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Quasilinear parabolic problems in the Lebsgue-Sobolev space with variable exponent and L^1 data

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Abstract

In this work, we study the existence of an initial boundary problem of a quasilinear parabolic problem with variable exponent and L^1-data of the type

$$\begin{cases} (b(u))_t - \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) + \lambda |u|^{p(x)-2} u = f(x,t,u) & \text{in} \quad Q = \Omega \times]0,T[, \\ u = 0 & \text{on} \quad \Sigma = \partial\Omega \times]0,T[, \\ b(u)(t=0) = b(u_0) & \text{in} \quad \Omega, \end{cases}$$

where $\lambda > 0$ and T is positive constant. The main contribution of our work is to prove the existence of a renormalized solution. The functional setting involves Lebesgue– Sobolev spaces with variable exponents.

Keywords: Quasilinear parabolic problems, variable exponent, truncations, renormalized solutions, L^1 data
2020 MSC: 35K59

1 Introduction

Variable-exponent Lebesgue and Sobolev spaces are the natural extensions of the classical constant exponent Lp-spaces. This kind of theory finds many applications, for example in nonlinear elastic mechanics (see [32]), electrorheological fluids (see [29]), or image restoration (see [22]). In recent years, there are a lot of interest in the study of various mathematical problems with variable exponent (see for example [12, 27, 24, 31] and references therein), the problems with variable exponent are interesting in applications and raise many difficult mathematical problems, some of the models leading to these problems of this type are the models of motion of electrorheological fluids, the mathematical models of stationary thermo-rheological viscous fows of non-Newtonian fluids and in the mathematical description of the processes filtration of an ideal barotropic gas through a porousmedium we refer the reader for example to [13].

In the classical case ($p(.) = 2$ or $p(.) = p$ (a constant)), we recall that the notion of renormalized solutions was introduced by Di Perna and Lions [14] in their study of the Boltzmann equation. This notion was then adapted to

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the study of some nonlinear elliptic problems with Dirichlet boundary conditions by Boccardo, Giachetti, Diaz, and Murat [10] and Lions and Murat (see Lions book on the Navier-Stokes equations [21]). For the corresponding parabolic equations with L 1 data, existence and uniqueness of renormalized solutions is established in Blanchard and Murat [7], see also Lions [21] for some time dependent problems motivated by the Navier-Stokes equations. For more recent results, see the papers [9, 23]. We also refer to the papers cited so far for a more complete account on the history of renormalized solutions and a long list of relevant references. Finally, let us mention that an equivalent notion of solutions, called entropy solutions, was introduced independently by Bénilan and al. [6].

In two papers (see [28, 5]) they have already studied the ellipsis problem corresponding to the $p(x)-$Laplacian equations and also the more general elliptic equations with variable exponents that include Order terms. In particular, we have generated an existential and uniqueness result for renormalization problem solutions with L1 and measure data.

It has been studied by many authors under various conditions on the data the existence and uniqueness of the renormalized solution for parabolic equations with L^1-data in the classical Sobolev spaces (see [3, 7, 25]). In Sobolev space with variable exponents, the authors [27] have proved the existence of renormalized solutions for a class of nonlinear parabolic systems with variable exponents and, for the corresponding parabolic equations with L^1 data. The main contribution of this work is evidence of the existence of renormalized solutions without the coercivity condition on nonlinearity that allows them to use Gagliardo-Nirenberg Theorem in proof, the authors in [12] have proved the existence and uniqueness of renormalized solution to nonlinear parabolic equations with variable exponents and, in [31] have proved an existence and uniqueness results renormalized solutions and entropy solutions for nonlinear parabolic equations with variable exponents and L^1 data. And moreover, we obtain the equivalence of renormalized solutions and entropy solutions. On the other hand in [24] S.Ouaro and all obtains existence and uniqueness of entropy solutions to nonlinear parabolic equation with variable exponent and L^1-data. The functional setting involves Lebesgue and Sobolev spaces with variable exponents.

Recently A. Aberqi and all in [1] studied the existence and the uniqueness of renormalized solution in the framework of Musielak Orlicz spaces. In 2021, Mohamed Badr Benboubker and all [5] provides the existence of renormalized solutions for our strongly nonlinear elliptic Neumann problem, the authors in [27] have proved the existence result of a renormalized solution to a class of nonlinear parabolic systems, which has a variable exponent Laplacian term and a Leary lions operator with data belong to $L1$. And in 2020, F. Souilah, and all [28] provides the existence of a renormalized solution for quasilinear parabolic problem with variable exponents and measure data.

In the present paper, we establish the existence of a renormalized solution for a class of a quasilinear parabolic problem of type

$$\begin{cases} (b(u))_t - \operatorname{div}\mathcal{A}(x,t,\nabla u) + \gamma(u) = f(x,t,u) & \text{in} \quad Q = \Omega\times]0,T[, \\ u = 0 & \text{on} \quad \Sigma = \partial\Omega\times]0,T[, \\ b(u)(t=0) = b(u_0) & \text{in} \quad \Omega. \end{cases} \tag{1.1}$$

In the problem (1.1), Ω be a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with lipshitz boundedary $\partial\Omega$ and $Q = \Omega \times]0,T[$ for any fixed T is a positive real number. Let $p : \overline{\Omega} \longrightarrow [1,+\infty)$ be a continuous real-valued function and let $p^- = \min_{x\in\overline{\Omega}} p(x)$ and $p^+ = \max_{x\in\overline{\Omega}} p(x)$ with $1 < p^- \leq p^+ < N$. Let $-\operatorname{div}\mathcal{A}(x,t,\nabla u) = -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is a Leary-Lions operator (see assumption (2.7)-(2.9)), respectively, $\gamma : \mathbb{R} \to \mathbb{R}$ with $\gamma(u) = \lambda |u|^{p(x)-2} u$ is a continuous increasing function for $\lambda > 0$ and $\gamma(0) = 0$ such that $\gamma(u)$ is assumed to belong to $L^1(Q)$. The function $f : Q \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function (see assumptions (2.11)-(2.12)). Finally the function $b : \mathbb{R} \to \mathbb{R}$ is a strictly increasing C^1-function lipchizienne with $b(0) = 0$ (see (2.10)), the data $f(x,t,u)$ and $b(u_0)$ is in $L^1(Q)$.

This paper is concerned with giving an accurate account of the existence of renormalized solutions for a large class of quasilinear parabolic problem of the type (1.1). We want to stress that, while the existence result follows a rather standard approximation argument, the proof of existence is not a direct extension of the result in classical sobolev space [17] due to the presence of the nonlinearity (it is non homogenous).

The paper is organized as follows: In section 2, we give some preliminaries and basic assumptions. In section 3, we give the definition of a renormalized solution of (1.1), and we establish (Theorem (3.3)) the existence of such a solution.

