} b^{\epsilon}_{2,n}\|_X \\ &\leq \|x\|_X \|b^{\epsilon}_{2,n}\|_X + \|a^{\epsilon}_{2,n}\|_X \|y\|_X + \|a^{\epsilon}_{2,n}\| \|b^{\epsilon}_{2,n}\|_X \\ &\leq \epsilon, \end{aligned} \tag{3.2}$$

we have $x_n y_n \to xy$. $\square$

Example 3.4. Let A be a real normed algebra. We show the space of nonempty closed bounded subsets of A with $\Omega(A)$. The operations of algebraic sum, multiplication by real number and the partial order are defined similarly to its quasilinear space sample. see [2]. The product on $\Omega(A)$ is defined as follows:

$$BC = \overline{\{bc : b \in B, c \in C\}}.$$

Then $\Omega(A)$ is a normed quasi-algebra, and

$$\|B\|_{\Omega} = \sup_{b \in B} \|b\|_A.$$

Example 3.5. Space of all symmetric intervals around zero in real numbers is another example of a normed quasi-algebra. It is subalgebra of

$$\Omega_c(\mathbb{R}) = \{[a,b] : a, b \in \mathbb{R}\}.$$

Here we have also suppose $\{a\}$ as an interval. It is easy to verify that $[-1,1]$ is an identity of this space. Therefore, it is not necessary that 1 is regular. This happens while the identity of $\Omega_c(\mathbb{R})$ is $\{1\}$.

A complete normed quasi-algebra is called a Banach quasi-algebra. Before stating the following definition, refer to [2] to get familiar with the concept of quasilinear operator.

Definition 3.6. Let X and Y be quasi-algebras. A mapping $\varphi : X \to Y$ is a quasi-homomorphism if it satisfies the following conditions:

1. φ is a quasilinear operator;
2. $\varphi(xy) \leq \varphi(x)\varphi(y)$.

We now extend the concept of n-homomorphism introduced for (complex) Banach algebras [13]. Let X and Y be quasi-algebras. The quasilinear mapping $\psi : X \to Y$ is called an n-quasi-homomorphism, if $\psi(x_1x_2\cdots x_n) \leq \psi(x_1)\psi(x_2)\cdots\psi(x_n)$, for all $x_1, x_2\cdots x_n \in X$. A 2-quasi-homomorphism is then a quasi-homomorphism between quasi-algebras. It is easy to see that any quasi-homomorphism $\varphi : X \to Y$ is an n-quasi-homomorphism for each $n \in \mathbb{N}$. Similar to what can be seen in [13], the relationship between notions of n-quasi-homomorphism and quasi-homomorphism can be checked.

Automatic continuity of n-homomorphisms considered for C*-algebras, factorizable Banach algebras and topological algebras in [14, 15, 16]. Interested researchers can try to obtain similar results for quasi-algebras. Also, by developing the concept of n-Jordan homomorphisms, which is introduced in [12], the similar issues in the new space, can be investigated. see [9, 10, 11].

Let X be a unital quasi-algebra. Then left and right product-invertible, product-invertible and product-inverse of an element are defined similarly to algebraic spaces. Also, we denote the set of all product-invertible elements of X by $G = G(X)$.

Lemma 3.7. Let X be a quasi-algebra with an identity 1. If $x \in G(X)$ then $x(y + z) = xy + xz$ for every $y, z \in X$.

Proof . Since $x^{-1}(xy + xz) \leq x^{-1}xy + x^{-1}xz$, thus

$$xy + xz = 1(xy + xz) = (xx^{-1})(xy + xz) = x(x^{-1}(xy + xz)) \leq x(y + z).$$

□

Corollary 3.8. Let X be a quasi-algebra with an identity 1. If $1 \in X_r$ then $x \in X_r$ for any $x \in G(X)$.

Proof . It follows from lemma 3.7 that $x(1 - 1) = x - x$. Since $x(1 - 1) = x0 = 0$, thus $x - x = 0$. □

Lemma 3.9. Let X be a proper Banach quasi-algebra. Then X_r is a real Banach algebra.

Proof . Suppose that x_i is a Cauchi sequence in X_r. Since X is complete, there exists $x \in X$ such that $x_i \to x$ and $-x_i \to -x$. Thus $x_i - x_i \to x - x$ and since $x_i - x_i \to 0$, so $x - x = 0$. Therefore $x \in X_r$. Proving that the space is an algebra is simple and follows from the minimality of regular elements. □

4 Quasi-invertible

Now, we try to generalize one of the rudimentary concepts of algebras to quasi-algebras such that this new concept does not contradict its counterpart in algebras. Also, this definition should be such that it can be used to generalize many propositions and results.

