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Some results on q-shift difference-differential polynomials sharing finite value

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Abstract

In this paper, we study the uniqueness of meromorphic functions with $q-$shift difference-differential polynomials $F = [P(f)\mathcal{L}(z, f)^s]^{(k)}$ and $G = [P(g)\mathcal{L}(z, g)^s]^{(k)}$, where $P(z)$ is a non-constant polynomial with degree n sharing a finite value. The results of this paper are an extension of the previous theorems given by Harina P. Waghamore and Rajeshwari S [19].

Keywords: Nevanlinna Theory, Sharing values, difference-differential polynomials, $q-$shift, Meromorphic functions.
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1 Introduction

In this article, by meromorphic functions, we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [8, 26, 23, 10]. Let E denote any set of positive real numbers of finite linear measure not necessarily the same at each occurrence. For a non-constant meromorphic function f, we denote by $T(r, f)$ the Nevanlinna characteristic of f and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ $(r \to \infty, r \notin E)$. We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$ and by $S(r)$ any quantity satisfying $S(r) = o\{T(r)\}$ $(r \to \infty, r \notin E)$.

We denote and define order of $f(z)$

$$\rho(f) = \lim_{r\to\infty} \sup \frac{\log T(r, f)}{\log r}.$$

If a non-constant meromorphic function $f(z)$ is of zero order, then $\rho(f) = 0$. Let f and g be two non-constant meromorphic functions. We say that f and g share the value a CM (counting multiplicities) if $f - a$ and $g - a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share the value a IM provided that $f - a$ and $g - a$ have the same zeros ignoring multiplicities.

Definition 1. [4] For a meromorphic function f and c, $q(\neq) \in \mathbb{C}$, let us now denote its q-shift $E_q f$ and q-difference operators $\Delta_q f$ respectively by $E_q f(z) = f(qz + c)$ and $\Delta_q f(z) = f(qz + c) - f(z)$.

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For further generalization of $\Delta_q f(z)$, we now define the q-difference operator of an meromorphic function f as as $\mathcal{L}(z,f) = b_1 f(qz+c) + b_0 f(z)$, where $b_1(\neq 0)$ and b_0 are complex constants. For $s \in \mathbb{N}$, let us define

$$\lambda_{b_0} = \begin{cases} 1, & if \quad b_0 \neq 0, \\ 0, & if \quad b_0 = 0, \end{cases}$$

In recent times, many mathematicians are working on difference equations, the difference product and the q-difference analogues the value distribution theory of entire and meromorphic functions in the complex plane (see [2, 3, 9, 15, 16, 17, 18]).

In 1959, Hayman [7] proved that $f^n f'$ takes every non-zero complex value infinitely often if $n \geq 3$. Yang and Hua [24] obtained some results about the uniqueness problems for entire functions. Since then the difference has become a subject of great interest (see [11, 12, 27, 28]).

Recently, the difference variant of the Nevanlinna theory has been established independently in [5],[6]. With the development of difference analogue of Nevanlinna theory, many authors gave attention to the uniqueness of difference and difference operator analogs of Nevanlinna theory. Halburd and Korhonen [5] established a difference analogue of the Logarithmic Derivative Lemma, and then applied it to prove a number of results on meromorphic solutions of complex difference equations.

In 2012, K. Liu, X. Liu and T. B. Cao [13] proved the following.

Theorem 1. Let f be a transcendental entire function of $\rho_2(f) < 1$. For $n \geq t(k+1)+1$, then $[P(f)f(z+c)]^{(k)} - \alpha(z)$ has infinitely many zeros.

Theorem 2. Let f be a transcendental entire function of $\rho_2(f) < 1$, not a periodic function with period c. If $n \geq (t+1)(k+1)+1$, then $[P(f)(\Delta_c f)^s]^{(k)} - \alpha(z)$ has infinitely many zeros.

