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Existence of solutions for stochastic functional integral equations via Petryshyn's fixed point theorem

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Abstract

The purpose of this paper is to analyze the solvability of a class of stochastic functional integral equations by utilizing the measure of non-compactness with Petryshyn's fixed point theorem in a Banach space. The results obtained in this paper cover numerous existing results concluded under some weaker conditions by many authors. An example is given to support our main theorem.

Keywords: Fixed point theorem, Measure of non-compactness (MNC), Functional integral equation (FIE)
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1 Introduction

Functional integral equations (FIEs) play an important role in expressing numerous scientific problems in mathematical form in applied analysis [1, 11, 25, 27]. Although there are several different numerical methods to discover the solutions of FIEs [4, 9, 24, 26], a solution method is still not desirable for most of them. Many authors have suggested different techniques for analysing the solvability of some integral equations [5, 14]. We also think on the approach established on the utilisation of Darbo's fixed point theorem in [6, 7] etc.

Such types of integral equations have also been solved via using Petryshyn's fixed point theorem (PFPT) in [3, 8, 15, 16, 17, 18, 29]. When integral equations are combined with stochastic ideas, it gives a new direction to this subject. It is necessary to confirm whether a solution exists or not for the problem of stochastic integral equations (SIEs). Stochastic integral equations and differential equations have been introduced and discussed in [12, 13, 25, 28, 31, 32]. Recently, many researchers have proposed different techniques to confirm the solvability of such equations.

Some techniques are discussed in [2, 21] by using Darbo's fixed point theorems. This paper deals with a method to examine the solvability of a stochastic FIE with the help of the MNC and PFPT.

In this paper, a method is proposed to investigate the existence of a solution of functional stochastic integral equations in which the measurement of inflexibility and the Petryshyn's fixed point theorem are used. Compared to

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the work of other researchers, in this paper, we try to provide a more general method for ensuring the existence of the solution of the stochastic FIE, which we think is as follows:

$$u(\varphi) = \left(F\Big(\varphi, u(\tau(\varphi)), u(\theta_1(\varphi)), \int_0^{\varphi} p(\varphi, s, u(\theta_2(s)))dB(s)\Big)\right) \times \left(G\Big(\varphi, u(\upsilon(\varphi)), u(\mu_1(\varphi)), \int_0^{a} q(\varphi, s, u(\mu_2(s)))dB(s)\Big)\right) \tag{1.1}$$

where $\varphi \in I_a = [0, a]$ and unknown function u, along with additional known functions are stochastic procedures specified on the probability space $(\Omega, \mathcal{F}, P)$. Furthermore, $B(\varphi)$ is the Brownian motion.

2 Preliminaries and notations

2.1 Stochastic calculus

Definition 2.1. [19] $B(\varphi)$(Brownian motion) is a stochastic approach, with the following notations:

1. For every $0 \le u_1 < u_2 < ... < u_n$, the increments $B(u_1), B(u_2) - B(u_1), ..., B(u_n) - B(u_{n-1})$ are freed of the way;
2. $B(u) - B(s)$ has a normal distribution with variance $u - s$ and mean 0;
3. $B(u)$, for $0 \le< u$ are continuous functions of u.

This description possesses established the start of motion from point u, and by counting the need for $P(B(0) = 0) = 0$, we get the standard Brownian motion description, in which motion begins at 0.

Definition 2.2. [19] An approach H is called adjusted to the filtration $\tilde{\mathcal{F}} = (\tilde{\mathcal{F}}_{\varphi})$, if for all φ, $H(\varphi)$ is $\tilde{\mathcal{F}}_{\varphi}$-measurable.

Theorem 2.1. [19] If H be a continuous adjusted approach, then the It$\hat{o}$ integral $\int_0^T H(\varphi)dB(\varphi)$ exists.

2.2 Preliminaries

In this manuscript, assume the following notations:

- $\mathcal{Q}$: A real Banach space;
- B_r: The open ball with center 0 and radius r;
- $\partial\bar{B}_r$: The sphere around 0 with radius $r > 0$ in $\mathcal{Q}$.

Definition 2.3. [20] Let J be a bounded subset of $\mathcal{Q}$, and assume that $\alpha(J)$ denotes the Kuratowski MNC of J, that is,

$$\alpha(J) = \inf\{\varsigma > 0 : J \text{ may be covered by finitely many sets by diameter } \le \varsigma\}. \tag{2.1}$$

Definition 2.4. [10] The Hausdorff MNC is:

$$\psi(J) = \inf\{\varsigma > 0 : \exists \text{ a finite } \varsigma\text{-net for } J \text{ in } \mathcal{Q}\}, \tag{2.2}$$

where by a finite ς-net for J in $\mathcal{Q}$ we indicate, as general, a set $\{u_1, u_2, ..., u_m\} \subset \mathcal{Q}$ such that $B_\kappa(\mathcal{Q}; u_1), B_\kappa(\mathcal{Q}; u_2)$ $, ..., B_\kappa(\mathcal{Q}; u_m)$ cover J. These MNC are mutually similar in the way

$$\psi(J) \le \alpha(J) \le 2\psi(J),$$

for any bounded set $J \subset \mathcal{Q}$.

