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Convolution properties for some subclasses of meromorphic p-valent functions of complex order associated with q-derivative

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Abstract

In this present investigation, for functions of the form $f(z)=\frac{1}{z^p}+\sum_{k=1-p}^{\infty}a_kz^k$, which are analytic in the punctured unit disk $\mathbb{U}^*=\{z\in\mathbb{C}:\ \ 0<|z|<1\}$, we introduce a new subclass of meromorphically p-valent functions and investigate convolution properties, Coefficient estimates and contianment for this subclass.

Keywords: q-derivative, meromorphic function, coefficient bound, extreme point, convex set, partial sum
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1 Introduction

Let Σ_p denote the class of mermorphic p-valent functions of the form

$$f(z)=\frac{1}{z^p}+\sum_{k=1-p}^{\infty}a_kz^k, \tag{1.1}$$

which are analytic and p-valent in the punctured unit disk

$$\mathbb{U}^*=\{z\in\mathbb{C}:\ 0<|z|<1\}.$$

If $f\in\Sigma_p$ is given by(1.1) and $g\in\Sigma_p$ given by

$$g(z)=\frac{1}{z^p}+\sum_{k=1-p}^{+\infty}b_kz^k,$$

then the Hadamard product (or convolution) $f*g$ of f and g is defined by

$$(f*g)(z)=\frac{1}{z^p}+\sum_{k=1-p}^{+\infty}a_kb_kz^k. \tag{1.2}$$

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For two function f and g analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$ and write $f \prec g$ or $f(z) \prec g(z) \quad (z \in \mathbb{U})$, if there exists a Schwarz function $\omega(z)$, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, shuch that $f(z) = \omega(g(z))$, (see [9, 11]). Gasper and Rahman in [3] defined the q-derivative of a function $f(z)$ of the from (1.1) by

$$D_q f(z) := \frac{f(qz) - f(q)}{(q-1)z}, \tag{1.3}$$

where $z \in \mathbb{U}^*$ and $0 < q < 1$. From (1.3) for a function $f(z)$ given by (1.1) we get

$$D_q f(z) = \frac{q^{-p} - 1}{(q-1)z^{p+1}} + \sum_{k=1}^{+\infty} [k]_q a_k z^{k-1}, \qquad z \in \mathbb{U}^*, \tag{1.4}$$

where

$$[k]_q := \frac{q^k - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{k-1}. \tag{1.5}$$

also $[k]_q \to 1$ as $q \to \bar{1}$.So we conclude $\lim_{q \to \bar{1}} D_q f(z) = f'(z), z \in \mathbb{U}^*$. Many important properties of certain subclasses of meromorphic p-valent functions were studied by several authors including Aouf and Srivastava [2], Joshi and Srivastava [4], Liu and Srivastava[7], Liu and Owa [6], Liu and Srivastava [8], Ravichandran, Sivaprasadkumar and Subramanian [12].

2 Preliminaries

Using the subclasses defined by Mostafa, Aouf, Zayed and Bulboaca in[10], Now we introduce new subclasess of mermorphic p-valent functions a and investigate convolution properties and cofficien estimates for these subclasses as follows:

Definition 2.1. For $0 \leq \lambda < 1$, $-1 \leq B < A \leq 1$, and $b \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, let $\Sigma_p \mathcal{S}^*_{q,\lambda}[b; A, B]$ be the subclass of Σ_p consisting of function $f(z)$ of the form(1.1) and satisfying the analytic criterion

$$q\frac{1-q^{-p}}{q-1} - \frac{1}{b}\left[\frac{zD_q f(z)}{(1-\lambda\frac{1-q^{-p}}{q-1})f(z) - \lambda z D_q f(z)} + \frac{1-q^{-p}}{q-1}\right] \prec \frac{1+Az}{1+Bz} \tag{2.1}$$

Also, let $\Sigma_p \mathcal{K}_{q,\lambda}[b; A, B]$ be the subclass of Σ_p consisting of function $f(z)$ of the form(1.1) and satisfying

$$q\frac{1-q^{-p}}{q-1} - \frac{1}{b}\left[\frac{zD_q(\frac{1-q^{-p}}{q-1} zD_q f(z))}{(1-\lambda\frac{1-q^{-p}}{q-1})(\frac{1-q^{-p}}{q-1} zD_q f(z)) - \lambda z D_q(\frac{1-q^{-p}}{q-1} zD_q f(z))} + \frac{1-q^{-p}}{q-1}\right] \prec \frac{1+Az}{1+Bz}. \tag{2.2}$$

