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Weighted spaces of holomorphic functions on the quarter plane and strip

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Abstract

In this paper, we investigate weighted spaces of holomorphic functions on some subsets of the complex plane such as the quarter plane and a strip in the upper half plane and we obtain isomorphic classification of these weighted spaces. Also, we characterize the boundedness of composition operators between weighted spaces of holomorphic functions on the quarter plane and strip. At last, we study a special subspace of holomorphic functions on the quarter plane.

Keywords: Weighted spaces, holmorphic functions, isomorphism of Banach spaces, quarter plane, strip
2020 MSC: 46E15, 46B03, 47B33

1 Introduction

We begin by recalling some definitions and notations on weighted spaces of holomorphic functions. Let $O \subseteq \mathbb{C}$ be an open subset and $v : O \longrightarrow (0, \infty)$ be a given function. For a holomorphic function $f : O \longrightarrow \mathbb{C}$ we define the weighted sup-norm

$$\|f\|_v = \sup_{z \in O} \mid f(z) \mid v(z)$$

and weighted spaces

$$H_v(O) = \{f : O \longrightarrow \mathbb{C} : \|f\|_v < \infty\}$$

and

$$H_{v_0}(O) = \{f \in H_v(O) :\mid f(z) \mid v(z) \text{ vanishes at infinity}\}.$$

Here $|f|v$ vanishes at infinity if for any $\varepsilon > 0$ there is a compact subset $K \subset O$ such that $|f(z)|v(z) < \varepsilon$ for all $z \in O \setminus K$. Weighted Banach spaces of holomorphic functions on the unit disc, upper halfplane, strip and complex plane with a deleted ray have been investigated from different aspects in large number of publications. In particular, isomorphic classification and operators between these spaces is interested in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] and references therein. In the present paper we study weighted spaces of holomorphic functions on the quarter plane and a strip in the upper halfplane. Let F be the quarter plane, i.e,

$$F = \{z \in \mathbb{C} : Rez > 0 \ \& \ Imz > 0\}.$$

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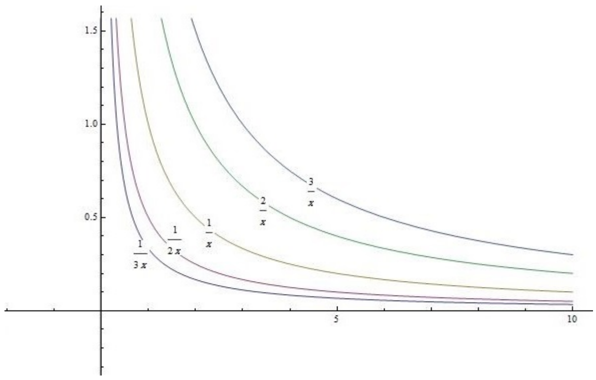

Figure 1: hyperbolas paths in F

By a standard weight υ on F we mean a continous function $\upsilon : F \longrightarrow (0,\infty)$ such that

$$\upsilon(\omega) = \upsilon(\sqrt{2(Re\ \omega)(Im\ \omega)}e^{i\frac{\pi}{4}}) \tag{1.1}$$

for all $\omega \in F$ and

$$\upsilon(se^{i\frac{\pi}{4}}) \leq \upsilon(te^{i\frac{\pi}{4}}) \ \text{ if } \ 0 < s \leq t \ \text{ and } \ \lim_{t\to 0^+} \upsilon(te^{i\frac{\pi}{4}}) = 0. \tag{1.2}$$

υ is concentrated on the angle bisector and goes to 0 towards the real and imaginary half axis. Relation (1.1) really makes sense, that is if $\omega = te^{i\frac{\pi}{4}}$ then

$$\sqrt{2(Re\ \omega)(Im\ \omega)}e^{i\frac{\pi}{4}} = te^{i\frac{\pi}{4}} = \omega.$$

If $\omega \in F$ and $Im\ \omega = \frac{c}{Re\ \omega}$ (where c is a positive constant) then

$$\begin{aligned}\upsilon(\omega) &= \upsilon(\sqrt{2(Re\ \omega)(Im\ \omega)}e^{i\frac{\pi}{4}}) \\ &= \upsilon(\sqrt{c} + \sqrt{c}\ i).\end{aligned}$$