Definition 4.1. Let X be a quasi-algebra with an identity 1. $x \in X$ is left (right) quasi-invertible if there exists $y \in X$, called the factor of left (right) invertibility, such that for any $z \leq x$, the following condition holds:

$$1 \leq yz \ (1 \leq zy).$$

Definition 4.2. $x \in X$ is quasi-invertible if it is both left and right quasi-invertible.

Here, it is not necessary that the left and right factors are equal.

Lemma 4.3. Every invertible element in a quasi-algebra X with an identity $1 \in X_r$, is quasi-invertible.

Proof . To prove, we use the minimal property of regular elements 2.5. □

The proof of the following lemma is simple.

Lemma 4.4. Let X be a quasi-algebra and x be a quasi-invertible element. every $y \in X$ with $y \leq x$ is quasi-invertible.

Example 4.5. In a unital algebra A, $x \in A$ is quasi-invertible if and only if it is invertible.

Example 4.6. Let A be a unital normed algebra. $x \in \Omega(A)$ is quasi-invertible if there is no $a \in A - G(A)$ in x.

The following theorem is simple and the proof of it is left to the reader

Lemma 4.7. Let X be a quasi-algebra with $1 \in X$ and $a, b \in X$:

1. If a is quasi-invertible and $\lambda \neq 0$, so is λa.
2. If ab is left quasi-invertible, so is b.
3. If ab is right quasi-invertible, so is a.
4. If a is right quasi-invertible and b is invertible, then ab is right quasi-invertible.
5. If b is left quasi-invertible and a is invertible, then ab is left quasi-invertible.

Theorem 4.8. Suppose that X is a proper unital Banach quasi-algebra with $1 \in X_r$. let $x \in X_r$ be such that $h_X(x, 1) < 1$. Then x is invertible.

Proof . Since X_r is itself a unital Banach algebra, by lemma 3.9, we conclude this from preliminary Banach algebras [1]. □

Definition 4.9. Let X be a preordered set. A subset $B \subseteq X$ is said to be coinitial, if it satisfied the following condition: For every $a \in X$, there exists some $b \in B$ such that $b \leq a$.

Theorem 4.10. Suppose that X is a non-zero unital quasi-algebra. let $X_r = \{0\}$ be coinitial. Then none of the elements of the space is left (right) quasi-invertible.

Proof . Let $x \in X$ be left quasi-invertible. So there exists $y \in X$ such that for any $z \leq x$, we have $1 \leq yz$. Since $\{0\}$ is coinitial, thus $0 \leq x$ and hence $1 \leq 0$. Therefore $1 = 0$. This means $X = \{0\}$, which is a contradiction. □

Lemma 4.11. Let X be a unital normed quasi-algebra, $0 \leq r < 1$ and let

$$A := \{a \in G(X) : \|a - 1\|_X \leq r\}.$$

Then the set $\{\|b^{-1}\|_X : b \in A\}$ is bounded.

Proof . Suppose that $b \in A$. Using lemma 3.7, we have

$$\|b^{-1}\|_X \leq \|b^{-1} - 1\|_X + \|1\|_X = \|b^{-1}(1 - b)\|_X + 1 \leq r\|b^{-1}\|_X + 1.$$

whence $(1 - r)\|b^{-1}\|_X \leq 1$. □

Theorem 4.12. Let X be a proper unital Banach quasi-algebra with an identity $1 \in X_r$. Let X_r be coinitial and every bounded subset of it has an upper bound. Then $a \in X$ is quasi-invertible if $h_X(a, 1) < 1$.

Proof . There exist a_1 with $\|a_1\|_X < 1$, such that:

$$a \leq 1 + a_1.$$

Necessarily $t \leq a$ for some $t \in X_r$ because the set of regular elements is coinitial. So that

$$h_X(t, 1) = \|t - 1\|_X \leq \|a - 1\|_X \leq \|a_1\|_X < 1.$$

It follows from theorem 4.8 that t is invertible. By a similar proof, for all $t' \leq a$ we have

$$\|t' - 1\|_X \leq \|a_1\|_X < 1.$$

So the set $B := \{t^{-1} : t \in X_r, t \leq a\}$ is bounded with its norm by lemma 4.11. Taking b the upper bounded of it. Then for any $x \leq a$ we have:

$$1 = rr^{-1} \leq xb$$

for some $r \leq x$ with $r \in X_r$, which complete the proof. □

We get the following result from theorem 2.9.

Corollary 4.13. Let X be a proper unital Banach quasi-algebra and Ω-space at the same time. Also, $1 \in X_r$ and X_r is coinitial. Then $a \in X$ is quasi-invertible if $h_X(a, 1) < 1$.