2 Assumptions on data and Preliminaries

2.1 Functional spaces

In this section, we first state some elementary results for the generalized Lebesgue spaces $L^{p(.)}(\Omega), W^{1,p(.)}(\Omega)$ and the generalized Lebesgue-Sobolev spaces $W_0^{1,p(.)}(\Omega)$ where Ω is an open subset of $\mathbb{R}^N$. We refer to Fan and Zhao [18] for further properties of Lebesgue Sobolev spaces with variable exponents. Let $p : \overline{\Omega} \longrightarrow [1, +\infty)$ be a continuous rel-valued function and let $p^- = \min_{x\in\overline{\Omega}} p(x)$, $p^+ = \max_{x\in\overline{\Omega}} p(x)$ with $1 < p(.) < N$. We denote the Lebesgue space with variable exponent $L^{p(.)}(\Omega)$ as the set of all measurable function $u : \Omega \longrightarrow \mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(u) = \int_\Omega |u|^{p(x)}\, dx; \tag{2.1}$$

is finite. If the exponent is bounded, i.e., if $p+ < +\infty$, then the expression

$$\|u\|_{L^{p(.)}(\Omega)} = \inf\left\{\mu > 0; \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}, \tag{2.2}$$

defines a norm in $L^{p(.)}(\Omega)$ called the Luxembourg norm. The space $(L^{p(.)}(\Omega); \|.\|_{p(.)})$ is a separable Banach space. Moreover, if $1 < p^- \le p+ < +\infty$, then $L^{p(.)}(\Omega)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{p\prime(.)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, for $x \in \Omega$. The following inequality will be used later:

$$\min\left\{\|u\|^{p^-}_{L^{p(.)}(\Omega)}, \|u\|^{p^+}_{L^{p(.)}(\Omega)}\right\} \le \int_\Omega |u(x)|^{p(x)}\, dx \le \max\left\{\|u\|^{p^-}_{L^{p(.)}(\Omega)}, \|u\|^{p^+}_{L^{p(.)}(\Omega)}\right\}. \tag{2.3}$$

Finally, we have the Holder type inequality

$$\left|\int_\Omega uv dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p^+}\right) \|u\|_{p(.)} \|v\|_{p'(.)}, \tag{2.4}$$

for all u$\in L^{p(.)}(\Omega)$ and v $\in L^{p'(.)}(\Omega)$. Let

$$W^{1,p(.)}(\Omega) = \left\{u \in L^{p(.)}(\Omega), |\nabla u| \in L^{p(.)}(\Omega)\right\}, \tag{2.5}$$

which is Banach space equiped with the following norm

$$\|u\|_{1,p(.)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)}. \tag{2.6}$$

The space $(W^{1,p(.)}(\Omega); \|.\|_{1,p(.)})$ is a separable and reflexive Banach space. An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(.)}$ of the space $L^{p(.)}(\Omega)$. We have the following result:

Proposition 2.1. [18] If $u_n, u \in L^{p(.)}(\Omega)$ and $p+ < +\infty$, the following properties hold true.

(i) $\|u\|_{p(.)} > 1 \Longrightarrow \|u\|^{p+}_{p(.)} < \rho_{p(.)}(u) < \|u\|^{p-}_{p(.)}$,

(ii) $\|u\|_{p(.)} < 1 \Longrightarrow \|u\|^{p-}_{p(.)} < \rho_{p(.)}(u) < \|u\|^{p+}_{p(.)}$,

(iii) $\|u\|_{p(.)} < 1$ (respectively $= 1, > 1$)$\Longleftrightarrow \rho_{p(.)}(u) < 1$ (respectively $= 1, > 1$),

(iv) $\|u_n\|_{p(.)} \longrightarrow 0$ (respectively $\longrightarrow +\infty$)$\Longleftrightarrow \rho_{p(.)}(u_n) < 1$(respectively $\longrightarrow +\infty$),

(v) $\rho_{p(.)}\left(\frac{u}{\|u\|_{p(.)}}\right) = 1$.

For a measurable function $u : \Omega \longrightarrow \mathbb{R}$, we introduce the following notation

$$\rho_{1,p(.)} = \int_\Omega |u|^{p(x)}\, dx + \int_\Omega |\nabla u|^{p(x)}\, dx.$$

Proposition 2.2. [18] If $u \in W^{1,p(.)}(\Omega)$ and $p+ < +\infty$, the following properties hold true.

(i)$\|u\|_{1,p(.)} > 1 \Longrightarrow \|u\|^{p+}_{1,p(.)} < \rho_{1,p(.)}(u) < \|u\|^{p-}_{1,p(.)}$,

(ii)$\|u\|_{1,p(.)} < 1 \Longrightarrow \|u\|^{p-}_{1,p(.)} < \rho_{1,p(.)}(u) < \|u\|^{p+}_{1,p(.)}$,

(iii)$\|u\|_{1,p(.)} < 1$ (respectively $= 1, > 1$)$\Longleftrightarrow \rho_{1,p(.)}(u) < 1$(respectively $= 1, > 1$).

Extending a variable exponent $p : \overline{\Omega} \longrightarrow [1, +\infty)$ to $\overline{Q} = [0, T] \times \overline{\Omega}$ by setting $p(x, t) = p(x)$ for all $(x, t) \in \overline{Q}$.

We may also consider the generalized Lebesgue space

$$L^{p(.)}(Q) = \left\{ u : Q \longrightarrow \mathbb{R}\text{measurable such that} \int_Q |u(x,t)|^{p(x)}\, d(x,t) < \infty \right\};$$

endowed with the norm

$$\|u\|_{L^{p(.)}(Q)} = \inf \left\{ \mu > 0; \int_Q \left| \frac{u(x,t)}{\mu} \right|^{p(x)} d(x,t) \leq 1 \right\};$$

which share the same properties as $L^{p(.)}(\Omega)$.

2.2 Assumptions

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 2)$, $T > 0$ is given and we set $Q = \Omega \times]0, T[$, and $\mathcal{A} : Q \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function such that for all $\xi, \eta \in \mathbb{R}^N, \xi \neq \eta$

$$\mathcal{A}(x, t, \xi).\xi \geqslant \alpha\, |\xi|^{p(x)}, \tag{2.7}$$

$$|\mathcal{A}(x, t, \xi)| \leqslant \beta \left[L(x, t) + |\xi|^{p(x)-1} \right], \tag{2.8}$$

$$(\mathcal{A}(x, t, , \xi) - \mathcal{A}(x, t, \eta)).(\xi - \eta) > 0, \tag{2.9}$$

where $1 < p- \leq p+ < +\infty$, α, β are positives constants and L is a nonnegative function in $L^{p'(.)}(Q)$ and $\gamma : \mathbb{R} \to \mathbb{R}$ is a continuous increasing function with $\gamma(0) = 0$. Let $b : \mathbb{R} \to \mathbb{R}$ is a strictly increasing C^1-function Lipschizienne with $b(0) = 0$ and for any ρ, τ are positives constants such that

$$\rho \leq b'(s) \leq \tau, \quad \forall s \in \mathbb{R}, \tag{2.10}$$

$f : Q \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that for any $\sigma > 0$, there exists $c \in L^{p'(.)}(Q)$ such that

$$|f(x, t, s)| \leq c(x, t) + \sigma |s|^{p(x)-1}, \tag{2.11}$$

for almost every $(x, t) \in (Q)$, $s \in \mathbb{R}$,

$$f(x, t, s)s \geq 0, \tag{2.12}$$

$$b(u_0) \in L^1(\Omega). \tag{2.13}$$

3 Main Results

In this section, we study the existence of renormalized solutions to problem (1.1).

