Int. J. Nonlinear Anal. Appl. In Press, 1–10
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.30206.4364

Existence of solutions for stochastic functional integral equations via Petryshyn's fixed point theorem

Ketki Singh[a], Harindri Chaudhary[a], Soniya Singh[b,*]

[a] *Department of Mathematics, Deshbandhu College, University of Delhi, New Delhi, India*

[b] *Department of Applied Mathematics and Scientific Computing, IIT Roorkee, Roorkee-247667, India*

(Communicated by Hamid Khodaei)

Abstract

The purpose of this paper is to analyze the solvability of a class of stochastic functional integral equations by utilizing the measure of non-compactness with Petryshyn's fixed point theorem in a Banach space. The results obtained in this paper cover numerous existing results concluded under some weaker conditions by many authors. An example is given to support our main theorem.

Keywords: Fixed point theorem, Measure of non-compactness (MNC), Functional integral equation (FIE)
2020 MSC: 35K90, 47H10

1 Introduction

Functional integral equations (FIEs) play an important role in expressing numerous scientific problems in mathematical form in applied analysis [1, 11, 25, 27]. Although there are several different numerical methods to discover the solutions of FIEs [4, 9, 24, 26], a solution method is still not desirable for most of them. Many authors have suggested different techniques for analysing the solvability of some integral equations [5, 14]. We also think on the approach established on the utilisation of Darbo's fixed point theorem in [6, 7] etc.

Such types of integral equations have also been solved via using Petryshyn's fixed point theorem (PFPT) in [3, 8, 15, 16, 17, 18, 29]. When integral equations are combined with stochastic ideas, it gives a new direction to this subject. It is necessary to confirm whether a solution exists or not for the problem of stochastic integral equations (SIEs). Stochastic integral equations and differential equations have been introduced and discussed in [12, 13, 25, 28, 31, 32]. Recently, many researchers have proposed different techniques to confirm the solvability of such equations.

Some techniques are discussed in [2, 21] by using Darbo's fixed point theorems. This paper deals with a method to examine the solvability of a stochastic FIE with the help of the MNC and PFPT.

In this paper, a method is proposed to investigate the existence of a solution of functional stochastic integral equations in which the measurement of inflexibility and the Petryshyn's fixed point theorem are used. Compared to

*Corresponding author
Email addresses: ketkisingh007@gmail.com (Ketki Singh), hchaudhary@db.du.ac.in (Harindri Chaudhary), sonia.iitd.21@gmail.com (Soniya Singh)

Received: March 2023 *Accepted:* May 2023

the work of other researchers, in this paper, we try to provide a more general method for ensuring the existence of the solution of the stochastic FIE, which we think is as follows:

$$u(\varphi) = \Big(F\Big(\varphi, u(\tau(\varphi)), u(\theta_1(\varphi)), \int_0^{\varphi} p(\varphi, s, u(\theta_2(s)))dB(s)\Big)\Big) \times \Big(G\Big(\varphi, u(\upsilon(\varphi)), u(\mu_1(\varphi)), \int_0^{a} q(\varphi, s, u(\mu_2(s)))dB(s)\Big)\Big) \tag{1.1}$$

where $\varphi \in I_a = [0, a]$ and unknown function u, along with additional known functions are stochastic procedures specified on the probability space $(\Omega, \digamma, P)$. Furthermore, $B(\varphi)$ is the Brownian motion.

2 Preliminaries and notations

2.1 Stochastic calculus

Definition 2.1. [19] $B(\varphi)$(Brownian motion) is a stochastic approach, with the following notations:

1. For every $0 \leq u_1 < u_2 < ... < u_n$, the increments $B(u_1), B(u_2) - B(u_1), ..., B(u_n) - B(u_{n-1})$ are freed of the way;
2. $B(u) - B(s)$ has a normal distribution with variance $u - s$ and mean 0;
3. $B(u)$, for $0 \leq< u$ are continuous functions of u.

This description possesses established the start of motion from point u, and by counting the need for $P(B(0) = 0) = 0$, we get the standard Brownian motion description, in which motion begins at 0.

Definition 2.2. [19] An approach H is called adjusted to the filtration $\tilde{\digamma} = (\tilde{\digamma}_{\varphi})$, if for all φ, $H(\varphi)$ is $\tilde{\digamma}_{\varphi}$-measurable.

Theorem 2.1. [19] If H be a continuous adjusted approach, then the Itô integral $\int_0^T H(\varphi)dB(\varphi)$ exists.

2.2 Preliminaries

In this manuscript, assume the following notations:
- $\mathcal{Q}$: A real Banach space;
- B_r: The open ball with center 0 and radius r;
- $\partial\bar{B}_r$: The sphere around 0 with radius $r > 0$ in $\mathcal{Q}$.

Definition 2.3. [20] Let J be a bounded subset of $\mathcal{Q}$, and assume that $\alpha(J)$ denotes the Kuratowski MNC of J, that is,

$$\alpha(J) = \inf\{\varsigma > 0 : J \text{ may be covered by finitely many sets by diameter} \leq \varsigma\}. \tag{2.1}$$

Definition 2.4. [10] The Hausdorff MNC is:

$$\psi(J) = \inf\{\varsigma > 0 : \exists \text{ a finite } \varsigma\text{-net for } J \text{ in } \mathcal{Q}\}, \tag{2.2}$$

where by a finite ς-net for J in $\mathcal{Q}$ we indicate, as general, a set $\{u_1, u_2, ..., u_m\} \subset \mathcal{Q}$ such that $B_\kappa(\mathcal{Q}; u_1), B_\kappa(\mathcal{Q}; u_2), ..., B_\kappa(\mathcal{Q}; u_m)$ cover J. These MNC are mutually similar in the way

$$\psi(J) \leq \alpha(J) \leq 2\psi(J),$$

for any bounded set $J \subset \mathcal{Q}$.