Theorem 2.2. [23] Let $J, \hat{J} \in \mathcal{Q}$ be bounded and $\lambda \in \mathbb{R}$. Then

1. $\psi(J \cup \hat{J}) = \max\{\psi(J), \psi(\hat{J})\}$,
2. $\psi(J + \hat{J}) \le \psi(J) + \psi(\hat{J})$,
3. $\psi(\lambda J) =\mid \lambda \mid \psi(J)$,
4. $\psi(J) \le \psi(\hat{J})$ where $J \subseteq \hat{J}$,
5. $\psi(\bar{J}) = \psi(Conv J) = \psi(J)$,

6. $\psi(J) = 0$ if and only if J is pre-compact.

Here, we will consider a MNC in the space $C[0, a]$ consisting of all continuous real-valued functions on $[0, a]$. The space $C[0, a]$ is equipped with the usual norm

$$\|u\| = \sup\{|u(\varphi)| : \varphi \in [0, a]\}.$$

The modulus of the continuity for $u \in C[0, a]$ is defined by

$$\jmath(u, \varsigma) = \sup\{|u(\varphi) - u(\varphi')| : |\varphi - \varphi'| \leq \varsigma\}.$$

Theorem 2.3. [10] The MNC (2.2) is equivalent to

$$\psi(J) = \lim_{\varsigma \to 0} \sup_{u \in J} \jmath(u, \varsigma), \tag{2.3}$$

for all bounded set $J \subset C[0, a]$.

Definition 2.5. [22] A continuous function $K : \mathcal{Q} \to \mathcal{Q}$ on a Banach space $\mathcal{Q}$ is called a $\hat{k}$-set contraction if for all $J \subset \mathcal{Q}$, $K(J)$ is bounded where J is bounded, and $\alpha(KJ) \leq \hat{k}\alpha(J), 0 < \hat{k} < 1$. If

$$\alpha(KJ) < \alpha(J), \text{ for all } J \subseteq \mathcal{Q},$$

then K is called a condensing (or a densifying) map.

Theorem 2.4. PFPT[23], see also [30] Let B_r be an open ball centered at the origin in $\mathcal{Q}$. If $K : B_r \to \mathcal{Q}$ is a densifying map that fulfill the boundary condition,

$$\text{If } K(u) = \hat{k}u, \text{ for some } u \text{ in } \partial B_r, \text{ then } \hat{k} \leq 1, \tag{P}$$

then, the set of fixed points of K in B_r is non-empty.

3 Main Results

In this section, we analyse the solvability of the Eq. (1.1) under the following hypothesis:

(C1) $u \in C(I_a, \mathbb{R}), F, G \in C(I_a \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and $p \in C(I_a \times I_a \times \mathbb{R}, \mathbb{R}), q \in C(I_a \times I_a \times \mathbb{R}, \mathbb{R})$. Also, τ, υ and $\theta_i, \mu_i : I_a \to I_a, i = 1, 2$ are continuous for each $\varphi \in I_a$.

(C2) There exist non-negative constants k, k' with $2k, 2k' < 1$ such that

$$|F(\varphi, u_1, u_2, u_3) - F(\varphi, \bar{u}_1, \bar{u}_2, \bar{u}_3)| \leq k(|u_1 - \bar{u}_1| + |u_2 - \bar{u}_2| + |u_3 - \bar{u}_3|),$$

$$|G(\varphi, u_1, u_2, u_3) - G(\varphi, \bar{u}_1, \bar{u}_2, \bar{u}_3)| \leq k'(|u_1 - \bar{u}_1| + |u_2 - \bar{u}_2| + |u_3 - \bar{u}_3|).$$

(C3) There exists $r_0 \geq 0$ so that $\sup\{(L_1) \times (L_2)\} \leq r_0$, where,

$$\begin{aligned}
&L_1 = \sup\{|F(\varphi, u_1, u_2, u_3)| : \text{for all } \varphi \in I_a, \text{ and } u_1, u_2 \in [-r_0, r_0], \ |u_3| \leq M\lambda\},\\
&M = \sup\{|p(\varphi, s, u)| : \text{for all } \varphi, s \in I_a, \quad \text{and} \quad u \in [-r_0, r_0]\},\\
&L_2 = \sup\{|G(\varphi, u_1, u_2, u_3)| : \text{for all } \varphi \in I_a, \text{ and } u_1, u_2 \in [-r_0, r_0], \ |u_3| \leq N\lambda\},\\
&N = \sup\{|q(\varphi, s, u)| : \text{for all } \varphi, s \in I_a, \text{ and } u \in [-r_0, r_0]\},\\
&\lambda = \sup\{|B(\varphi)| : \text{for all } \varphi \in I_a\}.
\end{aligned}$$

Theorem 3.1. Under the conditions (C1)-(C3), Eq. (1.1) possesses at least one solution in $\mathcal{Q} = C(I_a)$.