Definition 3.1. Let $2 - \dfrac{1}{N+1} < p^- \leq p^+ < N$ and $b(u_0) \in L^1(\Omega)$. A measurable function u defined on Q is a renormalized solution of problem (1.1) if ,

$$T_k(u) \in L^{p^-}(]0, T[; W_0^{1,p(.)}(\Omega)) \text{ for any } k > 0\ ,\ \gamma(u), f(x, t, u) \in L^1(Q)\,, \tag{3.1}$$

$$\text{and } b(u) \in L^\infty\left(]0, T[; L^1(\Omega)\right) \cap L^{q^-}(]0, T[; W_0^{1,q(.)}(\Omega)), \tag{3.2}$$

for all continuous functions $q(x)$on $\overline{\Omega}$ satisfying $q(x) \in \left[1, p(x) - \frac{N}{N+1}\right)$ for all $x \in \overline{\Omega}$,

$$\lim_{n\to\infty} \int_{\{n\leq|u|\leq n+1\}} \mathcal{A}(x,t,\nabla u)\nabla u dx dt = 0, \tag{3.3}$$

and for any non negative real number k we denote by $T_k(r) = \min(k, \max(r, -k))$ the truncation function at height k and if, for every function $S \in W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has compact support on $\mathbb{R}$, we have,

$$(B_S(u))_t - div(\mathcal{A}(x,t,\nabla u)S'(u)) + S''(u)\mathcal{A}(x,t,\nabla u)\nabla u + \gamma(u)S'(u) = f(x,t;u)S'(u) \text{ in } \mathcal{D}'(Q), \tag{3.4}$$

$$B_S(u)(t=0) = S(b(u_0)) \text{ in } \Omega, \tag{3.5}$$

where $B_S(z) = \int_0^t b'(r)S'(r)dr$.

The following remarks are concerned with a few comments on definition (3.1).

Remark 3.2. Note that, all terms in (3.4) are well defined. Indeed, let $k > 0$ such that $\operatorname{supp}(S') \subset [K, K]$, we have $B_S(u)$ belongs to $L^\infty(Q)$ because

$$|B_S(u)| \leq \int_0^u |b'(r)S'(r)|dr \leq \tau \|S'\|_{L^\infty(\mathbb{R})};$$

and $S(u) = S(T_k(u)) \in L^{p-}(]0,T[; W_0^{1;p(.)}(\Omega))$ and $\frac{\partial B_S(u)}{\partial t} \in \mathcal{D}'(Q)$. The term $S'(u)\mathcal{A}(x,t,\nabla T_k(u))$ identifes with $S'(T_k(u))\mathcal{A}(x,t,\nabla(T_k(u)))$ a.e. in Q, where $u = T_k(u)$ in $\{|u| \leq k\}$, assumptions (2.8) imply that

$$|S'(T_k(u))\mathcal{A}(x,t,\nabla T_k(u))| \leq \beta \|S'\|_{L^\infty(\mathbb{R})} \left[L(x,t) + |\nabla(T_k(u))|^{p(x)-1}\right] \text{ a.e in } Q. \tag{3.6}$$

Using (2.8) and (3.1), it follows that $S'(u)\mathcal{A}(x,t,\nabla u) \in (L^{p'(.)}(Q))^N$. The term $S''(u)\mathcal{A}(x,t,\nabla u)\nabla(u)$ identifies with $S''(u)\mathcal{A}(t,x,\nabla(T_k(u)))\nabla T_k(u)$ and in view of (2.8), (3.1) and (3.6). We obtain $S''(u)\mathcal{A}(x,t,\nabla u)\nabla(u) \in L^1(Q)$ and $S'(u)\gamma(u) \in L^1(Q)$.

Finally $f(x,t,u)\ S'(u) = f(x,t,T_k(u))S'(u)$ a.e in Q. Since $|T_k(u)| \leq k$ and $S'(u) \in L^\infty(Q)$, $c(x,t) \in L^{p'(.)}(Q)$, we obtain from (2.11) that $f(x,t,T_k(u))S'(u) \in L^1(Q)$. We also have $\frac{\partial B_S(u)}{\partial t} \in L^{(p^-)'}(]0,T[; W^{-1,p'(.)}(\Omega)) + L^1(Q)$ and $B_S(u) \in L^{p^-}(]0,T[; W_0^{1,p(.)}(\Omega)) \cap L^\infty(Q)$, which implies that $B_S(u) \in C(]0,T[; L^1(\Omega))$.

Theorem 3.3. Let $b(u_0) \in L^1(\Omega)$, assume that (2.7)-(2.13) hold true, then there exists at least one renormalized solution u of problem (1.1) (in the sens of Definition (3.1)).

Proof .[Proof of Theorem (3.3)] The above theorem is to be proved in five steps.

- **Step 1: Approximate problem and a priori estimates.**

Let us define the following approximation of b and f for $\varepsilon > 0$ fixed

$$b_\varepsilon(r) = T_{\frac{1}{\varepsilon}}(b(r)) \text{ a.e in } \Omega \text{ for } \varepsilon > 0, \ \ \forall r \in \mathbb{R}, \tag{3.7}$$

$$b_\varepsilon(u_0^\varepsilon) \text{ are a sequence of } C_c^\infty(\Omega) \text{ functions such that} \tag{3.8}$$

$$b_\varepsilon(u_0^\varepsilon) \to b(u_0) \text{ in } L^1(\Omega) \text{ as } \varepsilon \text{ tends to } 0.$$

$$f^\varepsilon(x,t,r) = f(x,t,T_{\frac{1}{\varepsilon}}(r)), \tag{3.9}$$

in view of (2.11) and (2.12), there exist $c_\varepsilon \in L^{p'(.)}(Q)$ and $\sigma_\varepsilon > 0$ such that

$$|f^\varepsilon(x,t,s)| \leq c_\varepsilon(x,t) + \sigma_\varepsilon |s|^{p(x)-1}, \tag{3.10}$$

for almost every $(x,t) \in (Q)$, $s \in \mathbb{R}$,

$$f^\varepsilon(x,t,s)s \geq 0, \tag{3.11}$$

Let us now consider the approximate problem

$$(b_\varepsilon(u^\varepsilon))_t - div\mathcal{A}(x,t,\nabla u^\varepsilon) + \gamma(u^\varepsilon) = f^\varepsilon(x,t,u^\varepsilon) \text{ in } Q, \tag{3.12}$$

$$u^\varepsilon = 0 \text{ on }]0,T[\times\partial\Omega, \tag{3.13}$$

$$b_\varepsilon(u^\varepsilon)(t=0) = b_\varepsilon(u_0^\varepsilon) \text{ in } \Omega. \tag{3.14}$$

As a consequence, proving existence of a weak solution $u^\varepsilon \in L^{p^-}(]0,T[;W_0^{1,p(.)}(\Omega))$ of (3.12)-(3.14) is an easy task (see [20]). We choose $T_k(u^\varepsilon)\chi_{(0,t)}$ as a test function in (3.12), we have

$$\int_\Omega B_k^\varepsilon(u^\varepsilon)(t)dx + \int_0^t\int_\Omega \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(u^\varepsilon) + \int_0^t\int_\Omega \gamma(u^\varepsilon)T_k(u^\varepsilon)dxds = = \int_0^t\int_\Omega f^\varepsilon(x,t,u^\varepsilon)T_k(u^\varepsilon)dxds + \int_\Omega B_k^\varepsilon(u_0^\varepsilon)dx, \tag{3.15}$$

for almost every t in $(0,T)$, and where

$$B_k^\varepsilon(r) = \int_0^r T_k(s)\frac{\partial b_\varepsilon(s)}{\partial s}ds.$$

Under the definition of $B_k^\varepsilon(r)$ the inequality

$$0 \le \int_\Omega B_k^\varepsilon(u_0^\varepsilon)(t)dx \le k|b_\varepsilon(u_0^\varepsilon)|dx, \quad k>0.$$