Theorem 2.2. [23] Let $J, \hat{J} \in \mathcal{Q}$ be bounded and $\lambda \in \mathbb{R}$. Then

1. $\psi(J \cup \hat{J}) = \max\{\psi(J), \psi(\hat{J})\}$,
2. $\psi(J + \hat{J}) \leq \psi(J) + \psi(\hat{J})$,
3. $\psi(\lambda J) =\mid \lambda \mid \psi(J)$,
4. $\psi(J) \leq \psi(\hat{J})$ where $J \subseteq \hat{J}$,
5. $\psi(\bar{J}) = \psi(Conv J) = \psi(J)$,

6. $\psi(J) = 0$ if and only if J is pre-compact.

Here, we will consider a MNC in the space $C[0,a]$ consisting of all continuous real-valued functions on $[0,a]$. The space $C[0,a]$ is equipped with the usual norm

$$\|u\| = \sup\{|u(\varphi)| : \varphi \in [0,a]\}.$$

The modulus of the continuity for $u \in C[0,a]$ is defined by

$$\jmath(u,\varsigma) = \sup\{|u(\varphi) - u(\varphi')| : |\varphi - \varphi'| \leq \varsigma\}.$$

Theorem 2.3. [10] The MNC (2.2) is equivalent to

$$\psi(J) = \lim_{\varsigma \to 0} \sup_{u \in J} \jmath(u,\varsigma), \tag{2.3}$$

for all bounded set $J \subset C[0,a]$.

Definition 2.5. [22] A continuous function $K : \mathcal{Q} \to \mathcal{Q}$ on a Banach space $\mathcal{Q}$ is called a $\hat{k}$-set contraction if for all $J \subset \mathcal{Q}$, $K(J)$ is bounded where J is bounded, and $\alpha(KJ) \leq \hat{k}\alpha(J), 0 < \hat{k} < 1$. If

$$\alpha(KJ) < \alpha(J), \text{ for all } J \subseteq \mathcal{Q},$$

then K is called a condensing (or a densifying) map.

Theorem 2.4. PFPT[23], see also [30] Let B_r be an open ball centered at the origin in $\mathcal{Q}$. If $K : B_r \to \mathcal{Q}$ is a densifying map that fulfill the boundary condition,

$$\text{If } K(u) = \hat{k}u, \text{ for some } u \text{ in } \partial B_r, \text{ then } \hat{k} \leq 1, \tag{P}$$

then, the set of fixed points of K in B_r is non-empty.

3 Main Results

In this section, we analyse the solvability of the Eq. (1.1) under the following hypothesis:

(C1) $u \in C(I_a,\mathbb{R}), F, G \in C(I_a \times \mathbb{R} \times \mathbb{R} \times \mathbb{R},\mathbb{R})$, and $p \in C(I_a \times I_a \times \mathbb{R},\mathbb{R}), q \in C(I_a \times I_a \times \mathbb{R},\mathbb{R})$. Also, τ, υ and $\theta_i, \mu_i : I_a \to I_a, i = 1,2$ are continuous for each $\varphi \in I_a$.

(C2) There exist non-negative constants k, k' with $2k, 2k' < 1$ such that

$$|F(\varphi,u_1,u_2,u_3) - F(\varphi,\bar{u}_1,\bar{u}_2,\bar{u}_3)| \leq k(|u_1 - \bar{u}_1| + |u_2 - \bar{u}_2| + |u_3 - \bar{u}_3|),$$

$$|G(\varphi,u_1,u_2,u_3) - G(\varphi,\bar{u}_1,\bar{u}_2,\bar{u}_3)| \leq k'(|u_1 - \bar{u}_1| + |u_2 - \bar{u}_2| + |u_3 - \bar{u}_3|).$$

(C3) There exists $r_0 \geq 0$ so that $\sup\{(L_1) \times (L_2)\} \leq r_0$, where,

$$\begin{aligned}
&L_1 = \sup\{|F(\varphi,u_1,u_2,u_3)| : \text{for all } \varphi \in I_a, \text{ and } u_1,u_2 \in [-r_0,r_0], \ |u_3| \leq M\lambda\},\\
&M = \sup\{|p(\varphi,s,u)| : \text{for all } \varphi, s \in I_a, \quad \text{and} \quad u \in [-r_0,r_0]\},\\
&L_2 = \sup\{|G(\varphi,u_1,u_2,u_3)| : \text{for all } \varphi \in I_a, \text{ and } u_1,u_2 \in [-r_0,r_0], \ |u_3| \leq N\lambda\},\\
&N = \sup\{|q(\varphi,s,u)| : \text{for all } \varphi, s \in I_a, \text{ and } u \in [-r_0,r_0]\},\\
&\lambda = \sup\{|B(\varphi)| : \text{for all } \varphi \in I_a\}.
\end{aligned}$$

Theorem 3.1. Under the conditions (C1)-(C3), Eq. (1.1) possesses at least one solution in $\mathcal{Q} = C(I_a)$.