It is easy to verify from (2.1) and (2.2) that

$$f \in \Sigma_p \mathcal{K}_{q,\lambda}[b; A, B] \Longleftrightarrow \frac{1-q^{-p}}{q-1} zD_q f(z) \in \Sigma_p \mathcal{S}^*_{q,\lambda}[b; A, B]. \tag{2.3}$$

we note that

1. For $p = 1$ we get $\Sigma_p \mathcal{S}^*_{q,\lambda}[b; A, B] = \Sigma_1 \mathcal{S}^*_{q,\lambda}[b; A, B] = \Sigma \mathcal{S}^*_{q,\lambda}[b; A, B]$, (see [1]).
2. For $p = 1$ we get $\Sigma_p \mathcal{K}_{q,\lambda}[b; A, B] = \Sigma_1 \mathcal{K}_{q,\lambda}[b; A, B] = \Sigma \mathcal{K}_{q,\lambda}[b; A, B]$, (see [1]).

3 Main Result

In this section we give some new subclasses of mermorphic p-valent functions.

3.1 subclasses $\Sigma_p\mathcal{S}^*_{q,\lambda}[b;A,B]$ and $\Sigma_p\mathcal{K}_{q,\lambda}[b;A,B]$

In the first theorem we give some necessary and sufficieint conditions for member of subclass $\Sigma_p\mathcal{S}^*_{q,\lambda}[b;A,B]$.

Theorem 3.1. If $f\in\Sigma_p$, then $f\in\Sigma_p\mathcal{S}^*_{q,\lambda}[b;A,B]$ if and only if

$$z^p\left[f(z)*\frac{1+\left\{\left(1-\lambda\frac{1-q^{-p}}{q-1}\right)M(\theta,p)-(q+\frac{\lambda}{q^p})\right\}z}{z^p(1-z)(1-qz)}\right]\neq 0 \tag{3.1}$$

for all $z\in\mathbb{U}^*$ and $\theta\in[0,2\pi)$. Where

$$M(\theta,p)=\frac{1+Be^{i\theta}}{q^pb[1-q\frac{1-q^{-p}}{q-1}+(A-Bq\frac{1-q^{-p}}{q-1})e^{i\theta}]} \tag{3.2}$$

Proof . It is easy to verify that for any $f\in\Sigma_p$ the next relations hold:

$$f(z)*\frac{1}{z^p(1-z)}=f(z), \tag{3.3}$$

and

$$f(z)*\left[\frac{1}{z^p(1-z)(1-qz)}-\frac{1+\frac{1-q}{q^{-p}-1}}{z^{p-1}(1-z)(1-qz)}\right]=\frac{1-q}{q^{-p}-1}zD_qf(z). \tag{3.4}$$

First, if $f\in\Sigma_p\mathcal{S}^*_{q,\lambda}[b;A,B]$, in order to prove that (3.1) holds we will write (2.1) by using the definition of the subordination, that is

$$\frac{zD_qf(z)}{(1-\lambda\frac{1-q^{-p}}{q-1})f(z)-\lambda zD_qf(z)}=\frac{\left[b(q\frac{1-q^{-p}}{q-1}-1)-\frac{1-q^{-p}}{q-1}\right]}{1+B\omega(z)}+\frac{\left[b(qB\frac{1-q^{-p}}{q-1}-A)-B\frac{1-q^{-p}}{q-1}\right]\omega(z)}{1+B\omega(z)}, \tag{3.5}$$

where ω is a Schwarz function, hence

$$\begin{aligned}&z\Big\{[1+Be^{i\theta}]zD_qf(z)-[b(q\frac{1-q^{-p}}{q-1}-1)-\frac{1-q^{-p}}{q-1}]\\&-[b(qB\frac{1-q^{-p}}{q-1}-A)-B\frac{1-q^{-p}}{q-1}]e^{i\theta}[(1-\lambda\frac{1-q^{-p}}{q-1})f(z)-\lambda zD_qf(z)]\Big\}\neq 0\end{aligned} \tag{3.6}$$

for all $z\in\mathbb{U}^*$ and $\theta\in[0,\pi)$. Using (3.3) and (3.4), the relation (3.6) may be written as