Using (2.7), $f^\varepsilon(x,t,u^\varepsilon)T_k(u^\varepsilon) \ge 0$, and we have $\gamma(u^\varepsilon) = \lambda|u^\varepsilon|^{p(x)-1}u^\varepsilon \ge 0$ because $1 < p^- \le p(x) \le +\infty$ and the definition of $B_k^\varepsilon(r)$ in (3.15), we obtain

$$\int_\Omega B_k^\varepsilon(u^\varepsilon)(t)dx + \alpha\int_{E_k} |\nabla u^\varepsilon|^{p(x)}\,dxds \le k\,\|b_\varepsilon(u_0^\varepsilon)\|_{L^1(Q)}, \tag{3.16}$$

where $E_k = \{(x,t)\in Q : |u^\varepsilon| \le k\}$, using $B_k^\varepsilon(u^\varepsilon)(t) \ge 0$ and inequality (2.3) in (3.16) , we get

$$\alpha\int_0^T \min\left\{\|\nabla T_k(u^\varepsilon)\|_{L^{p(x)}(\Omega)}^{p-}, \|\nabla T_k(u^\varepsilon)\|_{L^{p(x)}(\Omega)}^{p+}\right\} \le \alpha\int_{\{(x,t)\in Q:\ |u^\varepsilon|\le k\}} |\nabla u^\varepsilon|^{p(x)}\,dxdt \le C, \tag{3.17}$$

then is $T_k(u^\varepsilon)$ is bounded in $L^{p-}(]0,T[;W_0^{1,p(x)}(\Omega))$. In the other hand, we obtain

$$k\int_{\{(t,x)\in Q:|u^\varepsilon|>k\}} |\gamma(u^\varepsilon)|\,dxdt \le k\,\|b_\varepsilon(u_0^\varepsilon)\|_{L^1(Q)}, \tag{3.18}$$

and

$$k\int_{\{(x,t)\in Q:|u^\varepsilon|>k\}} |f^\varepsilon(x,t,u^\varepsilon)|\,dxdt \le k\,\|b_\varepsilon(u_0^\varepsilon)\|_{L^1(Q)}. \tag{3.19}$$

Now, let $T_1(s-T_k(s)) = T_{k,1}(s)$ and we take $T_{k,1}(b_\varepsilon(u^\varepsilon))$ as test function in (3.12). Reasoning as above, using that $\nabla T_{k,1}(s) = \nabla s\chi_{\{k\le|s|\le k+1\}}$ and appling young's inequality, we obtain

$$\begin{aligned}\alpha\int_{\{k\le|b_\varepsilon(u^\varepsilon)|\le k+1\}} b'_\varepsilon(u^\varepsilon)\,|\nabla(u^\varepsilon)|^{p(x)}\,dxdt \quad &\le \quad k\int_{|b_\varepsilon(u_0^\varepsilon)|>k} |b_\varepsilon(u_0^\varepsilon)|\,dx + Ck\int_{|b_\varepsilon(u^\varepsilon)|>k} |\gamma(u^\varepsilon)|\,dxdt \\ &\quad + Ck\int_{|b_\varepsilon(u^\varepsilon)|>k} |f^\varepsilon(x,t,u^\varepsilon)|\,dxdt \le C_1,\end{aligned}$$

inequality (2.3) implies that

$$\int_0^T \alpha\chi_{\{k\leq|b_\varepsilon(u^\varepsilon)|\leq k+1\}} \min\left\{\|\nabla(b_\varepsilon(u^\varepsilon))\|^{p-}_{L^{p(x)}(\Omega)}, \|\nabla(b_\varepsilon(u^\varepsilon))\|^{p+}_{L^{p(x)}(\Omega)}\right\} \leq \alpha \int_{\{k\leq|b_\varepsilon(u^\varepsilon)|\leq k+1\}} b'_\varepsilon(u^\varepsilon)\,|\nabla(u^\varepsilon)|^{p(x)}\,dxdt \leq C_1. \tag{3.20}$$

On know that the property of $B_k^\varepsilon(u^\varepsilon), (B_k^\varepsilon(u^\varepsilon)\geq 0, B_k^\varepsilon(u^\varepsilon)) \geq \rho(|s|-1)$, we obtain

$$\begin{aligned}\int_\Omega |B_k^\varepsilon(u^\varepsilon)(t)|\,dx \leq k\int_\Omega |b_\varepsilon(u^\varepsilon)(t)|\,dx &\leq \rho\left(\int_\Omega |1|\,dx + k\,\|b_\varepsilon(u_0^\varepsilon)\|_{L^1(\Omega)}\right)\\ &\leq \rho\left(meas(\Omega) + k\,\|b_\varepsilon(u_0^\varepsilon)\|_{L^1(\Omega)}\right).\end{aligned} \tag{3.21}$$

From the estimation (3.17), (3.20), (3.21) and the properties of B_k^ε and $b_\varepsilon(u_0^\varepsilon)$, we deduce that

$$b_\varepsilon(u^\varepsilon) \text{ is bounded in } L^\infty\left(]0,T[;L^1(\Omega)\right); \tag{3.22}$$

and

$$b_\varepsilon(u^\varepsilon) \text{ is bounded in } L^{p-}(]0,T[\,;W_0^{1,p(x)}(\Omega)); \tag{3.23}$$

by Lemma 2.1 in [12] and by (3.20), (3.21) and $2-\dfrac{1}{N+1}<p(.)<N$, we obtain

$$b_\varepsilon(u^\varepsilon) \text{ is bounded in } L^{q-}(]0,T[\,;W_0^{1,q(x)}(\Omega)), \tag{3.24}$$

for all continuous variable exponents $q\in C(\overline{\Omega})$ satisfying $1\leq q(x)<\dfrac{N(p(x)-1)+p(x)}{N+1}$, for all $x\in\Omega$ and

$$T_k(u^\varepsilon) \text{ is bounded in } L^{p^-}\left(]0,T[;W_0^{1,p(.)}(\Omega)\right). \tag{3.25}$$

By (3.18) and (3.19), we may conclude that

$$\gamma(u^\varepsilon) \text{ is bounded in } L^1\left(]0,T[;L^1(\Omega)\right), \tag{3.26}$$

and

$$f^\varepsilon(x,t,u^\varepsilon) \text{ is bounded in } L^1\left(]0,T[;L^1(\Omega)\right), \tag{3.27}$$

independently of ε. Proceeding as in [7, 8] that for any $S\in W^{2,\infty}(\mathbb{R})$ such that S' is compact (supp $S'\subset[-k,k]$)

$$S(u^\varepsilon) \text{ is bounded in } L^{p-}\left(]0,T[;W_0^{1,p(.)}(\Omega)\right), \tag{3.28}$$

and

$$\left(S(u^\varepsilon)\right)_t \text{ is bounded in } L^1(Q)+L^{(p-)'}\left(]0,T[;W^{-1,p'(.)}(\Omega)\right). \tag{3.29}$$

In fact, as a consequence of (3.25), by Stampacchia's Theorem, we obtain (3.28). To show that (3.29) holds true, we multiply the equation (3.12) by $S'(u^\varepsilon)$ to obtain

$$\left(B_S(u^\varepsilon)\right)_t = div(S'(u^\varepsilon)\,\mathcal{A}(x,t,\nabla u^\varepsilon)) - \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla\left(S'(u^\varepsilon)\right) - \gamma(u^\varepsilon)\,S'(u^\varepsilon) + f^\varepsilon(x,t,u^\varepsilon)S'(u^\varepsilon) \text{ in } \mathcal{D}'(Q). \tag{3.30}$$

Since supp(S') and supp(S'') are both included in $[-k;k]; u^\varepsilon$ may be replaced by $T_k(u^\varepsilon)$ in $\{|u^\varepsilon|\leq k\}$. On the other hand we have

$$|S'(u^\varepsilon)\,\mathcal{A}(x,t,\nabla u^\varepsilon)| \leq \beta\,\|S'\|_{L^\infty}\left[L(x,t)+|\nabla T_k(u^\varepsilon)|^{p(x)-1}\right]. \tag{3.31}$$

As a consequence, each term in the right hand side of (3.30) is bounded either in $L^{(p-)'}\left(]0,T[;W^{-1,p'(.)}(\Omega)\right)$ or in $L^1(Q)$, and we then obtain (3.29).