This means any weight on F is a constant function whenever imaginary parts of points varies on the hyberbolas paths. See figure 1.
In the present paper we intend to transfer results concerning the isomorphic classification of $H_\upsilon(G)$ and $H_{\upsilon_0}(G)$ (here G is the upper halfplane) to the $H_\upsilon(F),\ H_{\upsilon_0}(F)$ and $H_\upsilon(S),\ H_{\upsilon_0}(S)$ where

$$S = \{z \in \mathbb{C} : 0 < Imz < \pi\}$$

is a strip in the upper halfplane and a standard weight υ on S is a continous function $\upsilon : S \longrightarrow (0,\infty)$ such that

$$\upsilon(\omega) = \upsilon(Re\ \omega + \log(\sin(Im\ \omega)) + i\frac{\pi}{2}) \tag{1.3}$$

for all $\omega \in S$ and

$$\upsilon(s + i\frac{\pi}{2}) \leq \upsilon(t + i\frac{\pi}{2}) \quad \text{if} \quad -\infty < s \leq t \ \text{ and } \ \lim_{t\to -\infty} \upsilon(t + i\frac{\pi}{2}) = 0. \tag{1.4}$$

A standard weight υ on S is concentrated on the horizental line $t + i\frac{\pi}{2}$ and tends to 0 as the real part of points goes to $-\infty$. Realation (1.3) is well defined, since if $\omega = t + i\frac{\pi}{2}$, then

$$Re\ \omega + \log(\sin(Im\ \omega)) + i\frac{\pi}{2} = t + \log 1 + i\frac{\pi}{2} = \omega.$$

If $\omega \in S$ and $Im\ \omega = \arcsin(ce^{-Re\ \omega})$ ($c > 0$ and $Re\ \omega$ must be such that $ce^{-Re\ \omega} \in [-1,1]$) then

$$\begin{aligned}\upsilon(\omega) &= \upsilon(Re\ \omega + \log(\sin(\arcsin(cRe\ \omega))) + i\frac{\pi}{2}) \\ &= \upsilon(Re\ \omega + \log(ce^{-Re\ \omega}) + i\frac{\pi}{2}) \\ &= \upsilon(\log c + i\frac{\pi}{2})\end{aligned}$$

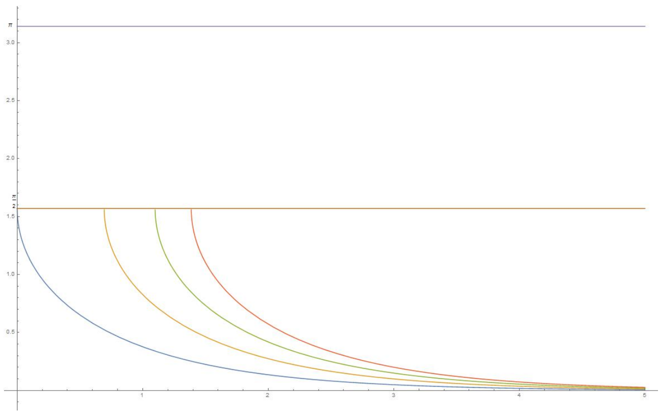

Figure 2: Ghraph of $\arcsin(ce^{-x})$ for $c = 1, 2, 3, 4$ and $x \geq 0$ in S

Above calculation shows that any weight on S is a constant function if imaginary parts of points located on the paths $y = \arcsin(ce^{-x})$. In figure 2 we have drawn $y = \arcsin(ce^{-x})$ $(c = 1, 2, 3, 4$ and $x \geq 0)$ and lines $t + i\frac{\pi}{2},\ t + i\pi$ for $t \geq 0$.

Definition 1.1. (a) A standard weight $\upsilon : F \longrightarrow (0, \infty)$ satisfies $(*)_F$ if

$$\sup_{k\in\mathbb{Z}} \frac{\upsilon(2^{\frac{k}{2}} e^{i\frac{\pi}{4}})}{\upsilon(2^{\frac{k-1}{2}} e^{i\frac{\pi}{4}})} < \infty$$

and satisfies $(**)_F$ if

$$\inf_{n\in\mathbb{N}} \sup_{k\in\mathbb{Z}} \frac{\upsilon(2^{\frac{k-1}{2}} e^{i\frac{\pi}{4}})}{\upsilon(2^{\frac{k-1+n}{2}} e^{i\frac{\pi}{4}})} < 1.$$

(b) A standard weight $\upsilon : S \longrightarrow (0, \infty)$ satisfies $(*)_S$ if

$$\sup_{k\in\mathbb{Z}} \frac{\upsilon(k + i\frac{\pi}{2})}{\upsilon(k - 1 + i\frac{\pi}{2})} < \infty$$

and satisfies $(**)_S$ if

$$\inf_{n\in\mathbb{N}} \sup_{k\in\mathbb{Z}} \frac{\upsilon(k - 1 + i\frac{\pi}{2})}{\upsilon(k - 1 + n + i\frac{\pi}{2})} < 1.$$

Example 1.2. (a) For any ω in F put $\upsilon_1(\omega) = (Im\ \omega^2)^{\frac{1}{2}} = \sqrt{2(Re\ \omega)(Im\ \omega)}$ and $\upsilon_2(\omega) = \min(\upsilon_1(\omega), 1)$. It is easy to see that υ_1 satisfies$(*)_F$ and $(**)_F$ while υ_2 satisfies only $(*)_F$.

