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A note on mj-clean rings

Mehrdad Esfandiar[a], Hamid Haj Seyyed javadi[b,*], Ahmad Moussavi[c]

[a]*Department of Mathematics, Shahed University, Tehran, Iran*

[b]*Department of Computer Engineering, Shahed University, Tehran, Iran*

[c]*Department of Pure Mathematics, Tarbiat Modares, Tehran,Iran*

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Abstract

In this paper, we examine the notions of mj-clean ring and strongly mj-clean ring. And we will provide some of its basic properties. We examine the relationship of mj-clean ring with m-clean ring and j-clean ring. We prove that R is strongly mj-clean ring if and only if $M_n(R)$ is strongly mj-clean ring. We prove that mj-clean ring is Dedekind-finite; i.e., $ab = 1$ implies that $ba = 1$.

Keywords: mj-clean ring, strongly mj-clean ring, Dedekind finite
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1 Introduction

The notion of clean ring was first titled by W.K. Nicolson in his study of "Lifting idempotents and exchange rings" in [10]. A ring is clean if each element can be written as a sum of a unit and an idempotent element. A ring is strongly clean if each of its elements can be written as a sum of an idempotent and a unit which are commutative. After the introduction of clean rings by Nicholson, many authors introduced new constructions of clean rings, includings j-clean rings and m-clean rings [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13]. In this paper, we examine the notions of mj-clean ring and strongly mj-clean ring. Throughout this paper all rings are associative with unity. We denote the Jacobson radical of the ring R by $radR$. The right and left annihilators of a set $X \subseteq R$ are denoted by $r(X)$ and $l(X)$ respectively.We denote the group of units of the ring R by $U(R)$. An element x of a ring R is j-clean if $x = e + w$ where $e^2 = e \in R$ and $w \in radR$; if further $ew = we$, the element x is called strongly j-clean. The ring R is called j-clean (strongly j-clean) if each of its element is j-clean (strongly j-clean). Let $m \geqslant 2$ be a positive integer. An element x of a ring R is m-clean if $x = e + w$, where w is a unit and e is m-potent (that is $e^m = e$) element of R; if further $ew = we$, the element x is called strongly m-clean. The ring R is called m-clean (strongly m-clean) if each of its element is m-clean (strongly m-clean).

2 mj-clean ring

Definition 2.1. Let $m \geqslant 2$ be a positive integer. An element x of a ring R is said to be mj-clean if x can be written as $x = e + w$, where $w \in radR$ and e is m-potent element of R; if $ew = we$, in this case x is called strongly mj-clean.

*Corresponding author
Email addresses: mehrdad.esfandiar@shahed.ac.ir (Mehrdad Esfandiar), h.s.javadi@shahed.ac.ir (Hamid Haj Seyyed javadi), moussavi.a@modares.ac.ir (Ahmad Moussavi)

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A ring R is called mj-clean (strongly mj-clean) if every elements of R is mj-clean (strongly mj-clean).

2.1 Example

1. It is clear from definition of mj-clean ring, that j-clean ring is an mj-clean ring.
2. Let R be a ring in which every element is m-potent, in this case R is mj-clean ring.
3. Any quotient of a strongly mj-clean ring is strongly mj-clean.
4. Any direct product of strongly mj-clean ring is strongly mj-clean.

Lemma 2.2. Homomorphic image of strongly mj-clean rings is strongly mj-clean.

Proof . Let $\phi : R \to S$ be a ring epimorphism, where R is a strongly mj-clean ring. For any $s \in R$, there exist $r \in R$ such that $\phi(r) = s$. since R is strongly mj-clean, we can write $r = e + w$, where $w \in radR$, e is an m-potent element of R and $ew = we$. Now $\phi(r) = \phi(e + w) = \phi(e) + \phi(w)$ and $\phi(e) = \phi(e^m) = \phi(e)^m$, and for any $s' \in S$, there exist $r' \in R$ such that $\phi(r') = s'$, then $1 - s'\phi(w) = \phi(1) - \phi(r')\phi(w) = \phi(1 - r'w) \in U(S)$, so $\phi(w) \in radS$. □

Proposition 2.3. Let R be a ring. The following are equivalent:

1. R is m-potent.
2. R is strongly mj-clean and $radR = 0$

Proof . (1) $\Longrightarrow$ (2) Clearly, R is strongly mj-clean. For any $x \in radR$, $x^m = x$, $x - x^m = 0 \Longrightarrow x(1 - x^{m-1}) = 0$ that $(1 - x^{m-1}) \in U(R)$ and so $x = 0$. This implies that $radR = 0$.
(2) $\Longrightarrow$ (1) is trivial. □

Proposition 2.4. Let R be a ring. The following are equivalent:

1. R is mj-clean.
2. $\frac{R}{radR}$ is m-potent.

Proof . The proof is clear. □

Corollary 2.5. Let R be a local ring. The following are equivalent:

1. R is strongly mj-clean.
2. $\frac{R}{radR} \cong \mathbb{Z}_p$

Proof . The proof is clear. □

Theorem 2.6. Let R be a ring. And $x \in R$ be a mj-clean, then $1 + x$ is m-clean.