Proof . To show this result, we utilize the Theorem 2.4 as our general concept. Define $P, Q : B_{r_0} \to \mathcal{Q}$ and K as $Ku = (Pu) \times (Qu)$, where

$$Pu(\varphi) = \left(F\Big(\varphi, u(\tau(\varphi)), u(\theta_1(\varphi)), \int_0^{\varphi} p(\varphi,s,u(\theta_2(s)))dB(s)\Big)\right), \tag{3.1}$$

and

$$Qu(\varphi) = \Big(G\Big(\varphi, u(\upsilon(\varphi)), u(\mu_1(\varphi)), \int_0^a q(\varphi, s, u(\mu_2(s)))dB(s)\Big)\Big), \tag{3.2}$$

for all $\varphi \in I_a$. Now, we prove P is continuous on B_{r_0}. Take $\varsigma > 0$ and arbitrary $u, v \in B_{r_0}$ such that $\| u - v \| \leq \varsigma$. Then for $\varphi \in I_a$, we obtain

$$\begin{aligned}
&|(Pu)(\varphi) - (Pv)(\varphi)| \\
=&\Big|\Big(F\Big(\varphi, u(\tau(\varphi)), u(\theta_1(\varphi)), \int_0^\varphi p(t, s, u(\theta_2(s)))dB(s)\Big)\Big) - \Big(F\Big(\varphi, v(\tau(\varphi)), v(\theta_1(\varphi)), \int_0^\varphi p(\varphi, s, y(\theta_2(s)))dB(s)\Big)\Big)\Big| \\
\leq& k|u(\tau(\varphi)) - v(\tau(\varphi))| + k|u(\theta_1(\varphi))) - v(\theta_1(\varphi))| + k\int_0^\varphi |p(\varphi, s, u(\theta_2(s))) - p(\varphi, s, v(\theta_2(s)))|dB(s).
\end{aligned}$$

So,

$$|(Pu)(\varphi) - (Pv)(\varphi)| \leq (2k)\|u - v\| + k\lambda\jmath(p, \varsigma).$$

Similarly, we have

$$\begin{aligned}
&|(Qu)(\varphi) - (Qv)(\varphi)| \\
=&\Big|\Big(G\Big(\varphi, u(\upsilon(\varphi)), u(\mu_1(\varphi)), \int_0^a q(\varphi, s, u(\mu_2(s)))dB(s)\Big)\Big) - \Big(G\Big(\varphi, v(\upsilon(\varphi)), v(\mu_1(\varphi)), \int_0^a q(\varphi, s, y(\mu_2(s)))dB(s)\Big)\Big)\Big| \\
\leq&(2k')\|u - v\| + k'\lambda\jmath(q, \varsigma),
\end{aligned}$$

where for all $\varsigma > 0$ we define

$$\begin{aligned}
\jmath(p, \varsigma) &= \sup\{|p(\varphi, s, u) - p(\varphi, s, \bar{u})| : \varphi \in I_a, s \in [0, a], u, \bar{u} \in [-r_0, r_0], \|u - \bar{u}\| \leq \varsigma\}, \\
\jmath(q, \varsigma) &= \sup\{|q(\varphi, s, u) - q(\varphi, s, \bar{u})| : \varphi \in I_a, s \in [0, a], u, \bar{u} \in [-r_0, r_0], \|u - \bar{u}\| \leq \varsigma\}.
\end{aligned}$$

Since $p(\varphi, s, u)$ and $q(\varphi, s, u)$ are uniformly continuous on $[0, a] \times [0, a] \times \mathbb{R}$, we obtain $\jmath(p, \varsigma) \to 0$ and $\jmath(q, \varsigma) \to 0$ as $\varsigma \to 0$. So, P and Q are continuous on B_{r_0}. Hence, K is also continuous on B_{r_0}.

Now, we show that P and Q fulfill the condensing condition with respect to ψ in B_{r_0}. Select an arbitrary $\varsigma > 0$. Taking $u \in J$, where J is a bounded subset of $\mathcal{Q}$. Further, for $\varphi_1, \varphi_2 \in I_a$, we can have $\varphi_1 \leq \varphi_2$ with $\varphi_2 - \varphi_1 \leq \varsigma$. We get

$$\begin{aligned}
|(Pu)(\varphi_2) - (Pu)(\varphi_1)| =&\Big|\Big(F\Big(\varphi_2, u(\tau(\varphi_2)), u(\theta_1(\varphi_2)), \int_0^{\varphi_2} p(\varphi_2, s, u(\theta_2(s)))dB(s)\Big)\Big) \\
&- \Big(F\Big(\varphi_1, u(\tau(\varphi_1))), u(\theta_1(\varphi_1)), \int_0^{\varphi_1} p(\varphi_1, s, u(\theta_2(s)))dB(s)\Big)\Big)\Big| \\
\leq&\Big|F\Big(\varphi_2, u(\tau(\varphi_2)), u(\theta_1(\varphi_2)), \int_0^{\varphi_2} p(\varphi_2, s, u(\theta_2(s)))dB(s)\Big) \\
&- F\Big(\varphi_2, u(\tau(\varphi_1))), u(\theta_1(\varphi_1)), \int_0^{\varphi_1} p(\varphi_1, s, u(\theta_2(s)))dB(s)\Big)\Big| \\
&+ \Big|F\Big(\varphi_2, u(\tau(\varphi_1)), u(\theta_1(\varphi_1)), \int_0^{\varphi_1} p(\varphi_1, s, u(\theta_2(s)))dB(s)\Big) \\
&- F\Big(\varphi_1, u(\tau(\varphi_1)), u(\theta_1(\varphi_1)), \int_0^{\varphi_1} p(\varphi_1, s, u(\theta_2(s)))dB(s)\Big)\Big| \\
\leq& k|u(\tau(\varphi_2)) - u(\tau(\varphi_1))| + k|u(\theta_1(\varphi_2)) - u(\theta_1(\varphi_1))| \\
&+ k\Big|\int_0^{\varphi_2} p(\varphi_2, s, u(\theta_2(s)))dB(s) - \int_0^{\varphi_1} p(\varphi_1, s, u(\theta_2(s)))dB(s)\Big| + \jmath^1_{r_0}(F, \varsigma) \\
\leq& k\jmath(u, \jmath(\tau, \varsigma)) + k\jmath(u, \jmath(\theta_1, \varsigma)) + k\int_0^{\varphi_1} |p(\varphi_2, s, u(\theta_2(s))) - p(\varphi_1, s, u(\theta_2(s)))|dB(s) \\
&+ k\int_{\varphi_1}^{\varphi_2} |p(\varphi_2, s, u(\theta_2(s)))|dB(s) + \jmath^1_{r_0}(F, \varsigma) \\
\leq& k\jmath(u, \jmath(\tau, \varsigma)) + k\jmath(u, \jmath(\theta_1, \varsigma)) + k\lambda\jmath^1_{r_0}(p, \varsigma) + kM\jmath(B, \varsigma) + \jmath^1_{r_0}(F, \varsigma).
\end{aligned}$$