$$\begin{aligned}&z\Bigg\{\Bigg(-\frac{1-q^{-p}}{q-1}-B\frac{1-q^{-p}}{q-1}z-\lambda\frac{1-q^{-p}}{q-1}\times\Bigg(\left[b(q\frac{1-q^{-p}}{q-1}-1)-\frac{1-q^{-p}}{q-1}\right]+\left[b(qB\frac{1-q^{-p}}{q-1}-A)-B\frac{1-q^{-p}}{q-1}\right]e^{i\theta}\Bigg)\Bigg)\\&f(z)*\left[\frac{1}{z^p(1-z)(1-qz)}-\frac{1+\frac{1-q}{q^{-p}-1}}{z^{p-1}(1-z)(1-qz)}\right]+\Bigg\{\left(\lambda\frac{1-q^{-p}}{q-1}-1\right)\left[b(q\frac{1-q^{-p}}{q-1}-1)-\frac{1-q^{-p}}{q-1}\right]\\&+\left[b(qB\frac{1-q^{-p}}{q-1}-A)-B\frac{1-q^{-p}}{q-1}\right]e^{i\theta}\Bigg\}\left[f(z)*\frac{1}{z^p(1-z)}\right]\Bigg\}\neq 0,\end{aligned} \tag{3.7}$$

which is equivalent to

$$z\left[f(z)*\frac{1+\left\{\left(1-\lambda\frac{1-q^{-p}}{q-1}\right)\frac{1+Be^{i\theta}}{q^pb[1-q\frac{1-q^{-p}}{q-1}+(A-Bq\frac{1-q^{-p}}{q-1})e^{i\theta}]}-(q+\frac{\lambda}{q^p})\right\}z}{z^p(1-z)(1-qz)}\right]\neq 0, \tag{3.8}$$

where $z\in\mathbb{U}$, $\theta\in[0,2\pi)$ and thus the first part of Theorem (3.1) was proved. Reversely, suppose that $f\in\Sigma_p$ satisfy the condition (3.1). Like it was previously shown, the assumption (3.1) is equivalent to (3.6), hence

$$\frac{zD_qf(z)}{(1-\lambda\frac{1-q^{-p}}{q-1})f(z)-\lambda zD_qf(z)}\neq\frac{\left[b(q\frac{1-q^{-p}}{q-1}-1)-\frac{1-q^{-p}}{q-1}\right]}{1+Be^{i\theta}}+\frac{\left[b(qB\frac{1-q^{-p}}{q-1}-A)-B\frac{1-q^{-p}}{q-1}\right]e^{i\theta}}{1+Be^{i\theta}}, \tag{3.9}$$

for all $z \in \mathbb{U}$ and $\theta \in [0, 2\pi)$. Denoting

$$\varphi(z) = \frac{zD_q f(z)}{(1 - \lambda \frac{1-q^{-p}}{q-1}) f(z) - \lambda z D_q f(z)}$$

and

$$\psi(z) = \frac{\left[b(q\frac{1-q^{-p}}{q-1} - 1) - \frac{1-q^{-p}}{q-1}\right] + \left[b(qB\frac{1-q^{-p}}{q-1} - A) - B\frac{1-q^{-p}}{q-1}\right] z}{1 + Bz},$$

The relation (3.9) means that

$$\varphi(\mathbb{U}) \cap \psi(L(\mathbb{U})) = \emptyset$$

and

$$(L(z) = \Psi(z) - \left[b(q\frac{1-q^{-p}}{q-1} - 1) - \frac{1-q^{-p}}{q-1}\right]).$$

Thus, the simply connected domain is included in a connected component of $\mathbb{C}\backslash\psi(L(\mathbb{U}))$. Therefore, using the fact that $\varphi(0) = \psi(L(0))$ and the p-valent function ψ, it follows that $\varphi(z) \prec \psi(z)$, which implies that $f \in \Sigma_p \mathcal{S}^*_{q,\lambda}[b; A, B]$. Thus, the proof of Theorem (3.1) is completed. □

Theorem 3.2. If $f \in \Sigma_p$, then $f \in \Sigma_p \mathcal{K}_{q,\lambda}[b; A, B]$ if and only if

$$z^p\left[f(z) * \frac{1 - \frac{1-q^{p+2}}{1-q^p} z + \left[\frac{q-q^p}{1-q^p} z + qz^2 \frac{q^{p+1}-1}{1-q^p}\right] \left\{\left(1 - \lambda\frac{1-q^{-p}}{q-1}\right) M(\theta, p) - (q + \frac{\lambda}{q^p})\right\}}{z^p(1-z)(1-qz)(1-q^2 z)}\right] \neq 0 \tag{3.10}$$

for all $z \in \mathbb{U}^*$ and $\theta \in [0, 2\pi)$, where $M(\theta, p)$ is given by (3.2).