Proof . To show this result, we utilize the Theorem 2.4 as our general concept. Define $P, Q : B_{r_0} \to \mathcal{Q}$ and K as $Ku = (Pu) \times (Qu)$, where

$$Pu(\varphi) = \left(F\left(\varphi, u(\tau(\varphi)), u(\theta_1(\varphi)), \int_0^\varphi p(\varphi, s, u(\theta_2(s)))dB(s)\right)\right), \tag{3.1}$$

and

$$Qu(\varphi) = \Big(G\Big(\varphi, u(v(\varphi)), u(\mu_1(\varphi)), \int_0^a q(\varphi, s, u(\mu_2(s)))dB(s)\Big)\Big), \tag{3.2}$$

for all $\varphi \in I_a$. Now, we prove P is continuous on B_{r_0}. Take $\varsigma > 0$ and arbitrary $u, v \in B_{r_0}$ such that $\| u - v \| \leq \varsigma$. Then for $\varphi \in I_a$, we obtain

$$\begin{aligned}
&|(Pu)(\varphi) - (Pv)(\varphi)| \\
=&\Big|\Big(F\Big(\varphi, u(\tau(\varphi)), u(\theta_1(\varphi)), \int_0^\varphi p(t, s, u(\theta_2(s)))dB(s)\Big)\Big) - \Big(F\Big(\varphi, v(\tau(\varphi)), v(\theta_1(\varphi)), \int_0^\varphi p(\varphi, s, y(\theta_2(s)))dB(s)\Big)\Big)\Big| \\
\leq& k|u(\tau(\varphi)) - v(\tau(\varphi))| + k|u(\theta_1(\varphi))) - v(\theta_1(\varphi))| + k\int_0^\varphi |p(\varphi, s, u(\theta_2(s))) - p(\varphi, s, v(\theta_2(s)))|dB(s).
\end{aligned}$$

So,

$$|(Pu)(\varphi) - (Pv)(\varphi)| \leq (2k)\|u - v\| + k\lambda\jmath(p, \varsigma).$$

Similarly, we have

$$\begin{aligned}
&|(Qu)(\varphi) - (Qv)(\varphi)| \\
=&\Big|\Big(G\Big(\varphi, u(v(\varphi)), u(\mu_1(\varphi)), \int_0^a q(\varphi, s, u(\mu_2(s)))dB(s)\Big)\Big) - \Big(G\Big(\varphi, v(v(\varphi)), v(\mu_1(\varphi)), \int_0^a q(\varphi, s, y(\mu_2(s)))dB(s)\Big)\Big)\Big| \\
\leq&(2k')\|u - v\| + k'\lambda\jmath(q, \varsigma),
\end{aligned}$$

where for all $\varsigma > 0$ we define

$$\begin{aligned}
\jmath(p, \varsigma) &= \sup\{|p(\varphi, s, u) - p(\varphi, s, \bar{u})| : \varphi \in I_a, s \in [0, a], u, \bar{u} \in [-r_0, r_0], \|u - \bar{u}\| \leq \varsigma\}, \\
\jmath(q, \varsigma) &= \sup\{|q(\varphi, s, u) - q(\varphi, s, \bar{u})| : \varphi \in I_a, s \in [0, a], u, \bar{u} \in [-r_0, r_0], \|u - \bar{u}\| \leq \varsigma\}.
\end{aligned}$$

Since $p(\varphi, s, u)$ and $q(\varphi, s, u)$ are uniformly continuous on $[0, a] \times [0, a] \times \mathbb{R}$, we obtain $\jmath(p, \varsigma) \to 0$ and $\jmath(q, \varsigma) \to 0$ as $\varsigma \to 0$. So, P and Q are continuous on B_{r_0}. Hence, K is also continuous on B_{r_0}.

Now, we show that P and Q fulfill the condensing condition with respect to ψ in B_{r_0}. Select an arbitrary $\varsigma > 0$. Taking $u \in J$, where J is a bounded subset of $\mathcal{Q}$. Further, for $\varphi_1, \varphi_2 \in I_a$, we can have $\varphi_1 \leq \varphi_2$ with $\varphi_2 - \varphi_1 \leq \varsigma$. We get