Similarly,

$$\begin{aligned}|(Qu)(\varphi_2)-(Qu)(\varphi_1)| =&\bigg|\bigg(G\Big(\varphi_2, u(\upsilon(\varphi_2))), u(\mu_1(\varphi_2))), \int_0^{\varphi_2} q_2(\varphi_2, s, u(\mu_2(s)))dB(s)\Big)\bigg)\\ &-\bigg(G\Big(\varphi_1, u(\upsilon(\varphi_1))), u(\mu_1(\varphi_1)), \int_0^{\varphi_1} q_2(\varphi_1, s, u(\mu_2(s)))dB(s)\Big)\bigg)\bigg|\\ \le& k'\jmath(u,\jmath(\upsilon,\varsigma))+k'\jmath(u,\jmath(\mu_1,\varsigma))+k'\lambda\jmath^1_{r_0}(q,\varsigma)+k'N\jmath(B,\varsigma)+\jmath^1_{r_0}(G,\varsigma),\end{aligned}$$

where:

$$\begin{aligned}&\jmath^1_{r_0}(p,\varsigma)=\sup\{|p(\varphi,s,u)-p(\bar\varphi,s,u)|:|\varphi-\bar\varphi|\le\varsigma,\varphi,\bar\varphi,s\in I_a,\ u\in[-r_0,r_0]\},\\ &\jmath^1_{r_0}(q,\varsigma)=\sup\{|q(\varphi,s,u)-q(\bar\varphi,s,u)|:|\varphi-\bar\varphi|\le\varsigma,\varphi,\bar\varphi,s\in I_a,u\in[-r_0,r_0]\},\\ &\jmath^1_{r_0}(F,\varsigma)=\sup\{F(\varphi,u_1,u_2,u_3)-F(\bar\varphi,u_1,u_2,u_3)|:|\varphi-\bar\varphi|\le\varsigma,\varphi,\bar\varphi\in I_a,\ u_1,u_2\in[-r_0,r_0],|u_3|\le M\lambda\}\\ &\jmath^1_{r_0}(G,\varsigma)=\sup\{G(\varphi,u_1,u_2,u_3)-G(\bar\varphi,u_1,u_2,u_3)|:|\varphi-\bar\varphi|\le\varsigma,\varphi,\bar\varphi\in I_a,\ u_1,u_2\in[-r_0,r_0],|u_3|\le N\lambda\}\\ &\jmath(B,\varsigma)=\sup\{|B(\varphi)-B(\bar\varphi)|:|\varphi-\bar\varphi|\le\varsigma,\varphi,\bar\varphi\in[-r_0,r_0]\}.\end{aligned}$$

From the above relations, we have

$$\jmath(Pu,\varsigma)\le k\jmath(u,\jmath(\tau,\varsigma))+k\jmath(u,\jmath(\theta_1,\varsigma))+k\lambda\jmath^1_{r_0}(p,\varsigma)+kM\jmath(B,\jmath(\varphi,\varsigma))+\jmath^1_{r_0}(F,\varsigma)$$

and

$$\jmath(Qu,\varsigma)\le k'\jmath(u,\jmath(\upsilon,\varsigma))+k'\jmath(u,\jmath(\mu_1,\varsigma))+k'\lambda\jmath^1_{r_0}(q,\varsigma)+k'N\jmath(B,\jmath(\varphi,\varsigma))+\jmath^1_{r_0}(G,\varsigma).$$

Applying limit as $\varsigma\to 0$, we have

$$\psi(PJ)\le(2k)\psi(J).$$

Also,

$$\psi(QJ)\le(2k')\psi(J).$$

Hence, K is a condensing map. Suppose that $u\in\partial\bar B_{r_0}$. If $Ku=\hat k x$, then $\hat k r_0=\hat k\|x\|=\|Ku\|$ and by (C3) we obtain

$$\begin{aligned}|Ku(\varphi)|=&\bigg|\bigg(F\Big(\varphi, u(\tau(\varphi)), u(\theta_1(\varphi)), \int_0^{\varphi} p(\varphi,s,u(\theta_2(s)))dB(s)\Big)\bigg)\\ &\times\bigg(G\Big(\varphi,\varphi(\upsilon(\varphi)), u(\mu_1(\varphi)), \int_0^{a} q(\varphi,s,u(\mu_2(s)))dB(s)\Big)\bigg)\bigg|\le r_0,\end{aligned}$$

for all $\varphi\in I_a$. Hence, $\|Ku\|\le r_0$, so this shows $\hat k\le 1$. The proof is complete. □

Corollary 3.2. [21] Let

(K1) $F,G\in C([0,a]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and there exists the constant $h>0$ such that $|F(\varphi,0,0)|\le h$, $|G(\varphi,0,0)|\le h$.
(K2) For some continuous functions $a_1,a_2,a_3,a_4:[0,a]\to[0,a]$, such that $|F(\varphi,u_1,u_2)-F(\varphi,\bar u_1,\bar u_2)|\le a_1(\varphi)|u_1-\bar u_1|+a_2(\varphi)|u_2-\bar u_2|$, $|G(\varphi,u_1,u_2)-G(\varphi,\bar u_1,\bar u_2)|\le a_3(\varphi)|u_1-\bar u_1|+a_4(\varphi)|u_2-\bar u_2|$.
(K3) there exists the constant $k>0$ such that for all $\varphi\in[0,a]$, $a_1(\varphi),a_2(\varphi),a_3(\varphi),a_4(\varphi)\le k$.
(K4) $p,q\in C([0,1]\times[0,1]\times\mathbb{R},\mathbb{R})$ and there exist constants $\gamma_1,\gamma_2>0$ such that $|p(\varphi,s,u)|\le\gamma_1+\gamma_2|u|$, $|q(\varphi,s,u)|\le\gamma_1+\gamma_2|u|$, for all $\varphi,s\in[0,1]$ and $u\in\mathbb{R}$.
(K5) $4\zeta\eta<1$, where $\zeta=k\gamma_1\beta+h$ and $\eta=k(\gamma_2\beta+1)$ and $\beta=\sup\{B(\varphi);\varphi\in[0,a]\}$.