Proof . From (2.3) it follows that $f \in \Sigma_p \mathcal{K}_{q,\lambda}[b; A, B]$ if and only if $\Phi_q(z) := \frac{q-1}{1-q^{-p}} zD_q f(z) \in \Sigma_p \mathcal{S}^*_{q,\lambda}[b; A, B]$. Then, accoding to Theorem (3.1), the function Φ_q belongs to $\Sigma_p \mathcal{S}^*_{q,\lambda}[b; A, B]$ if and only if

$$z[\Phi_q(z) * g(z)] \neq 0, \tag{3.11}$$

for all $z \in \mathbb{U}$ and $\theta \in [0, 2\pi)$, where

$$g(z) = \frac{1 + \left\{\left(1 - \lambda\frac{1-q^{-p}}{q-1}\right) \frac{1+Be^{i\theta}}{q^p b[1-q\frac{1-q^{-p}}{q-1}+(A-Bq\frac{1-q^{-p}}{q-1})e^{i\theta}]} - (q + \frac{\lambda}{q^p})\right\} z}{z^p(1-z)(1-qz)}. \tag{3.12}$$

A simple computation shows that

$$\begin{aligned} D_q g(z) = \frac{g(qz) - g(z)}{(q-1)z} &= \frac{(1-q^p) - (1-q^{p+2})z}{q^p(q-1)z^{p+1}(1-z)(1-qz)(1-q^2 z)} \\ &+ \frac{[(q-q^p)z + qz^2(q^{p+1}-1)]\left(\left(1 - \lambda\frac{1-q^{-p}}{q-1}\right) \frac{1+Be^{i\theta}}{q^p b[1-q\frac{1-q^{-p}}{q-1}+(A-Bq\frac{1-q^{-p}}{q-1})e^{i\theta}]}\right.}{q^p(q-1)z^{p+1}(1-z)(1-qz)(1-q^2 z)} \\ &- \frac{\left.(q + \frac{\lambda}{q^p})\right)}{q^p(q-1)z^{p+1}(1-z)(1-qz)(1-q^2 z)} \end{aligned} \tag{3.13}$$

and therefore

$$\begin{aligned} \frac{1-q}{1-q^{-p}} zD_q g(z) &= \frac{1 - \frac{1-q^{p+2}}{1-q^p} z + [\frac{q-q^p}{1-q^p} z + \frac{q^{p+1}-1}{1-q^p} qz^2][\left(1 - \lambda\frac{1-q^{-p}}{q-1}\right) \frac{1+Be^{i\theta}}{q^p b[1-q\frac{1-q^{-p}}{q-1}+(A-Bq\frac{1-q^{-p}}{q-1})e^{i\theta}]}}{(1-z)(1-qz)(1-q^2 z)} \\ &- \frac{(q + \frac{\lambda}{q^p}))]}{(1-z)(1-qz)(1-q^2 z)} \end{aligned}$$

Using the above relation and the identity

$$[\frac{q-1}{1-q^{-p}}zD_qf(z)]*g(z)=f(z)*[\frac{q-1}{1-q^{-p}}zD_qg(z)] \tag{3.14}$$

it is easy to check that (3.11) is equivalent to (3.10). □

Theorem 3.3. If $f\in\Sigma_p$, then $f\in\Sigma_p\mathcal{S}^*_{q,\lambda}[b;A,B]$ if and only if

$$1+\sum_{k=1}^{\infty}\Big[\frac{(1-\frac{\lambda}{q^p}[k]_q)\left[(1-q\frac{1-q^{-p}}{q-1})+(A-qB\frac{1-q^{-p}}{q-1})e^{i\theta}\right]bq^p}{q^pb\left[\left(1-q\frac{1-q^{-p}}{q-1}\right)+(A-qB\frac{1-q^{-p}}{q-1})e^{i\theta}\right]}+\frac{\left(1-\lambda\frac{1-q^{-p}}{q-1}\right)(1+Be^{i\theta})[k]_q}{q^pb\left[\left(1-q\frac{1-q^{-p}}{q-1}\right)+(A-qB\frac{1-q^{-p}}{q-1})e^{i\theta}\right]}\Big]a_kz^{k+p}\neq 0$$

for all $z\in\mathbb{U}^*$ and $\theta\in[0,2\pi)$.