Proof . Let $x \in R$ be mj-clean. There exist an m-potent $e \in R$ and $w \in radR$ such that $x = e + w$. Hence, $1 + x = e + 1 + w$. We see $1 + w \in U(R)$. Thus, $1 + x \in R$ is m-clean. □

Theorem 2.7. Let R be a mj-clean ring. Then R is m-clean ring.

Proof . For every $x \in R$, $x - 1$ is mj-clean. Then there exist an m-potent $e \in R$ and a $w \in radR$ such that $x - 1 = e + w$. Hence $x = 1 + x - 1 = e + 1 + w$. We see $1 + w \in U(R)$. Thus, R is m-clean ring. □

Note: The upper theorem reverse is not true. For example, suppose $\mathbb{Q}$ is the ring of rational numbers. Then $\mathbb{Q}$ is m-clean ring, but $\mathbb{Q}$ is not mj-clean ring.

Lemma 2.8. Let R be a ring, and let $x = e + w$ be a strongly mj-clean decomposition of x in R. Then $l(x) \subseteq l(e)$ and $r(x) \subseteq r(e)$.

Proof . Let $r \in l(x)$, then $rx = 0$. Write $x = e + w$ that $e^m = e$, $w \in radR$, and $ew = we$. Then $re = -rw$; hence $re = -rwe^{m-1}$. It follows that $re(1+e^{m-2}w) = 0$, where $1+e^{m-2}w \in U(R)$, and so $re = 0$. that is $r \in l(e)$. Therefor, $l(x) \subseteq l(e)$. A similar argument shows that $r(x) \subseteq r(e)$. □

Theorem 2.9. Let R be a ring, and let $e \in R$ be an m-potent. Then $x \in e^{m-1}Re^{m-1}$ is strongly mj-clean in R if and only if x is strongly mj-clean in $e^{m-1}Re^{m-1}$.

Proof . At first we note that e^{m-1} is an idempotent element of R because e is an m-potent of R. If $r \in rad(e^{m-1}Re^{m-1})$. Then $r \in rad(R)$. suppose that $x = f + w$, that $f^m = f \in e^{m-1}Re^{m-1}$, $w \in rad(e^{m-1}Re^{m-1})$ and $fw = wf$. Obviously, $w \in rad(e^{m-1}Re^{m-1}) \subseteq radR$. Hence, $x \in e^{m-1}Re^{m-1}$ is strongly mj-clean in R.

Conversely, suppose $x \in R$, $x = f + w$, $f^m = f$, $w \in radR$, $fw = wf$. As $x \in e^{m-1}Re^{m-1}$, we see that:

$$\begin{aligned} 1 - e^{m-1} &\in l(x) \cap r(x) \\ &\subseteq l(f) \cap r(f) \\ &= R(1 - f^{m-1}) \cap (1 - f^{m-1})R \\ &= (1 - f^{m-1})R(1 - f^{m-1}). \end{aligned}$$

Hence, $f^{m-1}(1 - e^{m-1}) = 0 = (1 - e^{m-1})f^{m-1}$ and $f^{m-1} = f^{m-1}e^{m-1} = e^{m-1}f^{m-1}$ and $f = fe^{m-1} = e^{m-1}f$. We observe that $x = e^{m-1}fe^{m-1} + e^{m-1}we^{m-1}$ that $(e^{m-1}fe^{m-1})^m = e^{m-1}fe^{m-1}$ and so $e^{m-1}we^{m-1} \in rad(e^{m-1}Re^{m-1})$. Clearly, $e^{m-1}fe^{m-1}$ and $e^{m-1}we^{m-1}$ commute together. □

Theorem 2.10. Let $\{e_1, e_2, ..., e_n\}$ be set of m-potent elements of a ring R such that e_i^{m-1} and e_j^{m-1} are mutually orthogonal for all $i \neq j$, where $1 \leq i, j \geq n$. Suppose that $1 = e_1 + ... + e_n$ and each $e_i^{m-1}Re_i^{m-1}$ is strongly mj-clean for every $i = 1, ..., n$. Then R is strongly mj-clean.