Then

$$u(\varphi)=\Big(F\big(\varphi,u(\tau(\varphi)),\int_0^{\varphi}p(\varphi,s,u(\theta_2(s)))dB(s)\big)\Big)\times\Big(G\big(v,u(\upsilon(\varphi)),\int_0^{a}q(\varphi,s,u(\mu_2(s)))dB(s)\big)\Big),\qquad(3.3)$$

possesses at least one solution in $\mathcal{Q}=C(I_a)$.

Proof . It can be proved that if $F(\varphi, u_1, u_2, u_3) = F(\varphi, u_1, u_3)$ and $G(\varphi, u_1, u_2, u_3) = G(\varphi, u_1, u_3)$, then Eq. 1.1 will be the Eq. 3.3. We check that (C2) is completed by (K3) and (K4). Now, we prove that (C3) is also satisfied. Let $\|u\| \le r_0$, $r_0 > 0$ and putting $M = N = \gamma_1 + \gamma_2 r_0$, then

$$\begin{aligned}
|u(\varphi)| &= \Big|F\big(\varphi, u(\tau(\varphi)), \int_0^{\varphi} p(\varphi, s, u(\theta_1(s)))dB(s)\big) \times G\big(\varphi, u(\upsilon(\varphi)), \int_0^{a} q(\varphi, s, u(\mu_2(s)))dB(s)\big)\Big| \\
&\le \Big(\Big|F\big(\varphi, u(\tau(\varphi)), \int_0^{\varphi} p(\varphi, s, u(\theta_1(s)))dB(s)\big)\Big| - |F(\varphi,0,0)| + |F(\varphi,0,0)|\Big) \\
&\times \Big(\Big|G\big(\varphi, u(\upsilon(\varphi)), \int_0^{a} q(\varphi, s, u(\mu_2(s)))dB(s)\big)\Big| - |G(\varphi,0,0)| + |G(\varphi,0,0)|\Big) \\
&\le \Big(a_1(\varphi)|u(\tau(\varphi))| + a_2(\varphi)\int_0^{\varphi} |p(\varphi, s, u(\theta_1(s)))|dB(s) + l\Big) \\
&\times \Big(a_3(\varphi)|u(\upsilon(\varphi))| + a_4(\varphi)\int_0^{1} |q(t, s, u(\mu_2(s)))|dB(s) + l\Big) \\
&\le \big(k\|u\| + kM\eta + h\big).\big(k\|u\| + kN\eta + h\big) \\
&\le \big((k + k\gamma_2\beta)\|u\| + k\gamma_1\beta + h\big)^2 \\
&\le \big(\eta\|u\| + \zeta\big)^2
\end{aligned}$$

for all $\varphi \in I_a$. Hence, r_0 in (C3) is the real number that fulfills in the following conditions

$$\sup_{\varphi \in I_a} |u(\varphi)| \le \big(\eta r_0 + \zeta\big)^2 \le r_0. \tag{3.4}$$

The inequality (3.4), possesses a solution in $[r_1, r_2]$, where

$$r_1 = \frac{1 - 2\eta\zeta - \sqrt{1 - 4\eta\zeta'}}{2\eta^2}$$

and

$$r_2 = \frac{1 - 2\eta\zeta + \sqrt{1 - 4\eta\zeta}}{2\eta'^2}.$$

Under the (K5), $1 - \sqrt{1 - 4\eta\zeta} < 1$. So, $r_0 = r_1$ is a positive real number. Now, the whole result got from Theorem 3.1. □

Corollary 3.3. Let

(D1) $f \in C(I_a \times \mathbb{R}, \mathbb{R}), F \in C(I_a \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and there exists constants $\hat{\mu}, \hat{\gamma}, \hat{\kappa}$ such that $|f(\varphi, 0)| \le \hat{\mu}$, $|F(\varphi, u(\tau(\varphi)), 0, 0| \le \hat{\gamma} + \hat{\kappa}|u(\varphi)|$,

(D2) there exist the functions $a_1, a_2, a_3 : I_a \to I_a$ such that $|f(\varphi, u) - f(\varphi, \bar{u})| \le a_1(\varphi)|u - \bar{u}|$, $|F(\varphi, u_2, u_1) - F(\varphi, \bar{u}_2, u_1)| \le a_2(\varphi)|u_2 - \bar{u}_2|$, $|F(\varphi, u_2, u_1) - F(\varphi, u_2, \bar{u}_1)| \le a_3(\varphi)|u_1 - \bar{u}_1|$, for all $u_1, \bar{u}_1, u_2, \bar{u}_2 \in \mathbb{R}$,, $\varphi \in I_a$ and let $k = \max\{|a_i(\varphi)| : \varphi \in I_a, i = 1, 2, 3\}$.

(D3) (Sub-linear condition) $p \in C([0, a] \times [0, a] \times \mathbb{R}, \mathbb{R})$ be a continuous derivative function and fulfills in sub-linear condition, so that there exist the constants γ_1 and γ_2 such that $|p(\varphi, s, u)| \le \gamma_1 + \gamma_2|u|$, for all $\varphi, s \in [0, a]$ and $u \in \mathbb{R}$.