Theorem 3. Let f be a transcendental meromorphic function of $\rho_2(f) < 1$. For $n \geq t(k+1)+5$, then $[P(f)f(z+c)]^{(k)} - \alpha(z)$ has infinitely many zeros.

Theorem 4. Let f be a transcendental meromorphic function of $\rho_2(f) < 1$. For $n \geq (t+2)(k+1)+3+s$, then $[P(f)(\Delta_c f)^s]^{(k)} - \alpha(z)$ has infinitely many zeros.

Theorem 5. Let $f(z)$ and $g(z)$ be transcendental entire functions of $\rho_2(f) < 1$, $n \geq 2k+m+6$. If $[f^n(f^m-1)f(z+c)]^{(k)}$ and $[g^n(g^m-1)g(z+c)]^{(k)}$ share the value 1 CM, then $f = tg$, where $t^{n+1} = t^m = 1$.

Theorem 6. The conclusion of Theorem 1.5 is also valid, if $n \geq 5k+4m+12$. If $[f^n(f^m-1)f(z+c)]^{(k)}$ and $[g^n(g^m-1)g(z+c)]^{(k)}$ share the value 1 IM.

In 2013, Harina P. Waghamore and Tanuja A [20] extend Theorem 5 and Theorem 6 to meromorphic functions.

Theorem 7. Let f and g be a transcendental meromorphic function with zero order. If $n \geq 4k+m+8$, $[f^n(f^m-1)f(qz+c)]^{(k)}$ and $[g^n(g^m-1)g(qz+c)]^{(k)}$ share the 1 CM, then $f = tg$, where $t^{n+1} = t^m = 1$.

Theorem 8. Let f and g be a transcendental meromorphic function with zero order. If $n \geq 5k+4m+17$, $[f^n(f^m-1)f(qz+c)]^{(k)}$ and $[g^n(g^m-1)g(qz+c)]^{(k)}$ share the 1 IM, then $f = tg$, where $t^{n+1} = t^m = 1$.

In 2016, Harina P. Waghamore and Rajeshwari S [19] we extend Theorem 7 and Theorem 8 to difference polynomials and obtain the following results.

Theorem 9. Let f and g be a transcendental meromorphic functions with zero order. If $n \geq 4k+8$, $[P(f)f(qz+c)]^{(k)}$ and $[P(g)g(qz+c)]^{(k)}$ share the 1 CM, then:

(i) $f \equiv tg$ for a constant t such that $t^d = 1$.

(ii) f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(w_1,w_2) = P(w_1)w_1(qz+c) - P(w_2)w_2(qz+c)$.

Theorem 10. Let f and g be a transcendental meromorphic functions with zero order. If $n \geq 10k+14$, $[P(f)f(qz+c)]^{(k)}$ and $[P(g)g(qz+c)]^{(k)}$ share the 1 IM, then the conclusion of theorem 1.9 still holds.

In this paper, we replace the term $f(qz+c)$ and $g(qz+c)$ in Theorem 9 and Theorem10 and obtained the following results.

Theorem 11. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of zero order, q_j and c_j are complex constants, $q_j \neq 0$ $(j = 1$ to $d)$ and let k, n be positive integers. Let $P(z) = a_n z^n + a_{n-1} z^{n-1} ... + a_1 z + a_0$ be a non-constant polynomial with degree n. If $n \geq 4k + 2\lambda_{b_0} + s(2k+1) + 4$, $[P(f)\mathcal{L}(z,f)^s)]^{(k)}$ and $[P(g)\mathcal{L}(z,g)^s)]^{(k)}$ share the 1 CM, then:

(i) $f(z) = tg(z)$ for a constant t such that such that $t^d = 1$, where $d = GCD(\lambda_0 + \lambda_1 +, \ldots, +\lambda_n)$,