Proof . If $f\in\Sigma_p$, then from Theorem (3.1) we have

$$z^p[f(z)*\frac{1+(1-\lambda\frac{1-q^{-p}}{q-1})\frac{1+Be^{i\theta}}{q^pb[1-q\frac{1-q^{-p}}{q-1}+(A-Bq\frac{1-q^{-p}}{q-1})e^{i\theta}]}-(q+\frac{\lambda}{q^p})z}{z^p(1-z)(1-qz)}]\neq 0 \tag{3.15}$$

for all $z\in\mathbb{U}^*$ and $\theta\in[0,2\pi)$, since

$$\frac{1}{z^p(1-z)(1-qz)}=\frac{1}{z^p}+\sum_{k=1-p}^{\infty}[k+p]_qz^k \tag{3.16}$$

it follows that

$$\frac{1+(1-\lambda\frac{1-q^{-p}}{q-1})\frac{1+Be^{i\theta}}{q^pb[1-q\frac{1-q^{-p}}{q-1}+(A-Bq\frac{1-q^{-p}}{q-1})e^{i\theta}]}-(q+\frac{\lambda}{q^p})z}{z^p(1-z)(1-qz)}$$
$$=\frac{1}{z^p}+\sum_{k=1-p}^{\infty}\Big[1+[\left(1-\lambda\frac{1-q^{-p}}{q-1}\right)\frac{1+Be^{i\theta}}{q^pb[1-q\frac{1-q^{-p}}{q-1}+(A-Bq\frac{1-q^{-p}}{q-1})e^{i\theta}]}-\frac{\lambda}{q^p}][k+p]_q\Big]z^k \tag{3.17}$$

and we may that easily check that (**??**) is equvalent to (3.15). □

3.2 Duality

In this section, we by using the definitions of the duality in[5], for a set $V\subset\mathbb{A}$,The dual set V, by $V^*$is defined as

$$V^*=\Big\{g\in\mathbb{A};\ \frac{1}{z}(f*g)(z)\neq 0\ for\ all f\in V\ and\ z\in\mathbb{U}\Big\}.$$

Now, for a set $W\subset\Sigma_p$, the dual W, denoted by W^*, is defined as

$$W^*=\Big\{g\in\Sigma_p;\ z^p(f*g)(z)\neq 0\ for\ all f\in W\ and\ z\in\mathbb{U}\Big\}.$$

The standard reference to duality for convolutions is the morograph by Rucheweyh [14], and his paper [13]. Assume that $f\in\Sigma_p$. By Theorem(3.1), $f\in\Sigma_p\mathcal{S}^*_{q,\lambda}[b;A,B]$ if and only if

$$z^2(f(z)*h_\theta(z))\neq 0,\qquad z\in\mathbb{U}^*, \tag{3.18}$$

where

$$h_\theta(z)=\frac{1+\left\{\left(1-\lambda\frac{1-q^{-p}}{q-1}\right)\frac{1+Be^{i\theta}}{q^pb[1-q\frac{1-q^{-p}}{q-1}+(A-Bq\frac{1-q^{-p}}{q-1})e^{i\theta}]}-(q+\frac{\lambda}{q^p})\right\}z}{z^p(1-z)(1-qz)} \tag{3.19}$$

and

$$M(\theta,p)=\frac{1+Be^{i\theta}}{q^pb[1-q\frac{1-q^{-p}}{q-1}+(A-Bq\frac{1-q^{-p}}{q-1})e^{i\theta}]}. \tag{3.20}$$

Moreover, for $f\in\Sigma_p$. By Theorem(3.1), $f\in\Sigma_p\mathcal{S}^*_{q,\lambda}[b;A,B]$ if and only if

$$z^2(f(z)*L_\sigma(z))\neq 0,\qquad z\in\mathbb{U}^*,$$

for all $z\in\mathbb{U}^*$ and $\theta\in[0,2\pi)$., where $M(\theta,p)$ is given by 1, where

$$L_\sigma(z)=\frac{1-\frac{1-q^{p+2}}{1-q^p}z+\left[\frac{q-q^p}{1-q^p}z+qz^2\frac{q^{p+1}-1}{1-q^p}\right]\left\{\left(1-\lambda\frac{1-q^{-p}}{q-1}\right)M(\theta,p)-(q+\frac{\lambda}{q^p})\right\}}{z^p(1-z)(1-qz)(1-q^2z)}.$$