Proof . By using upper theorem, the proof is clear. □

Theorem 2.11. If a ring R is strongly mj-clean then $M_n(R)$ is strongly mj-clean.

Proof . We observe that $I_n = E_{11} + ... + E_{nn}$, where I_n is the $n \times n$ identity matrix and E_{ii} is the $n \times n$ elementary matrix whose $(ii)''$th is 1 and all other entries are 0. We see that each E_{ii} is m-potent and E_{ii}^{m-1} are mutually orthogonal for all $i = 1, ..., n$. We also have $R \cong E_{ii}^{m-1}M_n(R)E_{ii}^{m-1}$. It is given that R is strongly mj-clean which implies that each $E_{ii}^{m-1}M_n(R)E_{ii}^{m-1}$ is strongly mj-clean. Consequently, by upper theorem, it follow that $M_n(R)$ is strongly mj-clean. □

Note: The upper theorem reverse is true. Let $M_n(R)$ be a strongly mj-clean. In this case, put $e = diag(1, 0, 0, .., 0) \in M_n(R)$. We will have $R \cong e^{m-1}M_n(R)e^{m-1}$. Therefor, R is strongly mj-clean.

Let R, S be two rings, and let M be an (R, S)-bimodule. In this case $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ will form a ring. We already know, $radT = \begin{pmatrix} radR & M \\ 0 & radS \end{pmatrix}$, $\frac{T}{radT} \cong \frac{R}{radR} \times \frac{S}{radS}$. Note here that, T is mj-clean ring if and only if R and S are mj-clean rings if and only if $\frac{R}{radR}$ and $\frac{S}{radS}$ are m-potent.

A Morita context is a 4-tuple $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$, where A, B are rings, ${}_AM_B$ and ${}_BN_A$ are bimodules, and there exist context products $M \times N \to A$ and $N \times M \to B$ written multiplicatively as $(w, z) \mapsto wz$ and $(z, w) \mapsto zw$, such that $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is an associative ring with the obvious matrix operation. The following lemmas have already been proved [12].

Lemma 2.12. Let $R := \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ be a Morita context. Then $radR = \begin{pmatrix} radA & M_0 \\ N_0 & radB \end{pmatrix}$, where $M_0 = \{x \in M : xN \subseteq radA\}$ and $N_0 = \{y \in N : yM \subseteq radB\}$.

Canonically, M/M_0 is an $(A/radA, B/radB)$-bimodule and N/N_0 is an $(B/radB, A/radA)$-bimodule, and this include a Morita context $\begin{pmatrix} A/radA & M/M_0 \\ N/N_0 & B/radB \end{pmatrix}$ where the context products are given by $(x+M_0)(y+N_0) = xy+radA$, $(y + N_0)(x + M_0) = yx + radB$ for all $x \in M$ and $y \in N$.

Lemma 2.13. Let $R := \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ be a Morita context, and let $\begin{pmatrix} A/radA & M/M_0 \\ N/N_0 & B/radB \end{pmatrix}$ be defined above. Then $R/radR \cong \begin{pmatrix} A/radA & M/M_0 \\ N/N_0 & B/radB \end{pmatrix}$.

Theorem 2.14. Let $R := \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ be a Morita context. Let A and B are mj-clean rings, $MN \subseteq radA$ and $NM \subseteq radB$. In This case R is mj-clean ring.

Proof . Because A and B are mj-clean rings, then $A/radA$ and $B/radB$ are m-potent. $MN \subseteq radA$ and $NM \subseteq radB$ gives $M = M_0$ and $N = N_0$. So, it follow that $R/radR \cong \begin{pmatrix} A/radA & M/M_0 \\ N/N_0 & B/radB \end{pmatrix}$. Thus, $A/radA$ and $B/radB$ being m-potent implies $R/radR$ is m-potent, and R is mj-clean ring. □

A ring R is said to be Dedekind-finite (or von Neumann-finite) if $ab = 1 \Longrightarrow ba = 1$.

Theorem 2.15. Let R be a mj-clean ring. Then R is Dedekind-finite.