$$\begin{aligned}
|(Pu)(\varphi_2) - (Pu)(\varphi_1)| =&\Big|\Big(F\Big(\varphi_2, u(\tau(\varphi_2)), u(\theta_1(\varphi_2)), \int_0^{\varphi_2} p(\varphi_2, s, u(\theta_2(s)))dB(s)\Big)\Big) \\
&- \Big(F\Big(\varphi_1, u(\tau(\varphi_1))), u(\theta_1(\varphi_1)), \int_0^{\varphi_1} p(\varphi_1, s, u(\theta_2(s)))dB(s)\Big)\Big)\Big| \\
\leq&\Big|F\Big(\varphi_2, u(\tau(\varphi_2)), u(\theta_1(\varphi_2)), \int_0^{\varphi_2} p(\varphi_2, s, u(\theta_2(s)))dB(s)\Big) \\
&- F\Big(\varphi_2, u(\tau(\varphi_1))), u(\theta_1(\varphi_1)), \int_0^{\varphi_1} p(\varphi_1, s, u(\theta_2(s)))dB(s)\Big)\Big| \\
&+ \Big|F\Big(\varphi_2, u(\tau(\varphi_1)), u(\theta_1(\varphi_1)), \int_0^{\varphi_1} p(\varphi_1, s, u(\theta_2(s)))dB(s)\Big) \\
&- F\Big(\varphi_1, u(\tau(\varphi_1)), u(\theta_1(\varphi_1)), \int_0^{\varphi_1} p(\varphi_1, s, u(\theta_2(s)))dB(s)\Big)\Big| \\
\leq& k|u(\tau(\varphi_2)) - u(\tau(\varphi_1))| + k|u(\theta_1(\varphi_2)) - u(\theta_1(\varphi_1))| \\
&+ k\Big|\int_0^{\varphi_2} p(\varphi_2, s, u(\theta_2(s)))dB(s) - \int_0^{\varphi_1} p(\varphi_1, s, u(\theta_2(s)))dB(s)\Big| + \jmath^1_{r_0}(F, \varsigma) \\
\leq& k\jmath(u, \jmath(\tau, \varsigma)) + k\jmath(u, \jmath(\theta_1, \varsigma)) + k\int_0^{\varphi_1} |p(\varphi_2, s, u(\theta_2(s))) - p(\varphi_1, s, u(\theta_2(s)))|dB(s) \\
&+ k\int_{\varphi_1}^{\varphi_2} |p(\varphi_2, s, u(\theta_2(s)))|dB(s) + \jmath^1_{r_0}(F, \varsigma) \\
\leq& k\jmath(u, \jmath(\tau, \varsigma)) + k\jmath(u, \jmath(\theta_1, \varsigma)) + k\lambda\jmath^1_{r_0}(p, \varsigma) + kM\jmath(B, \varsigma) + \jmath^1_{r_0}(F, \varsigma).
\end{aligned}$$

Similarly,

$$\begin{aligned}|(Qu)(\varphi_2)-(Qu)(\varphi_1)| =&\Bigg|\Bigg(G\Big(\varphi_2, u(\upsilon(\varphi_2))), u(\mu_1(\varphi_2))), \int_0^{\varphi_2} q_2(\varphi_2, s, u(\mu_2(s)))dB(s)\Big)\Bigg)\\ &-\Bigg(G\Big(\varphi_1, u(\upsilon(\varphi_1))), u(\mu_1(\varphi_1)), \int_0^{\varphi_1} q_2(\varphi_1, s, u(\mu_2(s)))dB(s)\Big)\Bigg)\Bigg|\\ \le& k'\jmath(u,\jmath(\upsilon,\varsigma))+k'\jmath(u,\jmath(\mu_1,\varsigma))+k'\lambda\jmath^1_{r_0}(q,\varsigma)+k'N\jmath(B,\varsigma)+\jmath^1_{r_0}(G,\varsigma),\end{aligned}$$

where:

$$\begin{aligned}&\jmath^1_{r_0}(p,\varsigma)=\sup\{|p(\varphi,s,u)-p(\bar\varphi,s,u)| : |\varphi-\bar\varphi|\le\varsigma, \varphi,\bar\varphi,s\in I_a,\ u\in[-r_0,r_0]\},\\ &\jmath^1_{r_0}(q,\varsigma)=\sup\{|q(\varphi,s,u)-q(\bar\varphi,s,u)| : |\varphi-\bar\varphi|\le\varsigma, \varphi,\bar\varphi,s\in I_a, u\in[-r_0,r_0]\},\\ &\jmath^1_{r_0}(F,\varsigma)=\sup\{F(\varphi,u_1,u_2,u_3)-F(\bar\varphi,u_1,u_2,u_3)| : |\varphi-\bar\varphi|\le\varsigma, \varphi,\bar\varphi\in I_a,\ u_1,u_2\in[-r_0,r_0], |u_3|\le M\lambda\}\\ &\jmath^1_{r_0}(G,\varsigma)=\sup\{G(\varphi,u_1,u_2,u_3)-G(\bar\varphi,u_1,u_2,u_3)| : |\varphi-\bar\varphi|\le\varsigma, \varphi,\bar\varphi\in I_a,\ u_1,u_2\in[-r_0,r_0], |u_3|\le N\lambda\}\\ &\jmath(B,\varsigma)=\sup\{|B(\varphi)-B(\bar\varphi)| : |\varphi-\bar\varphi|\le\varsigma, \varphi,\bar\varphi\in[-r_0,r_0]\}.\end{aligned}$$