Definition 3.4. We define W^* as follows:

$$\begin{aligned}W^*_\theta &= (\Sigma_p\mathcal{S}^*_{q,\lambda}[b;A,B])^*\\ &= \Big\{h_\theta(z)\in\Sigma_p;\ z^p(f(z)*h_\theta(z))(z)\neq 0\ ,\ f\in\Sigma_p\mathcal{S}^*_{q,\lambda}[b;A,B]\ ,\ \theta\in[0,2\pi)\Big\}.\end{aligned}$$

and

$$\begin{aligned}W^*_\zeta &= (\Sigma_p\mathcal{K}_{q,\lambda}[b;A,B])^*\\ &= \Big\{l_\zeta(z)\in\Sigma_p;\ z^p(f(z)*l_\zeta(z))(z)\neq 0\ ,\ f\in\Sigma_p\mathcal{K}_{q,\lambda}[b;A,B]\Big\}.\end{aligned}$$

Theorem 3.5. Let function $h_\theta(z)=\dfrac{1}{z^p}+\sum_{k=1-p}^{\infty}c_kz^k\in W^*_\theta$. The

$$\begin{aligned}|c_k| &\leq (1+q+q^2+\cdots+q^{k+p-1})\Big\{(1-\lambda\frac{1-q^{-p}}{q-1})\frac{1+B}{q^pb[1-q\frac{1-q^{-p}}{q-1}+(A-Bq\frac{1-q^{-p}}{q-1})]}\\ &\quad -(q+\frac{\lambda}{q^p})\Big\}-(1+q+q^2+\cdots+q^{k+p-1})\end{aligned}$$

Proof . Let $h_\theta\in W^*$. Then we have

$$\begin{aligned}h_\theta(z) &= \frac{1}{z^p(1-z)(1-qz)}+\frac{\Big\{(1-\lambda\frac{1-q^{-p}}{q-1})M(\theta,p)-(q+\frac{\lambda}{q^p})\Big\}z}{z^p(1-z)(1-qz)}\\ &= \frac{1}{z^p}(1+(1+q)z+(1+q+q^2)z^2+\cdots+q^{k+p-1})\\ &\quad +\frac{\Big\{(1-\lambda\frac{1-q^{-p}}{q-1})M(\theta,p)-(q+\frac{\lambda}{q^p})\Big\}}{z^{p-1}}(1+q+q^2)z^2+\cdots+q^{k+p-1})\\ &= \frac{1}{z^p}+\sum_{k=1-p}^{\infty}c_kz^k\end{aligned}$$

where

$$c_k = (1+q+q^2+\cdots+q^{k+p-1})\Big\{(1-\lambda\frac{1-q^{-p}}{q-1})M(\theta,p)-(q+\frac{\lambda}{q^p})\Big\}-(1+q+q^2+\cdots+q^{k+p-1})$$

and so

$$|c_k| \leq (1+q+q^2+\cdots+q^{k+p-1})\Big\{(1-\lambda\frac{1-q^{-p}}{q-1})M(\theta,p)-(q+\frac{\lambda}{q^p})\Big\}-(1+q+q^2+\cdots+q^{k+p-1})$$

where

$$M(\theta,p)=\frac{1+Be^{i\theta}}{q^pb[1-q\frac{1-q^{-p}}{q-1}+(A-Bq\frac{1-q^{-p}}{q-1})e^{i\theta}]}.$$

□

Corollary 3.6. Let $f(z) = \dfrac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k \in \Sigma_p$. if

$$\sum_{k=1-p}^{\infty} \Big[(1+q+q^2+\cdots+q^{k+p-1})\Big((1-\lambda\frac{1-q^{-p}}{q-1})|M(\theta,p)|-(q+\frac{\lambda}{q^p})\Big)+(q+\frac{\lambda}{q^p})\Big]$$
$$+(1+q+q^2+\cdots+q^{k+p})|c_k| \leq 1,$$

Then $f \in \Sigma_p \mathcal{K}_{q,\lambda}[b; A, B]$.

Proof . Let $h_\theta(z) = \dfrac{1}{z^p} + \sum_{k=1-p}^{\infty} c_k z^k \in W_\theta^*$. The we have

$$\begin{aligned} z^p|(f(z) * h_\theta(z))| &= |1+\sum_{k=1-p}^{\infty} a_k c_k z^k| \\ &\geq 1-\sum_{k=1-p}^{\infty} |a_k||c_k||z| \\ &> 1-\sum_{k=1-p}^{\infty} |a_k||c_k \\ &> 0. \end{aligned}$$

Thus $z^p(f(z) * h_\theta(z)) \neq 0$ and now form Theorem 2.1 we have $f \in \Sigma_p \mathcal{S}^*_{q,\lambda}[b; A, B]$. □

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