(ii) $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f,g) \equiv 0$, where

$$R(w_1, w_2) = P(w_1)\mathcal{L}(z, w_1)^s - P(w_2)\mathcal{L}(z, w_2)^s.$$

Theorem 12. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of zero order, q_j and c_j are complex constants, $q_j \neq 0$ $(j = 1$ to $d)$ and let k, n be positive integers. Let $P(z) = a_n z^n + a_{n-1} z^{n-1} ... + a_1 z + a_0$ be a non-constant polynomial with degree n. If $n \geq 10k + 8\lambda_{b_0} + s(2k+1) + 4$, $[P(f)\mathcal{L}(z,f)^s)]^{(k)}$ and $[P(g)\mathcal{L}(z,g)^s]^{(k)}$ share the 1 IM, then the conclusion of Theorem 1.11 still holds.

Example 1. Let $f(z) = \sin(z)$ and $g(z) = \cos(z)$, $q = 1$, $k = 0$, $c = 2\pi$. Hence we have $n \geq 8\lambda_{b_0} + s + 4$ and $[P(f)\mathcal{L}(z,f)^s)]^{(k)}$=$[P(g)\mathcal{L}(z,g)^s)]^{(k)}$. Therefore $[P(f)\mathcal{L}(z,f)^s)]^{(k)}$ and $[P(g)\mathcal{L}(z,g)^s)]^{(k)}$ share 1 CM. Clearly, we get $f = tg$ for a constant t such that such that $t^d = 1$, where $d = GCD(\lambda_0 + \lambda_1 +, \ldots, +\lambda_n)$.

Example 2. Let $P(z) = (z-1)^6(z+1)^6 z^{11}$, $f(z) = sin(z)$, $g(z) = cos(z)$. Take $s = 1 = q$, $c = 2\pi$, $k = 0$ then it is easy to verify that, $[P(f)\mathcal{L}(z,f)^s)]^{(k)}$ and $[P(g)\mathcal{L}(z,g)^s)]^{(k)}$ share 1 CM. Here f and g satisfy the algebraic equation $R(f,g) = 0$, i.e.,

$$P(f)\mathcal{L}(z,f)^s - P(g)\mathcal{L}(z,g)^s = 0.$$

2 Lemmas

In this section, we summarize some lemmas, which will be used to prove our main results.

Lemma 1. [14] Let $f(z)$ be a non-constant zero order meromorphic function and let q, c be a nonzero complex number. Then on a set of logarithmic density 1, we have

$$m\left(r, \frac{f(qz+c)}{f(z)}\right) = S(r,f).$$

Lemma 2. [22] Let $f(z)$ be a non-constant meromorphic function of zero order and let q, c be two nonzero complex constants. Then on a set of logarithmic density 1, we have

$$N(r, f(qz+c)) = N(r,f) + S(r,f).$$

$$N\left(r, \frac{1}{f(qz+c)}\right) = N\left(r, \frac{1}{f}\right) + S(r,f).$$

Lemma 3. [22] Let $f(z)$ be a non-constant meromorphic function of zero order and let q, c be two nonzero complex constants. Then on a set of logarithmic density 1, we have

$$T(r, f(qz+c)) = T(r,f) + S(r,f).$$

Lemma 4. [25] Let f(z) be a non-constant meromorphic function, then

$$T(r, P_n(f)) = T(r,f) + S(r,f).$$

Lemma 5. [11] Let $f(z)$ be a non-constant meromorphic function, and let p, k be a positive integers. Then

$$T(r,f^{(k)}) \leq T(r,f) + k\overline{N}(r,f) + S(r,f),$$
$$N_p\left(r,\frac{1}{f^{(k)}}\right) \leq T(r,f^{(k)}) - T(r,f) + N_{p+k}\left(r,\frac{1}{f}\right) + S(r,f),$$
$$N_p\left(r,\frac{1}{f^{(k)}}\right) \leq k\overline{N}(r,f) + N_{p+k}\left(r,\frac{1}{f}\right) + S(r,f).$$

Lemma 6. [24] Let F and G be non-constant meromorphic functions. If F and G share 1 CM, then one of the following three cases holds:
(i) $\max\{T(r,F),T(r,G)\} \leq N_2\left(r,\frac{1}{F}\right) + N_2(r,F) + N_2\left(r,\frac{1}{G}\right) + N_2(r,G) + S(r,F) + S(r,G)$.
(ii) $F \equiv G$.
(iii) $F.G \equiv 1$.