Lemma 4.14. Suppose that X is defined as in theorem 4.12. Let $t \in X$ be invertible and $a \in X$ be such that $h_X(a, t) < \frac{1}{\|t^{-1}\|_X}$. then a is quasi-invertible.

Proof . There exist a_1 and a_2 such that:

$$a \leq t + a_1, \qquad \|a_1\|_X < \frac{1}{\|t^{-1}\|_X}, \tag{4.1}$$

$$t \leq a + a_2, \qquad \|a_2\|_X < \frac{1}{\|t^{-1}\|_X}. \tag{4.2}$$

Since t is invertible,

$$t^{-1}a \leq 1 + t^{-1}a_1, \qquad \|t^{-1}a_1\|_X < 1,$$
$$1 \leq t^{-1}a + t^{-1}a_2, \qquad \|t^{-1}a_2\|_X < 1.$$

So $t^{-1}a$ is quasi-invertible by theorem 4.12 and then a is left quasi-invertible by lemma 4.7. We conclude a is right quasi-invertible too, if t^{-1} is multiplied from the right side in 4.1 and 4.2. therefore a is quasi-invertible. □

Lemma 4.15. Let X be a unital quasi-algebra and $a \in X$ be a left (right) quasi-invertible element with a_q^{-1} as its factor of left (right) invertibility. If $t \in X$ is an invertible element less and equal to a, then $\frac{1}{\|a_q^{-1}\|_X} \leq \frac{1}{\|t^{-1}\|_X}$.

Proof . Since a_q^{-1} is left factor of a and $t \leq a$, thus $1 \leq a_q^{-1}t$. So $t^{-1} \leq a_q^{-1}$ and therefore $\|t^{-1}\|_X \leq \|a_q^{-1}\|_X$. □

Definition 4.16. The left (right) factor y in 4.1 is left (right) quasi-inverse of x if x is a right (left) factor of it, too.

Example 4.17. In example 4.6, $\{b^{-1} : b \in x\}$ is the quasi-inverse of x.

It is easy to see that If x is left (right) quasi-inverse of y then y is right (left) quasi-inverse of x.

Theorem 4.18. Again, suppose that X is defined as in theorem 4.12 and $a \in X$ is quasi-invertible. Also suppose that every left quasi-invertible element x has a left quasi-inverse $x_{q,l}^{-1}$ and every right quasi-invertible element y has a right quasi-inverse $y_{q,r}^{-1}$. $b \in X$ is quasi-invertible if

$$\|a - b\|_X < \max\{\frac{1}{\|a_{q,l}^{-1}\|_X}, \frac{1}{\|a_{q,r}^{-1}\|_X}\}.$$

Proof . There is an element $t \in X_r$ with $t \leq a$. Since a is quasi-invertible so is t by lemma 4.4. Thus $1 \leq kt$ for some $k \in X_r$ and $k \leq t^{-1}_{q,l}$. kt is regular and so $kt = 1$. Therefore t is left invertible. Similarly, t is right invertible and hence it is invertible. It then follows from 4.15 that

$$\frac{1}{\|a^{-1}_{q,l}\|_X} \leq \frac{1}{\|t^{-1}\|_X},$$

and

$$\frac{1}{\|a^{-1}_{q,r}\|_X} \leq \frac{1}{\|t^{-1}\|_X}.$$

We see that

$$h_X(b,t) \leq \|b-t\|_X \leq \|b-a\|_X < \max\{\frac{1}{\|a^{-1}_{q,l}\|_X}, \frac{1}{\|a^{-1}_{q,r}\|_X}\} \leq \frac{1}{\|t^{-1}\|_X}.$$

The result now immediate from lemma 4.14. □

A meter h_X on a quasilinear space X is order-preserving whenever $a \leq b$ implies $h_X(a,c) \leq h_X(b,c)$ for any $a,b,c \in X$. It is easy to see that if h_X has this property then $h_X(a,c) \leq h_X(b,d)$ is implied from $a \leq b$ and $c \leq d$. In the previous theorem, if the meter is order-preserving, then the theorem can be concluded from

$$h_X(a,b) < \max\{\frac{1}{\|a^{-1}_{q,l}\|_X}, \frac{1}{\|a^{-1}_{q,r}\|_X}\},$$

and there is no need for the norm of their difference to be less than the right side.

Example 4.19. Let A be a unital Banach algebra. Then $\Omega(A)$ has all the properties needed to establish the above theorem. So if, for example, $X = \Omega_c(\mathbb{R})$ and $a > 0$, then $e \in \{[c,d] : \max\{|c-a|, |d-a|\} < a\}$ is quasi-invertible.