Now we look for an estimate on a sort of energy at infinity of the approximating solutions. For any integer $n \geq 1$, consider the Lipschitz continuous function θ_n defined through

$$\theta_n(s) = T_{n+1}(s) - T_n(s) = \begin{cases} 0 & \text{if } |s| \leq n, \\ (|s|-n)\,sign(s) & \text{if } n \leq |s| \leq n+1, \\ sign(s) & \text{if } |s| \geq n. \end{cases}$$

Remark that $||\theta_n||_{L^\infty} \leq 1$ for any $n \geq 1$ and that $\theta_n(s) \to 0$, for any s when n tends to infinity. Using the admissible test function $\theta_n(u^\varepsilon)$ in (3.12) leads to

$$\int_\Omega \widetilde{\theta_n}(u^\varepsilon)(t)\,dx + \int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla(\theta_n(u^\varepsilon))\,dxdt + \int_Q \gamma(u^\varepsilon)\,\theta_n(u^\varepsilon)dxdt = \int_Q f^\varepsilon(x,t,u^\varepsilon)\theta_n(u^\varepsilon)dxdt + \int_\Omega \widetilde{\theta_n}(u_0^\varepsilon)\,dx, \tag{3.32}$$

where $\widetilde{\theta_n}(r)(t) = \int_0^r \theta_n(s)\frac{\partial b_\varepsilon(s)}{\partial s}ds$, for almost any t in $]0,T[$ and where $\widetilde{\theta_n}(r) = \int_0^r \theta_n(s)ds \geq 0$. Hence, dropping a nonnegative term

$$\begin{aligned}\int_{\{n\leq|u^\varepsilon|\leq n+1\}} \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon dxdt &\leq \int_Q \gamma(u^\varepsilon)\,\theta_n(u^\varepsilon)dxdt + \int_Q f^\varepsilon(x,t,u^\varepsilon)\theta_n(u^\varepsilon)dxdt + \int_\Omega \widetilde{\theta_n}(u_0^\varepsilon)\,dx \\ &\leq \int_{\{|u^\varepsilon|\geq n\}} |\gamma(u^\varepsilon)|\,dxdt + \int_{\{|u^\varepsilon|\geq n\}} |f^\varepsilon(x,t,u^\varepsilon)|\,dxdt + \int_{\{|b_\varepsilon(u_0^\varepsilon)|\geq n\}} |b_\varepsilon(u_0^\varepsilon)|\,dx.\end{aligned} \tag{3.33}$$

- **Step 2: The limit of the solution of the approximated problem.**

Arguing again as in [[7],[8],[9]] estimates (3.28) and (3.29) imply that, for a subsequence still indexed by ε,

$$u^\varepsilon \text{ converge almost every where to } u \text{ in } Q, \tag{3.34}$$

using (3.12) ,(3.25) and (3.31), we get

$$T_k(u^\varepsilon) \text{ converge weakly to } T_k(u) \text{ in } L^{p-}\left(]0,T[\,, W_0^{1,p(.)}(\Omega)\right), \tag{3.35}$$

$$\chi_{\{|u^\varepsilon|\leq k\}}\mathcal{A}(x,t,\nabla u^\varepsilon) \rightharpoonup \eta_k \text{ weakly in } \left(L^{p'(.)}(Q)\right)^N, \tag{3.36}$$

as ε tends to 0 for any $k > 0$ and any $n \geq 1$ and where for any $k > 0$, η_k belongs to $\left(L^{p'(.)}(Q)\right)^N$. Since $\gamma(u^\varepsilon)$ is a continuous incrassing function, from the monotone convergence theorem and (3.18) and by (3.34), we obtain that

$$\gamma(u^\varepsilon) \text{ converge weakly to } \gamma(u) \text{ in } L^1(Q). \tag{3.37}$$

We now establish that $b(u)$ belongs to $L^\infty\left(]0,T[\,;L^1(\Omega)\right)$. Indeed using (3.15) and $|B_k^\varepsilon(s)| \geq |s| - 1$ leads to

$$\int_\Omega |b_\varepsilon(u^\varepsilon)|(t)dx \leq meas(\Omega) + k\,\|f^\varepsilon(x,t,u^\varepsilon)\|_{L^1(Q)} + k\,\|\gamma(u^\varepsilon)\|_{L^1(Q)} + k\,\|b_\varepsilon(u_0^\varepsilon)\|_{L^1(\Omega)}.$$

Using (3.18) and (3.8),(3.19) , we have u belongs to $L^\infty\left(]0,T[\,;L^1(\Omega)\right)$. We are now in a position to exploit (3.33). Since u^ε is bounded in $L^\infty\left(]0,T[\,;L^1(\Omega)\right)$, we get

$$\lim_{n\to+\infty}\left(\sup_\varepsilon meas\{|u^\varepsilon| \geq n\}\right) = 0. \tag{3.38}$$

The equi-integrability of the sequence $f^\varepsilon(x,t,u^\varepsilon)$ in $L^1(Q)$. We shall now prove that $f^\varepsilon(x,t,u^\varepsilon)$ converges to $f(x,t,u)$ strongly in $L^1(Q)$, by using Vitali's theorem. Since $f^\varepsilon(x,t,u^\varepsilon) \to f(x,t,u)$ a.e in Q it suffices to prove that

$f^\varepsilon(x,t,u^\varepsilon)$ are equi-integrable in Q. Let $\delta > 0$ and $\mathbf{A}$ be a measurable subset belonging to $\Omega\times]0,T[$, we define the following sets

$$G_\delta = \{(x,t) \in Q : \ |u_n| \le \delta\}; \tag{3.39}$$
$$F_\delta = \{(x,t) \in Q : \ |u_n| > \delta\}. \tag{3.40}$$