Proof . Let $a, b \in R$, where $ab = 1$. Since R is mj-clean ring, we can write $a = e + w$, $b = f + u$, where $e^m = e$, $f^m = f$, $u, w \in radR$. Now $ab = (e+w)(f+u) = 1$, $ef + eu + wf + wu = 1$, therefore $ef = 1 - eu - wf - wu \in U(R)$. There exist $v \in U(R)$ such that $ef = v$. We see that $(1-e^{m-1})ef = (1-e^{m-1})v = 0$, $v = e^{m-1}v$, $e^{m-1} = 1$, $e \in U(R)$. Therefore $a = e + w \in U(R)$, then $ba = 1$ and R is Dedekind finite. □

Note: Let R be a mj-clean ring. And let $a = e + w$ be a mj-clean element in R, where $e^m = e$, $w \in radR$. Then $a \in U(R)$ if and only if $e \in U(R)$.

3 strongly $g(x)$-mj-clean ring

Definition 3.1. Let R be a ring, and $C(R)$ denote the center of a ring R. Let $g(x) \in C(R)[x]$ be a fixed polynomial. An element $r \in R$ is strongly $g(x)$-mj-clean if $r = e + w$ where $g(e) = 0$ and $w \in radR$ and $ew = we$. R is strongly $g(x)$-mj-clean ring if every element of R is strongly $g(x)$-mj-clean.

Also an element $r \in R$ is strongly $g(x)$-m-clean if $r = e + w$ where $g(e) = 0$, $w \in U(R)$ and $ew = we$. R is strongly $g(x)$-m-clean ring if every element of R is strongly $g(x)$-m-clean. Clearly, every strongly $g(x)$-mj-clean ring is strongly $g(x)$-m-clean ring.

For a ring R, R is strongly mj-clean if and only if R is strongly $(x^m - x)$-mj-clean.

Theorem 3.2. Let R be a ring and $a \in C(R)$. Then R is strongly mj-clean ring and $a \in U(R)$ if and only if R is strongly $x(x^{m-1} - a^{m-1})$-mj-clean ring.

Proof . Let $r \in R$. Since R is strongly mj-clean and $a \in U(R)$, $\frac{r}{a} = e+w$ where $e^m = e$, $w \in radR$ and $ew = we$. Then $r = ea+wa$ where $wa \in radR$, ea is a root of $x(x^{m-1}-a^{m-1})$, because $ea((ea)^{m-1}-a^{m-1}) = ea((e^{m-1}-1)a^{m-1}) = 0$. We also have $eawa = waea$.

Conversely, let R is strongly $x(x^{m-1}-a^{m-1})$-mj-clean ring. Since 1 is strongly $x(x^{m-1}-a^{m-1})$-mj-clean, $1 = s+w$ where $s(s^{m-1} - a^{m-1}) = 0$ and $w \in radR$ and $ws = sw$. Since $s = 1 - w \in U(R)$ and $s(s^{m-1} - a^{m-1}) = 0$ so $s^{m-1} = a^{m-1} \in U(R)$ therefore $a \in U(R)$. Let $r \in R$. Since R is strongly $x(x^{m-1} - a^{m-1})$-mj-clean ring, $ra = e + w$ where $e(e^{m-1} - a^{m-1}) = 0$, $w \in radR$ and $ew = we$. Thus, $r = \frac{e}{a} + \frac{w}{a}$ where $\frac{w}{a} \in radR$ and $(\frac{e}{a})^m = \frac{e^m}{a^m} = \frac{e(e^{m-1}-a^{m-1}+a^{m-1})}{a^m} = \frac{e}{a}$. So R is strongly mj-clean ring. □

Theorem 3.3. Let R be strongly $g(x)$-mj-clean ring and strongly $h(x)$-mj-clean ring where $g(x), h(x) \in C(R)[x]$. Then R is $g(x)h(x)$-mj-clean ring.

Proof . The proof is clear. □

Theorem 3.4. Let R be a strongly $x(x^{m-1} - a^{m-1})$-mj-clean ring with $a \in C(R)$. Then for any $e = e^m \in R$, $e^{m-1}Re^{m-1}$ is strongly $x(x^{m-1} - e^{m-1}a^{m-1})$-$mj$-clean ring.

Proof . R is strongly $x(x^{m-1} - a^{m-1})$-mj-clean ring if and only if R is strongly mj-clean ring and $a \in U(R)$. If R is strongly mj-clean ring, then $e^{m-1}Re^{m-1}$ is strongly mj-clean. Again $e^{m-1}Re^{m-1}$ is strongly $x(x^{m-1} - e^{m-1}a^{m-1})$-$mj$-clean ring, because $e^{m-1}a^{m-1} \in U(e^{m-1}Re^{m-1})$. □

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