From the above relations, we have

$$\jmath(Pu,\varsigma)\le k\jmath(u,\jmath(\tau,\varsigma))+k\jmath(u,\jmath(\theta_1,\varsigma))+k\lambda\jmath^1_{r_0}(p,\varsigma)+kM\jmath(B,\jmath(\varphi,\varsigma))+\jmath^1_{r_0}(F,\varsigma)$$

and

$$\jmath(Qu,\varsigma)\le k'\jmath(u,\jmath(\upsilon,\varsigma))+k'\jmath(u,\jmath(\mu_1,\varsigma))+k'\lambda\jmath^1_{r_0}(q,\varsigma)+k'N\jmath(B,\jmath(\varphi,\varsigma))+\jmath^1_{r_0}(G,\varsigma).$$

Applying limit as $\varsigma\to 0$, we have

$$\psi(PJ)\le(2k)\psi(J).$$

Also,

$$\psi(QJ)\le(2k')\psi(J).$$

Hence, K is a condensing map. Suppose that $u\in\partial\bar B_{r_0}$. If $Ku=\hat k x$, then $\hat k r_0=\hat k\|x\|=\|Ku\|$ and by (C3) we obtain

$$\begin{aligned}|Ku(\varphi)| =& \Bigg|\Bigg(F\Big(\varphi, u(\tau(\varphi)), u(\theta_1(\varphi)), \int_0^{\varphi} p(\varphi, s, u(\theta_2(s)))dB(s)\Big)\Bigg)\\ &\times\Bigg(G\Big(\varphi, \varphi(\upsilon(\varphi)), u(\mu_1(\varphi)), \int_0^{a} q(\varphi, s, u(\mu_2(s)))dB(s)\Big)\Bigg)\Bigg|\le r_0,\end{aligned}$$

for all $\varphi\in I_a$. Hence, $\|Ku\|\le r_0$, so this shows $\hat k\le 1$. The proof is complete. □

Corollary 3.2. [21] Let

- (K1) $F,G\in C([0,a]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and there exists the constant $h>0$ such that $|F(\varphi,0,0)|\le h$, $|G(\varphi,0,0)|\le h$.
- (K2) For some continuous functions $a_1,a_2,a_3,a_4:[0,a]\to[0,a]$, such that $|F(\varphi,u_1,u_2)-F(\varphi,\bar u_1,\bar u_2)|\le a_1(\varphi)|u_1-\bar u_1|+a_2(\varphi)|u_2-\bar u_2|$, $|G(\varphi,u_1,u_2)-G(\varphi,\bar u_1,\bar u_2)|\le a_3(\varphi)|u_1-\bar u_1|+a_4(\varphi)|u_2-\bar u_2|$.
- (K3) there exists the constant $k>0$ such that for all $\varphi\in[0,a]$, $a_1(\varphi),a_2(\varphi),a_3(\varphi),a_4(\varphi)\le k$.
- (K4) $p,q\in C([0,1]\times[0,1]\times\mathbb{R},\mathbb{R})$ and there exist constants $\gamma_1,\gamma_2>0$ such that $|p(\varphi,s,u)|\le\gamma_1+\gamma_2|u|$, $|q(\varphi,s,u)|\le\gamma_1+\gamma_2|u|$, for all $\varphi,s\in[0,1]$ and $u\in\mathbb{R}$.
- (K5) $4\zeta\eta<1$, where $\zeta=k\gamma_1\beta+h$ and $\eta=k(\gamma_2\beta+1)$ and $\beta=\sup\{B(\varphi);\varphi\in[0,a]\}$.

Then

$$u(\varphi)=\Big(F\big(\varphi,u(\tau(\varphi)),\int_0^{\varphi} p(\varphi,s,u(\theta_2(s)))dB(s)\big)\Big)\times\Big(G\big(v,u(\upsilon(\varphi)),\int_0^{a} q(\varphi,s,u(\mu_2(s)))dB(s)\big)\Big), \tag{3.3}$$

possesses at least one solution in $\mathcal{Q}=C(I_a)$.

Proof . It can be proved that if $F(\varphi,u_1,u_2,u_3)=F(\varphi,u_1,u_3)$ and $G(\varphi,u_1,u_2,u_3)=G(\varphi,u_1,u_3)$, then Eq. 1.1 will be the Eq. 3.3. We check that (C2) is completed by (K3) and (K4). Now, we prove that (C3) is also satisfied. Let $\|u\|\le r_0$, $r_0>0$ and putting $M=N=\gamma_1+\gamma_2 r_0$, then