Lemma 7. [21] Let F and G be non-constant meromorphic function sharing the value 1 IM. Let

$$H = \frac{F''}{F'} - 2\frac{F'}{F-1} - \frac{G''}{G'} + 2\frac{G'}{G-1}.$$

If $H \not\equiv 0$, then

$$\begin{aligned}T(r,F) + T(r,G) \leq& 2\left(N_2\left(r,\frac{1}{F}\right) + N_2(r,f) + N_2\left(r,\frac{1}{G}\right) + N_2(r,G)\right)\\ &+ 3\left(\overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}(r,G) + \overline{N}(r,\frac{1}{G})\right) + S(r,F) + S(r,G).\end{aligned}$$

Lemma 8. Let $f(z)$ be a transcendental meromorphic function of zero order and $F = P(f)\mathcal{L}(z,f)^s$, $q_j(\neq 0)$ are complex constants, n, d be a positive integers. Then

$$(n-s)T(r,f) + S(r,f) \leq T(r,F).$$

Proof . From first fundamental theorem, Lemma 4 and Lemma 1, we obtain

$$\begin{aligned}(n+1)T(r,f) =& T(r,P(f(z))) + S(r,f) \leq T\left(r,\frac{f(z)F}{\mathcal{L}(z,f)^s}\right) + S(r,f),\\ \leq& T(r,F) + T\left(r,\frac{\mathcal{L}(z,f)^s}{f(z)}\right) + S(r,f),\\ \leq& T(r,F) + m\left(r,\frac{\mathcal{L}(z,f)^s}{f(z)}\right) + N\left(r,\frac{\mathcal{L}(z,f)^s}{f(z)}\right) + S(r,f),\\ \leq& T(r,F) + (s+1)T(r,f) + S(r,f),\end{aligned}$$

Therefore, $(n-s)T(r,f) + S(r,f) \leq T(r,F)$ on a set of logarithmic density 1. □

Lemma 9. Let $f(z)$ and $g(z)$ be a transcendental meromorphic function of zero order. If $n \geq 2k + 2\lambda + (k+1)(1+d) + d + 2$ and

$$[P(f)\mathcal{L}(z,f)^s)]^{(k)} = [P(g)\mathcal{L}(z,g)^s)]^{(k)}. \tag{2.1}$$

Then

(i) $f(z) = tg(z)$ for a constant t such that such that $t^d = 1$, where $d = GCD(\lambda_0 + \lambda_1 +, \ldots, +\lambda_n)$,

(ii) $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f,g) \equiv 0$, where

$$R(w_1,w_2) = P(w_1)\mathcal{L}(z,w_1)^s - P(w_2)\mathcal{L}(z,w_2)^s.$$

Proof . From (2.1), we have

$$P(f)\mathcal{L}(z,f)^s = P(g)\mathcal{L}(z,g)^s + \alpha(z),$$

where $\alpha(z)$ is a polynomial of degree at most $k-1$. Suppose $\alpha(z)\equiv 0$, then we get

$$\frac{P(f)\mathcal{L}(z,f)^s}{\alpha(z)} = \frac{P(g)\mathcal{L}(z,g)^s}{\alpha(z)} + 1$$