Using the generalized Hölder's inequality and Poincaré inequality, we have

$$\int_{\mathbf{A}} |f^\varepsilon(x,t,u^\varepsilon)|\, dxdt = \int_{\mathbf{A}\cap G_\delta} |f^\varepsilon(x,t,u^\varepsilon)|\, dxdt + \int_{\mathbf{A}\cap F_\delta} |f^\varepsilon(x,t,u^\varepsilon)|\, dxdt,$$

therfore

$$\begin{aligned}
\int_{\mathbf{A}} |f^\varepsilon(x,t,u^\varepsilon)|\, dxdt \le & \int_{\mathbf{A}\cap G_\delta} \left(c_\varepsilon(x,t) + \sigma_\varepsilon\, |u_n|^{p(x)-1}\right) dxdt + \int_{\mathbf{A}\cap F_\delta} |f^\varepsilon(x,t,u^\varepsilon)|\, dxdt \\
\le & \int_{\mathbf{A}} c_\varepsilon(x,t) dxdt + \sigma_\varepsilon \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) (meas(\mathbf{Q})+1)^{\frac{1}{p^-}} \\
& \left(\int_{Q_T} |\nabla T_\delta(u^\varepsilon)|^{(p(x)-1)p'(x)}\, dxdt\right)^{\frac{1}{p'^-}} + \int_{\mathbf{A}\cap F_\delta} |f^\varepsilon(x,t,u^\varepsilon)|\, dxdt \\
\le & K_1 + C_2 \left(\frac{k}{\alpha}\, \|b_\varepsilon(u_0^\varepsilon)\|_{L^1(\Omega)}\right)^{\frac{1}{2}} + \int_{\mathbf{A}\cap F_\delta} \frac{1}{|u^\varepsilon|}\, |u^\varepsilon f^\varepsilon(x,t,u^\varepsilon)|\, dxdt \\
\le & K_2 + \frac{1}{\delta}\left(\frac{1}{p^-} + \frac{1}{p'^-}\right) \left(\int_{\mathbf{A}\cap F_\delta} |u^\varepsilon|^{p(x)}\, dxdt\right)^{\frac{1}{p^-}} \left(\int_{\mathbf{A}\cap F_\delta} |f^\varepsilon(x,t,u^\varepsilon)|^{p'(x)(p(x)-1)}\, dxdt\right)^{\frac{1}{p'^-}} \\
\to & 0 \text{ when } meas(\mathbf{A}) \to \mathbf{0}.
\end{aligned}$$

Which shows that $f^\varepsilon(x,t,u^\varepsilon)$ is equi-integrable. By using Vitali's theorem, we get

$$f^\varepsilon(x,t,u^\varepsilon) \to f(x,t,u) \text{ strongly in } L^1(Q). \tag{3.41}$$

Using (3.37), (3.41) and the equi-integrability of the sequence $|b_\varepsilon(u_0^\varepsilon)|$ in $L^1(\Omega)$, we deduce that

$$\lim_{n\to+\infty} \left(\sup_\varepsilon \int_{\{n\le|u^\varepsilon|\le n+1\}} \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon dxdt\right) = 0. \tag{3.42}$$

- **Step 4: Strong convergence.**

The specific time regularization of $T_k(u)$ (for fixed $k \ge 0$) is defined as follows. Let $(v_0^\mu)_\mu$ be a sequaence in $L^\infty(\Omega) \cap W_0^{1,p(.)}(\Omega)$ such that$\|v_0^\mu\|_{L^\infty(\Omega)} \le k, \ \forall \mu > 0$, and $v_0^\mu \to T_k(u_0)$ a.e in Ω with $\frac{1}{\mu}\, \|v_0^\mu\|_{L^{p(.)}(\Omega)} \to 0$ as $\mu \to +\infty$.

For fixed $k \ge 0$ and $\mu > 0$, let us consider the unique solution $T_k(u)_\mu \in L^\infty(\Omega) \cap L^{p-}\left(]0,T[; W_0^{1,p(.)}(\Omega)\right)$ of the monotone problem

$$\frac{\partial T_k(u)_\mu}{\partial t} + \mu\left(T_k(u)_\mu - T_k(u)\right) = 0 \text{ in } \mathcal{D}'(Q), \tag{3.43}$$
$$T_k(u)_\mu(t=0) = v_0^\mu. \tag{3.44}$$

The behavior of $T_k(u)_\mu$ as $\mu \to +\infty$ is investigated in [13] and we just recall here that (3.43)-(3.44) imply that

$$T_k(u)_\mu \to T_k(u) \text{ strongly in } L^{p-}\left(]0,T[; W_0^{1,p(.)}(\Omega)\right) \text{ a.e in } Q \text{ as } \mu \to +\infty, \tag{3.45}$$

with $\|T_k(u)_\mu\|_{L^\infty(\Omega)} \le k$, for any μ, and $\frac{\partial T_k(u)_\mu}{\partial t} \in L^{(p-)'}\left(]0,T[; W^{-1,p'(.)}(\Omega)\right)$. The main estimate is the following

Lemma 3.4. Let S be an increasing $C^{\infty}(\mathbb{R})-$ function such that $S(r)=r$ for $r\leq k$, and $suppS'$ is compact. Then

$$\liminf_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_0^T\left\langle\frac{\partial u^{\varepsilon}}{\partial t}, S'(u^{\varepsilon})\left(T_k(u^{\varepsilon})_{\mu}-T_k(u)\right)\right\rangle dt\geq 0,$$

where here $\langle .,.\rangle$ denotes the duality pairing between $L^1(\Omega)+W^{-1,p'(.)}(\Omega)$ and $L^{\infty}(\Omega)\cap W_0^{1,p(.)}(\Omega)$.

Proof . See[9], Lemma 1. □

- **Step 4:**

Here, we are to prove that the weak limit η_k and we prove the weak L^1 convergence of the "truncted" energy $\mathcal{A}(x,t,\nabla T_k(u^{\varepsilon}))$ as ε tends to 0. In order to show this result we recall the lemma below.

Lemma 3.5. The subsequence of u^{ε} defined in step 3 satisfies

$$\limsup_{\varepsilon\to 0}\int_Q\mathcal{A}(x,t,\nabla u^{\varepsilon})\nabla T_k(u^{\varepsilon})dxdt\leq\int_Q\eta_k\nabla T_k(u)dxdt, \tag{3.46}$$

$$\lim_{\varepsilon\to 0}\int_Q\left[\mathcal{A}\left(x,t,\nabla u^{\varepsilon}_{\chi_{\{|u^{\varepsilon}|\leq k\}}}\right)-\mathcal{A}\left(x,t,\nabla u_{\chi_{\{|u|\leq k\}}}\right)\right]\times\left[\nabla u^{\varepsilon}_{\chi_{\{|u^{\varepsilon}|\leq k\}}}-\nabla u_{\chi_{\{|u|\leq k\}}}\right]dxdt=0 \tag{3.47}$$

$\eta_k=\mathcal{A}\left(x,t,\nabla u_{\chi_{\{|u|\leq k\}}}\right)$ a.e in Q, for any $k\geq 0$, as ε tends to 0.

$$\mathcal{A}(x,t,\nabla u^{\varepsilon})\nabla T_k(u^{\varepsilon})\to\mathcal{A}(x,t,\nabla u)\nabla T_k(u)\text{ weakly in }L^1(Q). \tag{3.48}$$

Proof . Let us introduce a sequence of increasing $C^{\infty}(\mathbb{R})$-functions S_n such that, for any $n\geq 1$

$$\begin{cases} S_n(r)=r\text{ if }|\mathrm{r}|\leq n;\\ \operatorname{supp}(S_n')\subset[-(n+1),(n+1)],\\ \|S_n''\|_{L^{\infty}(\mathbb{R})}\leq 1.\end{cases} \tag{3.49}$$

For fixed $k\geq 0$, we consider the test function $S_n'(u^{\varepsilon})\left(T_k(u_{\varepsilon})-(T_k(u))_{\mu}\right)$ in (3.12), we use the definition (3.49) of S_n' and we definie $W^{\varepsilon}_{\mu}=T_k(u_{\varepsilon})-(T_k(u))_{\mu}$, we get

$$\int_0^T\left\langle(u^{\varepsilon})_t, S_n'(u^{\varepsilon})W^{\varepsilon}_{\mu}\right\rangle dt+\int_Q S_n'(u^{\varepsilon})\mathcal{A}(x,t,\nabla u^{\varepsilon})\nabla W^{\varepsilon}_{\mu}dxdt+\int_Q S_n''(u^{\varepsilon})\mathcal{A}(x,t,\nabla u^{\varepsilon})\nabla u^{\varepsilon}W^{\varepsilon}_{\mu}dxdt+\int_Q\gamma(u^{\varepsilon})S_n'(v^{\varepsilon})W^{\varepsilon}_{\mu}dxdt$$
$$=\int_Q f^{\varepsilon}(x,t,u^{\varepsilon})S_n'(u^{\varepsilon})W^{\varepsilon}_{\mu}dxdt. \tag{3.50}$$