$$\begin{aligned}
|u(\varphi)| &= \Big|F\big(\varphi,u(\tau(\varphi)),\int_0^{\varphi} p(\varphi,s,u(\theta_1(s)))dB(s)\big)\times G\big(\varphi,u(\upsilon(\varphi)),\int_0^{a} q(\varphi,s,u(\mu_2(s)))dB(s)\big)\Big|\\
&\le \Big(\Big|F\big(\varphi,u(\tau(\varphi)),\int_0^{\varphi} p(\varphi,s,u(\theta_1(s)))dB(s)\big)\Big|-|F(\varphi,0,0)|+|F(\varphi,0,0)|\Big)\\
&\times \Big(\Big|G\big(\varphi,u(\upsilon(\varphi)),\int_0^{a} q(\varphi,s,u(\mu_2(s)))dB(s)\big)\Big|-|G(\varphi,0,0)|+|G(\varphi,0,0)|\Big)\\
&\le \Big(a_1(\varphi)|u(\tau(\varphi))|+a_2(\varphi)\int_0^{\varphi}|p(\varphi,s,u(\theta_1(s)))|dB(s)+l\Big)\\
&\times \Big(a_3(\varphi)|u(\upsilon(\varphi))|+a_4(\varphi)\int_0^{1}|q(t,s,u(\mu_2(s)))|dB(s)+l\Big)\\
&\le \big(k\|u\|+kM\eta+h\big).\big(k\|u\|+kN\eta+h\big)\\
&\le \big((k+k\gamma_2\beta)\|u\|+k\gamma_1\beta+h\big)^2\\
&\le \big(\eta\|u\|+\zeta\big)^2
\end{aligned}$$

for all $\varphi\in I_a$. Hence, r_0 in (C3) is the real number that fulfills in the following conditions

$$\sup_{\varphi\in I_a}|u(\varphi)|\le\big(\eta r_0+\zeta\big)^2\le r_0. \tag{3.4}$$

The inequality (3.4), possesses a solution in $[r_1,r_2]$, where

$$r_1=\frac{1-2\eta\zeta-\sqrt{1-4\eta\zeta'}}{2\eta^2}$$

and

$$r_2=\frac{1-2\eta\zeta+\sqrt{1-4\eta\zeta}}{2\eta'^2}.$$

Under the (K5), $1-\sqrt{1-4\eta\zeta}<1$. So, $r_0=r_1$ is a positive real number. Now, the whole result got from Theorem 3.1. □

Corollary 3.3. Let

(D1) $f\in C(I_a\times\mathbb{R},\mathbb{R}),F\in C(I_a\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, and there exists constants $\hat{\mu},\hat{\gamma},\hat{\kappa}$ such that $|f(\varphi,0)|\le\hat{\mu}$, $|F(\varphi,u(\tau(\varphi)),0,0|\le\hat{\gamma}+\hat{\kappa}|u(\varphi)|$,

(D2) there exist the functions $a_1,a_2,a_3:I_a\to I_a$ such that $|f(\varphi,u)-f(\varphi,\bar{u})|\le a_1(\varphi)|u-\bar{u}|$, $|F(\varphi,u_2,u_1)-F(\varphi,\bar{u}_2,u_1)|\le a_2(\varphi)|u_2-\bar{u}_2|$, $|F(\varphi,u_2,u_1)-F(\varphi,u_2,\bar{u}_1)|\le a_3(\varphi)|u_1-\bar{u}_1|$, for all $u_1,\bar{u}_1,u_2,\bar{u}_2\in\mathbb{R}$,, $\varphi\in I_a$ and let $k=\max\{|a_i(\varphi)|:\varphi\in I_a,i=1,2,3\}$.

(D3) (Sub-linear condition) $p\in C([0,a]\times[0,a]\times\mathbb{R},\mathbb{R})$ be a continuous derivative function and fulfills in sub-linear condition, so that there exist the constants γ_1 and γ_2 such that $|p(\varphi,s,u)|\le\gamma_1+\gamma_2|u|$, for all $\varphi,s\in[0,a]$ and $u\in\mathbb{R}$.

(D4) $k<\frac{1-\hat{\kappa}}{2(1+\lambda(\gamma_2))}$ and $\kappa=\sup\{B(\varphi);\varphi\in[0,a]\}$.

Then

$$u(\varphi)=f\big(\varphi,u(\tau(\varphi))\big)+F\big(\varphi,u(\theta_1(\varphi)),\int_0^{\varphi}p(\varphi,s,u(\theta_2(s)))dB(s)\big),\qquad \varphi\in I_a$$

possesses at least one solution in $\mathcal{Q}=C(I_a)$.