(D4) $k < \frac{1-\hat{\kappa}}{2(1+\lambda(\gamma_2))}$ and $\kappa = \sup\{B(\varphi); \varphi \in [0, a]\}$.

Then

$$u(\varphi) = f\big(\varphi, u(\tau(\varphi))\big) + F\big(\varphi, u(\theta_1(\varphi)), \int_0^{\varphi} p(\varphi, s, u(\theta_2(s)))dB(s)\big), \qquad \varphi \in I_a$$

possesses at least one solution in $\mathcal{Q} = C(I_a)$.

Proof . Let $r_0 = \frac{R_2}{1-R_1}$ where

$$R_1 = k + \hat{\kappa} + k\kappa(\gamma_2) \text{ and } R_2 = \hat{\mu} + \hat{\gamma} + k\kappa(\gamma_1).$$

From (D4), $R_1 = k + \hat{\kappa} + k\kappa\gamma_2 < 1 - (k + k\kappa\gamma_2) < 1$. So, r_0 is a positive real number. In addition, we check that (C2) is finished from (D2) and (D4). Now we prove that (C3) is also satisfied. Putting $M = \gamma_1 + \gamma_2 r_0$, then

$$\begin{aligned}|u(\varphi)| =& \Big|f\big(\varphi, u(\tau(\varphi))\big) + F\big(\varphi, u(\theta_1(\varphi)), \int_0^{\varphi} p(\varphi, s, u(\theta_2(s)))dB(s)\big)\Big| \\ \leq& \big|f\big(\varphi, u(\tau(\varphi))\big) - f(\varphi, 0)\big| + |f(\varphi, 0)| + \big|F\big(\varphi, u(\theta_1(\varphi)), \int_0^{\varphi} p(\varphi, s, u(\theta_2(s)))dB(s)\big) \\ &- F\big(\varphi, u(\theta_1(\varphi)), 0\big)\big| + \big|F\big(\varphi, u(\theta_1(\varphi)), 0\big)\big| \\ \leq& k|\varphi| + \hat{\mu} + k\kappa(\gamma_1 + \gamma_2|u|) + \hat{\gamma} + \hat{\kappa}|u| \\ \leq& (k + \hat{\kappa} + k\kappa(\gamma_2))|u| + \hat{\mu} + \hat{\gamma} + k\kappa(\gamma_1),\end{aligned}$$

for all $\varphi \in I_a$. Consequently,

$$\sup_{\varphi \in I_a} |u(\varphi)| \leq R_1 r_0 + R_2 = L_1 \frac{R_2}{1 - R_1} + R_2 = r_0.$$

Now, we get the complete result from Theorem 3.1. □

4 Examples

As applications and to establish the efficiency of the presented approach, an example is discussed in this section.

Example 4.1. Let the following stochastic FIE.

$$\begin{aligned}u(\varphi) =& \Big(\frac{1}{7}\cos(\frac{e^{-\sqrt{\varphi}}}{3+\varphi^2}) + \frac{1}{5}\frac{\sqrt{\varphi}}{1+\varphi+\varphi^2}\sin(u(\varphi) + \frac{1}{9}\int_0^1 \frac{|u(s)|}{4+|u(s)|}dB(s)\Big) \\ &\times \Big(\frac{1}{5}\cos(\frac{\varphi^2}{4}) + \frac{e^{-\varphi}}{2+2\varphi}\cos(u(\varphi) + \frac{1}{3}\int_0^1 \cos(\frac{|u(s)|}{5+|u(s)|})dB(s)\Big), \qquad \varphi \in [0,1].\end{aligned} \tag{4.1}$$

Here,

$$\theta_1(\varphi) = \theta_2(\varphi) = \mu_1(\varphi) = \mu_2(\varphi) = \varphi, \quad \text{for all} \quad \varphi \in [0,1],$$

$$\begin{aligned}&F(\varphi, u_1, u_2, u_3) = \frac{1}{7}\cos(\frac{e^{-\sqrt{\varphi}}}{3+\varphi^2}) + \frac{u_1}{5} + 0u_2 + \frac{u_3}{9}, \quad u_1 = \frac{\sqrt{\varphi}}{1+\varphi+\varphi^2}\sin(u(\varphi), \quad u_3 = \int_0^1 \frac{|u(s)|}{4+|u(s)|}dB(s) \\ &G(\varphi, u_1, u_2, u_3) = \frac{1}{5}\cos(\frac{\varphi^2}{4}) + \frac{u_1}{2} + 0u_2 + \frac{u_3}{3}, \quad u_3 = \int_0^1 \cos(\frac{|\varphi(s)|}{5+|\varphi(s)|})dB(s), \\ &p(\varphi, s, u) = \frac{|u(s)|}{4+|u(s)|}, \quad |p(\varphi, s, u)| \leq 1 \\ &q(\varphi, s, u) = \cos(\frac{|u(s)|}{5+|u(s)|}), \quad |q(\varphi, s, u)| \leq 1\end{aligned}$$

for all $\varphi \in [0,1]$. Observe that the assumptions (C1) and (C2) are fulfilled. It is easy to see that (C3) also fulfill. Taking $r_0 = (\frac{12}{35} + \frac{1}{9}\lambda)(\frac{7}{10} + \frac{1}{3}\lambda)$, then we get $M \leq 1$, $N \leq 1$ and

$$\begin{aligned}\sup\Big|&\Big(\frac{1}{7}\cos(\frac{e^{-\sqrt{\varphi}}}{3+v^2}) + \frac{1}{5}\frac{\sqrt{\varphi}}{1+\varphi+\varphi^2}\sin(u(\varphi) + \frac{1}{9}\int_0^{\varphi} \frac{|u(s)|}{4+|u(s)|}dB(s)\Big) \\ &\times \Big(\frac{1}{5}\cos(\frac{u^2}{4}) + \frac{e^{-u}}{2+2u}\cos(u(\varphi) + \frac{1}{3}\int_0^1 \cos(\frac{|u(s)|}{5+|u(s)|})dB(s)\Big)\Big| \leq r_0.\end{aligned}$$

for all $\varphi \in [0,1]$. Hence (C3) holds if,

$$(\frac{12}{35} + \frac{1}{9}\lambda)(\frac{7}{10} + \frac{1}{3}\lambda) \leq r_0. \tag{4.2}$$