Example 4.20. For any $a, b \in \mathbb{R}$, $\{[a',b'] : a \leq a' \leq b' \leq b\}$ is a closed and bounded subset of $\Omega_c(\mathbb{R})$. The space of all $\{[a',b'] : a \leq a' \leq b' \leq b\}$ is another example that has the conditions of theorem 4.18. Here the algebraic operations are defined pointwise and the partial order is defined by inclusion.

Now it is time to adapt the spectrum to the new concept of invertibility.

Definition 4.21. Let X be a quasi-algebra with an identity 1 and let $x \in X$. We define the quasi-spectrum $Sp_q(x)$ of x to be:

$$Sp_q(x) = \{\lambda \in \mathbb{R} : \lambda 1 - x \ is\ not\ quasi-invertible\}$$

Theorem 4.22. Let X be a Banach quasi-algebra that has the property of theorem 4.18 and let $x \in X$. Then $Sp_q(x)$ is a compact subset of the line segment $\{\lambda \in \mathbb{R} : |\lambda| \leq \|x\|_X\}$.

Proof . For each $\lambda \in \mathbb{R}$ with $|\lambda| > \|x\|_X$, we have:

$$h_X(1 - \lambda^{-1}x, 1) \leq \|\lambda^{-1}x\|_X < 1.$$

Hence $1 - \lambda^{-1}x$ is quasi-invertible and so is $\lambda 1 - x$, i.e. $\lambda \notin Sp_q(x)$. Also since the norm of a quasi-algebra is continuous, so the set of quasi-invertible elements is open in X by theorem 4.18, and then $Sp_q(x)$ is the inverse image of a closed subset, with respect to the continuous map $\lambda \mapsto \lambda 1 - x$, see 3.3. So it is compact in $\mathbb{R}$. □

Corollary 4.23. Suppose X is defined similarly to corollary 4.13 and B_X is the element of it introduced in 2.8. Then $Sp_q(\beta B_X) = [-|\beta|, |\beta|]$.

Proof . As above, $Sp_q(\beta B_X) \subseteq [-|\beta|, |\beta|]$. If $|\lambda| \leq |\beta|$ then $\lambda 1 \leq \beta B_X$. It follows that $0 \leq \lambda 1 - \beta B_X$. We shall show that $\lambda 1 - \beta B_X$ is not quasi-invertible. Assume to the cotrary that it is quasi-invertible. So 0 is quasi-invertible too, by lemma 4.4, a contradiction. □

Theorem 4.24. In a quasi-algebra X, if $x \leq y$ then $Sp_q(x) \subseteq Sp_q(y)$.

Proof . Suppose that $\lambda \notin Sp_q(y)$. Thus $\lambda 1 - y$ is quasi-invertible. It follows that $\lambda 1 - x$ is quasi-invertible too, because $\lambda 1 - x \leq \lambda 1 - y$. The last case is immediate from lemma 4.4. □

The following theorem is the generalization of the theorem named "Spectral mapping property for polynomials", which is proved only for a certain group of elements in quasi-algebras.

Theorem 4.25. Let X be the quasi-algebra defined in corollary 4.23. In addition, let $a \in X$ and $p(x) = \alpha_n x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_0$ be a real polynomial with $\alpha_1 \neq 0$. Then:

$$\{p(\lambda) : \lambda \in Sp_q(\beta B_X)\} \subseteq Sp_q(p(\beta B_X)).$$

Proof . We may suppose that $\alpha_1 = 1$, for otherwise, we divide the polynomial by α_1. Let $\lambda \in Sp_q(\beta B_X)$. Then $\|\lambda 1\|_X \leq \|\beta B_X\|_X$ by corollary 4.23. So $\lambda 1 \leq \beta B_X$ and therefore

$$\begin{aligned} &\lambda 1 - \beta B_X \\ &\quad \leq \alpha_n(\lambda^n 1 - (\beta B_X)^n) + \alpha_{n-1}(\lambda^{n-1} 1 - (\beta B_X)^{n-1}) + \cdots + \lambda 1 - \beta B_X \\ &\quad = p(\lambda) 1 - p(\beta B_X). \end{aligned} \tag{4.3}$$

Thus $p(\lambda) \in Sp_q(p(\beta B_X)$ by lemma 4.4, which completes the proof. □

5 Inquiry

In this paper we have not proved that if an element is left (right) quasi-invertible, then it has a left (right) quasi-inverse? Also, under what conditions if it has left and right inverses, these two are equal? This research can continue by answering these questions, as well as proving theorems for more general situations.

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