Therefore from Lemma 8, and the second fundamental Theorem, we have

$$\begin{aligned}(n-s)T(r,f) \leq & T\left(r, \frac{P(f)\mathcal{L}(z,f)^s}{\alpha(z)}\right) + S(r,f)\\ \leq & \overline{N}\left(r, \frac{P(f)\mathcal{L}(z,f)^s}{\alpha(z)}\right) + \overline{N}\left(r, \frac{\alpha(z)}{P(f)\mathcal{L}(z,f)^s}\right) + \overline{N}\left(r, \frac{\alpha(z)}{P(g)\mathcal{L}(z,g)^s}\right) + S(r,f)\\ \leq & \overline{N}(r, P(f)\mathcal{L}(z,f)^s) + \overline{N}\left(r, \frac{1}{P(f)\mathcal{L}(z,f)^s}\right) + \overline{N}\left(r, \frac{1}{P(g)\mathcal{L}(z,g)^s}\right) + S(r,f),\\ \leq & \overline{N}(r,f) + sT(r,f) + T(r,f) + \lambda_{b_0}T(r,f) + T(r,g) + \lambda T(r,g) + S(r,f),\\ \leq & [\lambda_{b_0} + s + 2]T(r,f) + [1+\lambda_{b_0}]T(r,g) + S(r,f).\end{aligned} \tag{2.2}$$

Similarly,

$$(n-s)T(r,g) \leq [\lambda_{b_0} + s + 2]T(r,g) + [1+\lambda]T(r,f) + S(r,g). \tag{2.3}$$

From (2.2) and (2.3), we obtain

$$[n - 2\lambda_{b_0} - 2s - 3]\{T(r,f) + T(r,g)\} \leq S(r,f) + S(r,g).$$

This is a contradiction to $n > 2k + 2\lambda_{b_0} + (k+1)(1+s) + s + 2$. Therefore $\alpha(z)\equiv 0$, which implies that

$$P(f)\mathcal{L}(z,f)^s = P(g)\mathcal{L}(z,g)^s. \tag{2.4}$$

That is

$$(a_nf^n + a_{n-1}f^{n-1} + \cdots + a_1f + a_0)(\mathcal{L}(z,f)^s) = (a_ng^n + a_{n-1}g^{n-1} + \cdots + a_1g + a_0)(\mathcal{L}(z,g)^s)^{v_j}).$$

Letting $h(z) = \frac{f(z)}{g(z)}$, we consider the following cases

Case 1. If $h(z)$ is a constant then substituting $f(z) = h(z)g(z)$ in (2.4), we have

$$(a_n(gh)^n + a_{n-1}(gh)^{n-1} + \cdots + a_1(gh) + a_0)(\mathcal{L}(z,g)^s\mathcal{L}(z,g)) = (a_ng^n + a_{n-1}g^{n-1} + \cdots + a_1g + a_0)(\mathcal{L}(z,g)^s).$$

This implies that

$$\mathcal{L}(z,g)^s[a_ng^n(h^{n+\lambda} - 1) + a_{n-1}g^{n-1}(h^{n+\lambda-1} - 1) + \cdots + a_0(h^\lambda - 1)] = 0 \tag{2.5}$$

where a_n is non-zero complex constant and $\mathcal{L}(z,g)^s \not\equiv 0$, since $g(z)$ is non-constant meromorphic function, then from (2.5)

$$a_ng^n(h^{n+\lambda} - 1) + a_{n-1}g^{n-1}(h^{n+\lambda-1} - 1) + \cdots + a_0(h^\lambda - 1) = 0. \tag{2.6}$$

If $a_n(\not\equiv 0)$ and $a_{n-1} = a_{n-2} = \cdots = a_1 = a_0 = 0$ then from (2.6) and g is non-constant meromorphic function, we get $h^{n+\lambda} - 1 = 0$ implies $h^{n+\lambda} = 1$. If $a_n(\not\equiv 0)$ and there exist $a_i \neq 0$ $[i \in \{0,1,2,...,n\}]$. Suppose that $h^{n+\lambda} \neq 1$, from (2.6), we have $T(r,g) = S(r,g)$. Which is contradiction with transcendental function g. Then $h^{n+\lambda} = 1$, similar to this discussion we can see that $h^{n+\lambda} = 1$, where $a_j \not\equiv 0$, for some $j = 0,1,2,...,n$. Thus we have $f(z) = tg(z)$, for a constant t such that $t^d = 1$, where $d = GCD(\lambda_0 + \lambda_1 + \cdots + \lambda_n)$.