Now we pass to the limit in (3.50) as $\varepsilon\to 0$, $\mu\to+\infty$, $n\to+\infty$ for k real number fixed. In order to perform this task, we prove below the following results for any $k\geq 0$:

$$\liminf_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_0^T\left\langle(u^{\varepsilon})_t, S_n'(u^{\varepsilon})W^{\varepsilon}_{\mu}\right\rangle dt\geq 0\text{ for any }n\geq k, \tag{3.51}$$

$$\lim_{n\to+\infty}\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q S_n''(u^{\varepsilon})\mathcal{A}(x,t,\nabla u^{\varepsilon})\nabla u^{\varepsilon}W^{\varepsilon}_{\mu}dxdt=0, \tag{3.52}$$

$$\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q \gamma(u^\varepsilon)S_n'(u^\varepsilon)W_\mu^\varepsilon dxdt = 0, \text{ for any } n \geq 1, \tag{3.53}$$

$$\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q f^\varepsilon(x,t,u^\varepsilon)S_n'(u^\varepsilon)W_\mu^\varepsilon dxdt = 0, \text{ for any } n \geq 1, \tag{3.54}$$

Proof .[Proof of (3.51)] In view of the definition W_μ^ε, we apply lemma (3.4) with $S = S_n$ for fixed $n \geq k$. As a consequence, (3.51) hold true. □

Proof .[Proof of (3.52)] For any $n \geq 1$ fixed, we have $supp(S_n'') \subset [-(n+1), -n] \cup [n, n+1]$, $\left\|W_\mu^\varepsilon\right\|_{L^\infty(Q)} \leq 2k$ and $\|S_n''\|_{L^\infty(\mathbb{R})} \leq 1$, we get

$$\left|\int_Q S_n''(u^\varepsilon)\mathcal{A}\left(x,t,\nabla u^\varepsilon\right)\nabla u^\varepsilon W_\mu^\varepsilon dxdt\right| \leq 2k \int_{\{n\leq|u^\varepsilon|\leq n+1\}} \mathcal{A}\left(x,t,\nabla u^\varepsilon\right)\nabla u^\varepsilon dxdt \tag{3.55}$$

for any $n \geq 1$, by (3.42) it possible to etablish (3.52) □

Proof .[Proof of (3.53)] For fixed $n \geq 1$ and in view (3.37) . Lebesgue's convergence theorem implies that for any $\mu > 0$ and any $n \geq 1$

$$\lim_{\varepsilon\to 0}\int_Q \gamma(u^\varepsilon)S_n'(u^\varepsilon)W_\mu^\varepsilon \, dxdt = \int_Q \gamma(u)S_n'(u)(T_k(u) - T_k\left(u\right)_\mu)dxdt. \tag{3.56}$$

Appealing now to (3.45) and passing to the limit as $\mu \to +\infty$ in (3.56) allows to conclude that (3.53) holds true. □

Proof .[Proof of (3.54)] By (3.9), (3.41) and Lebesgue's convergence theorem implies that for any $\mu > 0$ and any $n \geq 1$, it is possible to pass to the limit for $\varepsilon \to 0$

$$\lim_{\varepsilon\to 0}\int_Q f^\varepsilon(x,t,u^\varepsilon)S_n'(u^\varepsilon)W_\mu^\varepsilon \, dxdt = \int_Q f(x,t,u)S_n'(u)(T_k(u) - T_k\left(u\right)_\mu)dxdt,$$

using (3.45) permits to the limit as μ tends to $+\infty$ in the above equality to obtain (3.54). □

We now turn back to the proof of Lemma (3.5), due to (3.51)-(3.54), we are in a position to pass to the limit-sup when $\varepsilon \to 0$, then to the limit-sup when $\mu \to +\infty$ and then to the limit as $n \to +\infty$ in (3.50). Using the definition of W_μ^ε, we deduce that for any $k \geq 0$,

$$\lim_{n\to+\infty}\limsup_{\mu\to+\infty}\limsup_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)S_n'(u^\varepsilon)\nabla\left(T_k(u^\varepsilon) - T_k(u)_\mu\right)dxdt \leq 0.$$

Since $\mathcal{A}(x,t,\nabla u^\varepsilon)S_n'(u^\varepsilon)\nabla T_k(u^\varepsilon) = \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(u^\varepsilon)$ fo $k \leq n$, the above inequality implies that for $k \leq n$,

$$\limsup_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(u^\varepsilon)dxdt \leq \lim_{n\to+\infty}\limsup_{\mu\to+\infty}\limsup_{\varepsilon\to 0}\int_Q \mathcal{A}(t,x,\nabla u^\varepsilon)S_n'(u^\varepsilon)\nabla T_k(u)_\mu dxdt. \tag{3.57}$$

Due to (3.36), we have

$$\mathcal{A}(x,t,\nabla u^\varepsilon)S_n'(u^\varepsilon) \to \eta_{n+1}S_n'(u) \text{ weakly in } \left(L^{p'(\cdot)}\left(Q\right)\right)^N \text{ as } \varepsilon \to 0$$

and the strong convergence of $T_k(u)_\mu$ to $T_k(u)$ in $L^{p^-}(]0,T[;W_0^{1,p}(\Omega))$ as $\mu \to +\infty$, we get

$$\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)S_n'(u^\varepsilon)\nabla T_k(u)_\mu dxdt = \int_Q S_n'(u)\eta_{n+1}\nabla T_k(u)dxdt = \int_Q \eta_{n+1}\nabla T_k(u)dxdt, \tag{3.58}$$

as soon as $k \leq n$, since $S_n'(s) = 1$ for $|s| \leq n$. Now, for $k \leq n$, we have

$$S_n'(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)_{\chi_{\{|u^\varepsilon|\leq k\}}} = \mathcal{A}(x,t,\nabla u^\varepsilon)_{\chi_{\{|u^\varepsilon|\leq k\}}} \text{ a.e in } Q.$$

Letting $\varepsilon \to 0$, we obtain

$$\eta_{n+1}\chi_{\{|u|\leq k\}} = \eta_k \chi_{\{|u|\leq k\}} \text{ a.e in } Q - \{|u| = k\} \text{ for } k \leq n.$$

Recalling (3.57) and (3.58) allows to conclude that (3.46) holds true. □

Proof .[Proof of (3.47)] Let $k \geq 0$ be fixed. We use the monotone character (2.9) of $\mathcal{A}(x,t,\xi)$ with respest to ξ, we obtain

$$I^\varepsilon = \int_Q \left(\mathcal{A}(x,t,\nabla u^\varepsilon \chi_{\{|u^\varepsilon|\leq k\}}) - \mathcal{A}(x,t,\nabla u \chi_{\{|u|\leq k\}})\right)\left(\nabla u^\varepsilon \chi_{\{|u^\varepsilon|\leq k\}} - \nabla u \chi_{\{|u|\leq k\}}\right) dxdt \geq 0. \tag{3.59}$$