Proof . Let $r_0=\frac{R_2}{1-R_1}$ where

$$R_1=k+\hat{\kappa}+k\kappa(\gamma_2)\ \text{and}\ R_2=\hat{\mu}+\hat{\gamma}+k\kappa(\gamma_1).$$

From (D4), $R_1 = k + \hat{\kappa} + k\kappa\gamma_2 < 1 - (k + k\kappa\gamma_2) < 1$. So, r_0 is a positive real number. In addition, we check that (C2) is finished from (D2) and (D4). Now we prove that (C3) is also satisfied. Putting $M = \gamma_1 + \gamma_2 r_0$, then

$$\begin{aligned}
|u(\varphi)| =& \Big|f\big(\varphi, u(\tau(\varphi))\big) + F\Big(\varphi, u(\theta_1(\varphi)), \int_0^{\varphi} p(\varphi, s, u(\theta_2(s)))dB(s)\Big)\Big| \\
\le& \big|f\big(\varphi, u(\tau(\varphi))\big) - f(\varphi, 0)\big| + |f(\varphi,0)| + \Big|F\Big(\varphi, u(\theta_1(\varphi)), \int_0^{\varphi} p(\varphi, s, u(\theta_2(s)))dB(s)\Big) \\
& - F\big(\varphi, u(\theta_1(\varphi)), 0\big)\Big| + \big|F\big(\varphi, u(\theta_1(\varphi)), 0\big)\big| \\
\le& k|\varphi| + \hat{\mu} + k\kappa(\gamma_1 + \gamma_2|u|) + \hat{\gamma} + \hat{\kappa}|u| \\
\le& (k + \hat{\kappa} + k\kappa(\gamma_2))|u| + \hat{\mu} + \hat{\gamma} + k\kappa(\gamma_1),
\end{aligned}$$

for all $\varphi \in I_a$. Consequently,

$$\sup_{\varphi \in I_a} |u(\varphi)| \le R_1 r_0 + R_2 = L_1 \frac{R_2}{1 - R_1} + R_2 = r_0.$$

Now, we get the complete result from Theorem 3.1. □

4 Examples

As applications and to establish the efficiency of the presented approach, an example is discussed in this section.

Example 4.1. Let the following stochastic FIE.

$$\begin{aligned}
u(\varphi) =& \Big(\frac{1}{7}\cos(\frac{e^{-\sqrt{\varphi}}}{3+\varphi^2}) + \frac{1}{5}\frac{\sqrt{\varphi}}{1+\varphi+\varphi^2}\sin(u(\varphi) + \frac{1}{9}\int_0^1 \frac{|u(s)|}{4+|u(s)|}dB(s)\Big) \\
& \times \Big(\frac{1}{5}\cos(\frac{\varphi^2}{4}) + \frac{e^{-\varphi}}{2+2\varphi}\cos(u(\varphi) + \frac{1}{3}\int_0^1 \cos(\frac{|u(s)|}{5+|u(s)|})dB(s)\Big), \qquad \varphi \in [0,1].
\end{aligned} \tag{4.1}$$

Here,

$$\theta_1(\varphi) = \theta_2(\varphi) = \mu_1(\varphi) = \mu_2(\varphi) = \varphi, \quad \text{for all} \quad \varphi \in [0,1],$$

$$\begin{aligned}
& F(\varphi, u_1, u_2, u_3) = \frac{1}{7}\cos(\frac{e^{-\sqrt{\varphi}}}{3+\varphi^2}) + \frac{u_1}{5} + 0u_2 + \frac{u_3}{9}, \quad u_1 = \frac{\sqrt{\varphi}}{1+\varphi+\varphi^2}\sin(u(\varphi), \quad u_3 = \int_0^1 \frac{|u(s)|}{4+|u(s)|}dB(s) \\
& G(\varphi, u_1, u_2, u_3) = \frac{1}{5}\cos(\frac{\varphi^2}{4}) + \frac{u_1}{2} + 0u_2 + \frac{u_3}{3}, \quad u_3 = \int_0^1 \cos(\frac{|\varphi(s)|}{5+|\varphi(s)|})dB(s), \\
& p(\varphi, s, u) = \frac{|u(s)|}{4+|u(s)|}, \quad |p(\varphi, s, u)| \le 1 \\
& q(\varphi, s, u) = \cos(\frac{|u(s)|}{5+|u(s)|}), \quad |q(\varphi, s, u)| \le 1
\end{aligned}$$

for all $\varphi \in [0,1]$. Observe that the assumptions (C1) and (C2) are fulfilled. It is easy to see that (C3) also fulfill. Taking $r_0 = (\frac{12}{35} + \frac{1}{9}\lambda)(\frac{7}{10} + \frac{1}{3}\lambda)$, then we get $M \le 1$, $N \le 1$ and

$$\begin{aligned}
\sup \Big| & \Big(\frac{1}{7}\cos(\frac{e^{-\sqrt{\varphi}}}{3+v^2}) + \frac{1}{5}\frac{\sqrt{\varphi}}{1+\varphi+\varphi^2}\sin(u(\varphi) + \frac{1}{9}\int_0^{\varphi} \frac{|u(s)|}{4+|u(s)|}dB(s)\Big) \\
\times & \Big(\frac{1}{5}\cos(\frac{u^2}{4}) + \frac{e^{-u}}{2+2u}\cos(u(\varphi) + \frac{1}{3}\int_0^1 \cos(\frac{|u(s)|}{5+|u(s)|})dB(s)\Big)\Big| \le r_0.
\end{aligned}$$

for all $\varphi \in [0,1]$. Hence (C3) holds if,

$$(\frac{12}{35} + \frac{1}{9}\lambda)(\frac{7}{10} + \frac{1}{3}\lambda) \le r_0. \tag{4.2}$$

This shows that

$$r_0 = (\frac{12}{35} + \frac{1}{9}\lambda)(\frac{7}{10} + \frac{1}{3}\lambda)$$

and $\lambda = \sup\{|B(\varphi)|; \forall \varphi \in [0,1]\}$. Hence, from Theorem 3.1 equation (4.1) has at least one random solution in Banach space $[0,1]$.