Case 2. Suppose $h(z)$ is not cnstant, then $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f(z), g(z)) \equiv 0$, where

$$R(w_1,w_2) = P(w_1)\mathcal{L}(z,w_1)^s - P(w_2)\mathcal{L}(z,w_2)^s.$$

□

3 Proof of the Theorems

Theorem 1.11

Proof . Let $F = [P(f)\mathcal{L}(z,f)^s]^{(k)}$ and $G = [P(g)\mathcal{L}(z,g)^s]^{(k)}$. Thus F and G share the value 1 CM. From Lemma 5 and f is a transcendental meromorphic function, then

$$T(r,F) \le T(r, P(f)\mathcal{L}(z,f)^s) + k\overline{N}(r,f) + S(r, P(f)\mathcal{L}(z,f)^s). \tag{3.1}$$

combining (3.1) with Lemma 8, we have $S(r,F) = S(r,f)$. We also have $S(r,G) = S(r,g)$, from the same reason as above, from Lemma 5 we obtain

$$\begin{aligned} N_2\left(r, \frac{1}{F}\right) &= N_2\left(r, \frac{1}{[P(f)\mathcal{L}(z,f)^s]^{(k)}}\right) \\ &\le T(r,F) - T(r, P(f)\mathcal{L}(z,f)^s) + N_{k+2}\left(r, \frac{1}{P(f)\mathcal{L}(z,f)^s}\right) + S(r,f). \end{aligned} \tag{3.2}$$

Thus, from Lemma 8 and (3.2) we get

$$(n-s)T(r,f) \le T(r,F) - N_2\left(r, \frac{1}{F}\right) + N_{k+2}\left(r, \frac{1}{P(f)\mathcal{L}(z,f)^s}\right) + S(r,f). \tag{3.3}$$

We have

$$\overline{N}(r,F) \le (1+s)T(r,f) + S(r,f), \tag{3.4}$$

$$N_{k+2}\left(r, \frac{1}{F}\right) \le (k+2+\lambda_{b_0})T(r,f) + S(r,f), \tag{3.5}$$

$$N_{k+1}\left(r, \frac{1}{F}\right) \le (k+1+\lambda_{b_0})T(r,f) + S(r,f). \tag{3.6}$$

Similarly,

$$\overline{N}(r,G) \le (1+s)T(r,g) + S(r,g), \tag{3.7}$$

$$N_{k+2}\left(r, \frac{1}{G}\right) \le (k+2+\lambda_{b_0})T(r,g) + S(r,g), \tag{3.8}$$

$$N_{k+1}\left(r, \frac{1}{G}\right) \le (k+1+\lambda_{b_0})T(r,g) + S(r,g). \tag{3.9}$$

From Lemma 5, we obtain

$$\begin{aligned} N_2\left(r, \frac{1}{F}\right) \le& N_{k+2}\left(r, \frac{1}{F}\right) + k\overline{N}(r,F) + S(r,F), \\ \le& N_{k+2}\left(r, \frac{1}{P(f)\mathcal{L}(z,f)^s}\right) + k\overline{N}(r,f) + S(r,f), \\ \le& (k+2)N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{\mathcal{L}(z,f)^s}\right) + k\overline{N}(r,f) + S(r,f), \\ \le& (k+2+\lambda_{b_0} + k(1+s))T(r,f) + S(r,f). \end{aligned} \tag{3.10}$$

Similarly,

$$N_2\left(r, \frac{1}{G}\right) \le (k+2+\lambda_{b_0} + k(1+s))T(r,g) + S(r,g). \tag{3.11}$$