Inequality (3.59) is split into $I^\varepsilon = I_1^\varepsilon + I_2^\varepsilon + I_2^\varepsilon$ where

$$\begin{aligned} I_1^\varepsilon &= \int_Q \mathcal{A}(x,t,\nabla u^\varepsilon \chi_{\{|u^\varepsilon|\leq k\}})\nabla u^\varepsilon \chi_{\{|u^\varepsilon|\leq k\}} dxdt, \\ I_2^\varepsilon &= -\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon \chi_{\{|u^\varepsilon|\leq k\}})\nabla u \chi_{\{|u|\leq k\}} dxdt, \\ I_3^\varepsilon &= -\int_Q \mathcal{A}(x,t,\nabla u \chi_{\{|u|\leq k\}})\left(\nabla u^\varepsilon \chi_{\{|u^\varepsilon|\leq k\}} - \nabla u \chi_{\{|u|\leq k\}}\right) dxdt. \end{aligned}$$

We pass to the limit-sup as $\varepsilon \to 0$ in I_1^ε, I_2^ε and I_3^ε. Let us remark that we have $u^\varepsilon = T_k(u^\varepsilon)$ and $\nabla u^\varepsilon \chi_{\{|u^\varepsilon|\leq k\}} = \nabla T_k(u^\varepsilon)$ a.e in Q, and we can assume that k is such that $\chi_{\{|u^\varepsilon|\leq k\}}$ almost everywhere converges to $\chi_{\{|u|\leq k\}}$(in fact this is true for almost every k, see Lemma 3.2 in [11]). Using (3.46), we obtain

$$\begin{aligned} \lim_{\varepsilon\to 0} I_1^\varepsilon &= \lim_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(u^\varepsilon) dxdt \\ &\leq \int_Q \eta_k \nabla T_k(u) dxdt. \end{aligned} \tag{3.60}$$

In view of (3.35) and (3.36), we have

$$\begin{aligned} \lim_{\varepsilon\to 0} I_2^\varepsilon &= -\lim_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon \chi_{\{|u^\varepsilon|\leq k\}})\left(\nabla T_k(u)\right) dxdt \\ &= -\int_Q \eta_k \left(\nabla T_k(u)\right) dxdt. \end{aligned} \tag{3.61}$$

As a consequence of (3.35), we have for all $k > 0$

$$\lim_{\varepsilon\to 0} I_3^\varepsilon = -\int_Q \mathcal{A}(x,t,\nabla u \chi_{\{|u|\leq k\}})\left(\nabla T_k(u^\varepsilon) - \nabla T_k(u)\right) dxdt = 0. \tag{3.62}$$

Taking the limit-sup as $\varepsilon \to 0$ in (3.59) and using (3.60), (3.61) and (3.62) show that (3.47) holds true. □

Proof .[Proof of (3.48)] Using (3.47) and the usual Minty argument applies it follows that (3.48) holds true. □

- **Step 5:**

In this step we prove that u satisfies (3.3), (3.4) and (3.5) . For any fixed $n \leq 0$ one has

$$\int_{\{n\leq|u^\varepsilon|\leq n+1\}} \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon dxdt = \int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_{n+1}(u^\varepsilon) dxdt - \int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_n(u^\varepsilon) dxdt.$$

According to (3.36) and (3.48) one is at liberty to pass to the limit as ε tends to 0 for fixed $n \geq 1$ and to obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\{n\leq|u^\varepsilon|\leq n+1\}}\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon dxdt &= \int_Q \mathcal{A}(x,t,\nabla u)\nabla T_{n+1}(u)dxdt - \int_Q \mathcal{A}(x,t,\nabla u)\nabla T_n(u)dxdt\\ &= \int_{\{n\leq|u^\varepsilon|\leq n+1\}}\mathcal{A}(x,t,\nabla u)\nabla u dxdt. \end{aligned}\tag{3.63}$$

Taking that limit as n tends to $+\infty$ in (3.63) and using the estimate (3.42), that u satisfies (3.3).

Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact. Let k be a positive real number such that $\operatorname{supp}(S') \subset [-k,k]$. Multiplying of that approximate equation (3.12) by $S'(u^\varepsilon)$ leads to

$$(B_S(u^\varepsilon))_t - div(S'(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)) + S''(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla(u^\varepsilon) + \gamma(u^\varepsilon)S'(u^\varepsilon) = f^\varepsilon(x,t,u^\varepsilon)S'(u^\varepsilon) \text{ in } \mathcal{D}'(Q). \tag{3.64}$$

In what follows we pass to the limit as ε tends to 0 in each term of (3.64). Since S is bounded, and $S(u^\varepsilon)$ converges to $S(u)$ a.e in Q and in $L^\infty(Q)$ *-weak, then $(S(u^\varepsilon))_t$ converges to $(S(u^\varepsilon))_t$ in $\mathcal{D}'(Q)$ as ε tends to 0. Since $\operatorname{supp}(S') \subset [-k,k]$, we have $S'(u^\varepsilon)\mathcal{A}(t,x,\nabla u^\varepsilon) = S'(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\chi_{\{|u^\varepsilon|\leq k\}}$ a.e in Q. The pointwise convergence of u^ε to u as ε tends to 0, the bounded character of S and (3.48) of Lemma(3.5) imply that $S'(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)$ converges to $S'(u)\mathcal{A}(x,t,\nabla u)$ weakly in $\left(L^{p'(\cdot)}(Q)\right)^N$ as ε tends to 0, because $S'(u) = 0$ for $|u| \geq k$ a.e in Q. The pointwise convergence of u^ε to u, the bounded character of S', S'' and (3.48) of Lemma (3.5) allow to conclude that

$$S''(u^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(u^\varepsilon) \to S''(u)\mathcal{A}(x,t,\nabla u)\nabla T_k(u) \text{ weakly in } L^1(Q)$$

as $\varepsilon \to 0$. We use (3.37) we obtain that $\gamma(u^\varepsilon)S'(u^\varepsilon)$ converges to $\gamma(u)S'(u)$ in $L^1(Q)$, and we use (3.9), (3.35) and we obtain that $f^\varepsilon(x,t,u^\varepsilon)S'(u^\varepsilon)$ converges to $f(x,t,u)S'(u)$ in $L^1(Q)$. As a consequence of the above convergence result, we are in a position to pass to the limit as ε tends to 0 in equation (3.64) and to conclude that u satisfies (3.4). It remains to show that $S(u)$ satisfies the initial condition (3.5). To this end, firstly remark that, S being bounded, $S(u^\varepsilon)$ is bounded in $L^\infty(Q)$, $B_S(u^\varepsilon)$ is bounded in $L^\infty(Q)$. Secondly, (3.64) and the above considerations on the behavior of the terms of this equation show that $\frac{\partial B_S(u^\varepsilon)}{\partial t}$ is bounded in $L^1(Q) + L^{(p-)'}(]0,T[;W^{-1,p'(\cdot)}(\Omega))$. As a consequence, an Aubin's type lemma ([26], Corollary 4) implies that $B_S(u^\varepsilon)$ lies in a compact set of $C(]0,T[;L^1(\Omega))$. It follows that, on the one hand, $B_S(u^\varepsilon)(t=0)$ converges to $B_S(u)(t=0)$ strongly in $L^1(\Omega)$ Due to(3.8), we conclude that (3.5) holds true. As a conclusion of **Step 3** and **Step 5**, the proof of Theorem (3.3) is complete. □

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