This shows that

$$r_0 = (\frac{12}{35} + \frac{1}{9}\lambda)(\frac{7}{10} + \frac{1}{3}\lambda)$$

and $\lambda = \sup\{|B(\varphi)|; \forall\varphi \in [0,1]\}$. Hence, from Theorem 3.1 equation (4.1) has at least one random solution in Banach space $[0,1]$.

Example 4.2. Let the following stochastic FIE.

$$u(\varphi) = \left(\frac{e^{-2\varphi}}{3(1+\varphi)} + \frac{|u(\varphi)|\sin(\varphi)}{1+|u(\varphi)|} + \frac{\varphi^2}{2(1+\varphi^2)}\int_0^t \frac{s^2\sqrt{|u(s)|}}{1+s^2}dB(s)\right)$$
$$\times\left(\frac{1}{6}\varphi\sin(\frac{u(\varphi^3)}{1+\varphi}) + \frac{\varphi^2 u(1-\varphi)}{2+2u(1-\varphi)} + \frac{1}{7+\varphi}\int_0^1 \sqrt{\ln(1+|u(s)|)}dB(s)\right), \qquad \varphi\in[0,1]. \tag{4.3}$$

Here,

$$\tau(\varphi) = \theta_2(\varphi) = \mu_2(\varphi) = \varphi, \upsilon(\varphi) = \varphi^3, \mu_1(\varphi) = 1-\varphi \quad \text{for all} \quad \varphi\in[0,1],$$

$$F(\varphi,u_1,u_2,u_3) = \frac{e^{-2\varphi}}{3(1+\varphi)} + \frac{|u_1|\sin(\varphi)}{1+|u_1|} + \frac{\varphi^2}{2(1+\varphi^2)}u_3, \quad u_3 = \int_0^t \frac{s^2\sqrt{|u(s)|}}{1+s^2}dB(s)$$
$$G(\varphi,u_1,u_2,u_3) = \frac{1}{6}\varphi\sin(\frac{u_1}{1+\varphi}) + \frac{\varphi^2 u_2}{2+2u_2} + \frac{1}{7+\varphi}u_3, \quad u_3 = \int_0^1 \sqrt{\ln(1+|u(s)|)}dB(s),$$
$$p(\varphi,s,u) = \frac{s^2\sqrt{|u|}}{1+s^2}, \quad |p(\varphi,s,u)| \le \sqrt{|u|}$$
$$q(\varphi,s,u) = \sqrt{\ln(1+|u|)}, \quad |q(\varphi,s,u)| \le \sqrt{|u|}$$

Now, we can see that these functions satisfy the assumptions (C1) and (C2). We check that (C3) also holds. Suppose that $\|u\| \le r_0, r_0 > 0$, then we have

$$|u(\varphi)| = |\Big(\frac{e^{-2\varphi}}{3(1+\varphi)} + \frac{|u(\varphi)|\sin(\varphi)}{1+|u(\varphi)|} + \frac{\varphi^2}{2(1+\varphi^2)}\int_0^t \frac{s^2\sqrt{|u(s)|}}{1+s^2}dB(s)\Big)$$
$$\times\Big(\frac{1}{6}\varphi\sin(\frac{u(\varphi^3)}{1+\varphi}) + \frac{\varphi^2 u(1-\varphi)}{2+2u(1-\varphi)} + \frac{1}{7+\varphi}\int_0^1 \sqrt{\ln(1+|u(s)|)}dB(s)\Big)| \le r_0, \tag{4.4}$$

for all $\varphi\in[0,1]$. Hence (C3) holds if,

$$(\frac{4}{3} + \frac{1}{2}\sqrt{r_0}\lambda)(\frac{2}{3} + \frac{1}{7}\sqrt{r_0}\lambda) \le r_0. \tag{4.5}$$

This shows that

$$r_0 = \frac{1}{9}\frac{(11\lambda + \sqrt{9\lambda^2+1568})^2}{(\lambda^2-14)^2}$$

and $\lambda = \sup\{|B(\varphi)|; \forall\varphi\in[0,1]\}$. Hence, from Theorem 3.1 equation (4.3) has at least one random solution in Banach space $[0,1]$.

5 Conclusions

The significance of the existence of solutions is one of the problems of researchers in the study of functional stochastic integral equations. So far, various methods have been developed for this purpose.This paper is based on an additional general form of the functional stochastic integral equation, which also covers some other equivalent works. In the presented method, Petryshyn's fixed point theorem and the idea of MNC with weaker conditions are used.

Acknowledgements

We are grateful to the editor and anonymous referees for their careful reading and thoughtful feedback. Their comments and suggestions have led to a significant improvement in the clarity and presentation of this paper.

References

[1] E. Castillo, A. Iglesias, and R. Ruiz-Cobo, *Functional Equations in Applied Sciences*, Elsevier, 2004.

[2] A. Deep, S. Abbas, B. Singh, M.R. Alharthi, and K.S. Nisar, *Solvability of functional stochastic integral equations via Darbo's fixed point theorem,* Alexandria Eng. J. **60** (2021), 5631–5636.