If Lemma 6 is satisfied, which implies that

$$max\{T(r,F), T(r,G)\} \le N_2\left(r, \frac{1}{F}\right) + N_2(r,F) + N_2\left(r, \frac{1}{G}\right) + N_2(r,G) + S(r,F) + S(r,G).$$

Thus, combining (3.10) and (3.11), we obtain

$$\begin{aligned}(n-s)\{T(r,f)+T(r,g)\} \leq & 2[N(r.f)+N(r,g)]+2N_{k+2}\left(r,\frac{1}{P(f)\mathcal{L}(z,f)^s}\right)\\ & +2N_{k+2}\left(r,\frac{1}{P(g)\mathcal{L}(z,g)^s}\right)+S(r,f)+S(r,g),\\ \leq & 2(k+2+\lambda_{b_0}+k(1+s))\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).\end{aligned}$$

Which is contradiction with $n \geq 4k+2\lambda_{b_0}+s(2k+1)+4$. Hence $F=G$ or $FG=1$. From Lemma 9, we get $f(z)=tg(z)$, for a constant t such that $t^d=1$, where $d=GCD(\lambda_0+\lambda_1+,\ldots,+\lambda_n)$, and $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f(z),g(z)\equiv 0$, where

$$R(w_1,w_2)=P(w_1)\mathcal{L}(z,w_1)^s-P(w_2)\mathcal{L}(z,w_2)^s.$$

□

Theorem 1.12

Proof . Let $F=[P(f)\mathcal{L}(z,f)^s]^{(k)}$ and $G=[P(g)\mathcal{L}(z,g)^s]^{(k)}$. Let H be defined as in Lemma 7. Assume that $H\not\equiv 0$, from Lemma 5, we get

$$\begin{aligned}T(r,F)+T(r,G) \leq & 2\left[N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_2(r,F)+N_2(r,G)\right]\\ \leq & \left[\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)\right]+S(r,F)+S(r,G).\end{aligned} \tag{3.12}$$

Combining (3.10)-(3.11) and Lemma 5, we get

$$\begin{aligned}(n-d)[T(r,f)+T(r,g)] \leq & T(r,F)+T(r,G)-N_2\left(r,\frac{1}{F}\right)-N_2\left(r,\frac{1}{G}\right)+2N_{k+2}\left(r,\frac{1}{P(f)\mathcal{L}(z,f)^s}\right)\\ & +2N_{k+2}\left(r,\frac{1}{P(g)\mathcal{L}(z,g)^s}\right)+S(r,f)+S(r,g),\\ \leq & 2[N(r.f)+N(r,g)]+2N_{k+2}\left(r,\frac{1}{P(f)\mathcal{L}(z,f)^s}\right)\\ & +2N_{k+2}\left(r,\frac{1}{P(g)\mathcal{L}(z,g)^s}\right)3\left[\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)\right]+S(r,f)+S(r,g),\\ \leq & 2(k+2+\lambda_{b_0}+k(1+s))\{T(r,f)+T(r,g)\}+3(2k+2\lambda_{b_0})\{T(r,f)+T(r,g)\}\\ & +S(r,f)+S(r,g)\\ \leq & (10k+8\lambda_{b_0}+2ks+4)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).\end{aligned}$$

Which is contradiction with $n \geq 10k+8\lambda_{b_0}+s(2k+1)+4$. Thus we get $H\equiv 0$. Therefore, we get $F=G$ or $FG=1$. From Lemma 9, we get $f(z)=tg(z)$, for a constant t such that $t^d=1$, where $d=GCD(\lambda_0+\lambda_1+,\ldots,+\lambda_n)$, and $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f(z),g(z)\equiv 0$, where

$$R(w_1,w_2)=P(w_1)\mathcal{L}(z,w_1)^s-P(w_2)\mathcal{L}(z,w_2)^s.$$

□

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