Example 4.2. Let the following stochastic FIE.

$$\begin{aligned} u(\varphi) = &\left(\frac{e^{-2\varphi}}{3(1+\varphi)} + \frac{|u(\varphi)|\sin(\varphi)}{1+|u(\varphi)|} + \frac{\varphi^2}{2(1+\varphi^2)}\int_0^t \frac{s^2\sqrt{|u(s)|}}{1+s^2}dB(s)\right) \\ &\times \left(\frac{1}{6}\varphi\sin(\frac{u(\varphi^3)}{1+\varphi}) + \frac{\varphi^2 u(1-\varphi)}{2+2u(1-\varphi)} + \frac{1}{7+\varphi}\int_0^1 \sqrt{\ln(1+|u(s)|)}dB(s)\right), \qquad \varphi \in [0,1]. \end{aligned} \tag{4.3}$$

Here,

$$\tau(\varphi) = \theta_2(\varphi) = \mu_2(\varphi) = \varphi, \upsilon(\varphi) = \varphi^3, \mu_1(\varphi) = 1-\varphi \quad \text{for all} \quad \varphi \in [0,1],$$

$$\begin{aligned} F(\varphi,u_1,u_2,u_3) &= \frac{e^{-2\varphi}}{3(1+\varphi)} + \frac{|u_1|\sin(\varphi)}{1+|u_1|} + \frac{\varphi^2}{2(1+\varphi^2)}u_3, \quad u_3 = \int_0^t \frac{s^2\sqrt{|u(s)|}}{1+s^2}dB(s) \\ G(\varphi,u_1,u_2,u_3) &= \frac{1}{6}\varphi\sin(\frac{u_1}{1+\varphi}) + \frac{\varphi^2 u_2}{2+2u_2} + \frac{1}{7+\varphi}u_3, \quad u_3 = \int_0^1 \sqrt{\ln(1+|u(s)|)}dB(s), \\ p(\varphi,s,u) &= \frac{s^2\sqrt{|u|}}{1+s^2}, \quad |p(\varphi,s,u)| \le \sqrt{|u|} \\ q(\varphi,s,u) &= \sqrt{\ln(1+|u|)}, \quad |q(\varphi,s,u)| \le \sqrt{|u|} \end{aligned}$$

Now, we can see that these functions satisfy the assumptions (C1) and (C2). We check that (C3) also holds. Suppose that $\|u\| \le r_0, r_0 > 0$, then we have

$$\begin{aligned} |u(\varphi)| = |&\Big(\frac{e^{-2\varphi}}{3(1+\varphi)} + \frac{|u(\varphi)|\sin(\varphi)}{1+|u(\varphi)|} + \frac{\varphi^2}{2(1+\varphi^2)}\int_0^t \frac{s^2\sqrt{|u(s)|}}{1+s^2}dB(s)\Big) \\ &\times \Big(\frac{1}{6}\varphi\sin(\frac{u(\varphi^3)}{1+\varphi}) + \frac{\varphi^2 u(1-\varphi)}{2+2u(1-\varphi)} + \frac{1}{7+\varphi}\int_0^1 \sqrt{\ln(1+|u(s)|)}dB(s)\Big)| \le r_0, \end{aligned} \tag{4.4}$$

for all $\varphi \in [0,1]$. Hence (C3) holds if,

$$(\frac{4}{3} + \frac{1}{2}\sqrt{r_0}\lambda)(\frac{2}{3} + \frac{1}{7}\sqrt{r_0}\lambda) \le r_0. \tag{4.5}$$

This shows that

$$r_0 = \frac{1}{9}\frac{(11\lambda + \sqrt{9\lambda^2 + 1568})^2}{(\lambda^2 - 14)^2}$$

and $\lambda = \sup\{|B(\varphi)|; \forall\varphi \in [0,1]\}$. Hence, from Theorem 3.1 equation (4.3) has at least one random solution in Banach space $[0,1]$.

5 Conclusions

The significance of the existence of solutions is one of the problems of researchers in the study of functional stochastic integral equations. So far, various methods have been developed for this purpose.This paper is based on an additional general form of the functional stochastic integral equation, which also covers some other equivalent works. In the presented method, Petryshyn's fixed point theorem and the idea of MNC with weaker conditions are used.

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