[3] A. Deep, Deepmala, and R. Ezzati, *Application of Petryshyn's fixed point theorem to solvability for functional integral equations,* Appl. Math. Comput. **395** (2021), 125878.

[4] A. Deep, Deepmala, and M. Rabbani, *A numerical method for solvability of some non-linear functional integral equations,* Appl. Math. Comput. **402** (2021), 125637.

[5] A. Deep, Deepmala, and J. Rezaee Roshan, *Solvability for generalized nonlinear functional integral equations in Banach spaces with applications,* J. Integr. Equations Appl. **33** (2021), 19–30.

[6] A. Deep, D. Dhiman, B. Hazarika, and S. Abbas, *Solvability for two dimensional functional integral equations via Petryshyn's fixed point theorem,* Rev. Real Acad. Ciencias Exactas Fis. Nat. Ser. A Mat. **115** (2021), 1–17.

[7] A. Deep, Deepmala, and B. Hazarika, *An existence result for Hadamard type two dimensional fractional functional integral equations via measure of noncompactness*, Chaos Solitons Fractals. **147** (2021), 110874.

[8] A. Deep, A. Kumar, B. Hazarika, and S. Abbas, *An existence result for functional integral equations via Petryshyn's fixed point theorem,* J. Integr. Equations Appl. **34** (2022), 165–181.

[9] A. Deep, A. Kumar, S. Abbas, and M. Rabbani, *Solvability and numerical method for non-linear Volterra integral equations by using Petryshyn's fixed point theorem,* Int. J. Nonlinear Anal. Appl. **13** (2022), no. 1, 1–28.

[10] L.S. Goldenstein and A.S. Markus, *On a measure of noncompactness of bounded sets and linear operators,* Studies in Algebra and Mathematical Analysis, Kishinev, 1965, pp. 45–54.

[11] G. Gripenberg, S.-O. Londen and O. Staffans, *Volterra integral and Functional Equations,* Cambridge University Press. 1990.

[12] Y. Guo, M. Chen, X.B. Shu, and F. Xu, *The existence and Hyers-Ulam stability of solution for almost periodical fractional stochastic differential equation with fBm*, Stochastic Anal. Appl. **39** (2021), 643—666.

[13] I. Ito, *On the existence and uniqueness of solutions of stochastic integral equations of the Volterra type,* Kodai Math. J. **2** (1979), 158–170.

[14] M. Kazemi, *On existence of solutions for some functional integral equations in Banach algebra by fixed point theorem* Int. J. Nonlinear Anal. Appl. **13** (2022), 451–456.

[15] M. Kazemi, A. Deep, and J. Nieto, *An existence result with numerical solution of nonlinear fractional integral equations*, Math. Methods Appl. Sci. **46** (2023), 10384–10399.

[16] M. Kazemi, A. Deep and A. Yaghoobnia, *Application of fixed point theorem on the study of the existence of solutions in some fractional stochastic functional integral equations*, Math. Sci. (2022). https://doi.org/10.1007/s40096-022-00489-7

[17] M. Kazemi and R. Ezzati, *Existence of solutions for some nonlinear Volterra integral equations via Petryshyn's fixed point theorem,* Int. J. Nonlinear Anal. Appl. **9** (2018), 1–12.

[18] M. Kazemi and A.R. Yaghoobnia, *Application of fixed point theorem to solvability of functional stochastic integral equations*, Appl. Math. Comput. **417** (2022), 126759.

[19] F.C. Klebaner, *Introduction to Stochastic Calculus with Applications,* World Scientific Publishing Company, 2012.

[20] K. Kuratowski, *Sur les espaces complets,* Fund. Math. **1** (1930) 301–309.

[21] F. Mirzaee and N. Samadyar, *Extension of Darbo fixed-point theorem to illustrate existence of the solutions of some nonlinear functional stochastic integral equations,* Int. J. Nonlinear Anal. Appl. **11** (2020), 413–421.

[22] R.D. Nussbaum, *The fixed point index and asymptotic fixed point theorems for k-set-contractions,* Bull. Am. Math. Soc. 75(1969) 490–495.

[23] W. Petryshyn, *Structure of the fixed points sets of k-set-contractions,* Arch. Ration. Mech. Anal. **40** (1971),

312–328.

[24] M. Rabbani, A. Deep, and Deepmala, *On some generalized non-linear functional integral equations of two variables via measures of noncompactness and numerical method to solve it,* Math. Sci. **15** (2021), 317–324.

[25] A.N.V. Rao and C.P. Tsokos, *On a class of stochastic functional integral equations,* Colloq. Math. **35** (1976), 141–146.

[26] M.T. Rashed, *Numerical solutions of functional integral equations,* Appl. Math. Comput. **156** (2004), 507–512.

[27] P.K. Sahoo and P. Kannappan, *Introduction to Functional Equations*, CRC Press, 2011.

[28] L. Shu, X.B. Shu, Q. Zhu, and F. Xu *Existence and exponential stability of mild solutions for second-order differential neutral stochastic differential equations with random impulses*, J. Appl. Anal. Comput. **11** (2021), 59–80.

[29] S. Singh, B. Singh, K.S. Nisar, A. Hyder, and M. Zakarya, *Solvability for generalized nonlinear two dimensional functional integral equations via measure of noncompactness,* Adv. Differ. Equ. **372** (2021).

[30] S. Singh, B. Watson and P. Srivastava, *Fixed Point Theory and Best Approximation: The KKM-Map Principle*, Springer Science and Business Media, 2013.

[31] R. Subramaniam, K. Balachandran, and J.K. Kim, *Existence of random solutions of a general class of stochastic functional integral equations,* Stochastic Anal. Appl. **21** (2003), 1189–1205.

[32] J. Turo, *Existence and uniqueness of random solutions of nonlinear stochastic functional integral equations,* Acta Sci. Math. **44** (1982), 321–328.