(b) For any $\omega \in S$ put $\upsilon_3(\omega) = e^{Re\ \omega} \sin(Im\ \omega)$. Then υ_3 satisfies both $(*)_S$ and $(**)_S$. Indeed

$$\sup_{k\in\mathbb{Z}} \frac{\upsilon_3(k + i\frac{\pi}{2})}{\upsilon_3(k - 1 + i\frac{\pi}{2})} = e \quad \text{and} \quad \inf_{n\in\mathbb{N}} \sup_{k\in\mathbb{Z}} \frac{\upsilon_3(k - 1 + i\frac{\pi}{2})}{\upsilon_3(k - 1 + n + i\frac{\pi}{2})} = 0.$$

If we put $\upsilon_4(\omega) = \min(\upsilon_3(\omega), 1)$ for any $\omega \in S$, then υ_4 only satisfies $(*)_S$. Exactly, we have

$$\sup_{k\in\mathbb{Z}} \frac{\upsilon_4(k + i\frac{\pi}{2})}{\upsilon_4(k - 1 + i\frac{\pi}{2})} = \inf_{n\in\mathbb{N}} \sup_{k\in\mathbb{Z}} \frac{\upsilon_4(k - 1 + i\frac{\pi}{2})}{\upsilon_4(k - 1 + n + i\frac{\pi}{2})} = 1.$$

As the secondary aim of this paper, we study the boundedness of composition operators on weighted spaces of holomorphic functions on the quarter plane. It is necessary to recall the definition of standard weight on G from [3]. Let u be a continuous function on G satisfying $u(\omega) > 0$ for all $\omega \in G$. Also u only depends on the imaginary part and satisfies

$$\lim_{r\to 0} u(ir) = 0 \text{ and } u(\omega_1) \leq u(\omega_2) \text{ whenever } 0 < Im\ \omega_1 \leq\ Im\ \omega_2.$$

Such u is called a standard weight on G. We conclude this section by recalling two following concepts.

Definition 1.3. (a) Let U, V be open subsets of $\mathbb{C}$. An isomorphism $\psi : U \longrightarrow V$ is a holomorphic(analytic) map wich has a holomorphic inverse map $\psi^{-1} = g : V \longrightarrow U$, that is, $\psi \circ g = id_V$ and $g \circ \psi = id_U$.

(b) Let O be an open subset of $\mathbb{C}$. A function $f : O \longrightarrow \mathbb{C}$ is called 2π-periodic if $f(\omega + 2\pi) = f(\omega)$ for all $\omega \in O$. For example $f(\omega) = e^{i\omega}$ is a 2π-periodic function on the upper halfplane.

2 Main results

2.1 Isomorphism Classifications of $H_\upsilon(F), H_{\upsilon_0}(F), H_\upsilon(S)$ and $H_{\upsilon_0}(S)$.

Before arriving to the main result of this section in order to avoid any misleading of the readers it worth to mention the following remark .

Remark 2.1. Whenever we consider an isomorphism between open subsets of $\mathbb{C}$, we mean Definition 1.3 (a) and when we talk about Banach spaces by an isomorphism we mean isomorphism between Banch spaces in its usual meaing. Also, in this case isomorphism is in terms of the topology of Banach spaces.

In the following theorems, we peresent isomorphism classifications of $H_\upsilon(F), H_{\upsilon_0}(F), H_\upsilon(S)$ and $H_{\upsilon_0}(S)$.

Theorem 2.2. Let υ be a weight on F satisfying $(*)_F$. Then

(i) $H_\upsilon(F)$ is isomorphic to ℓ_∞ if and only if υ satisfies $(**)_F$.

(ii) $H_{\upsilon_0}(F)$ is isomorphic to c_0 if and only if υ satisfies $(**)_F$.

(iii) $H_\upsilon(F)$ is isomorphic to $H_\infty(\mathbb{D})$ if and only if υ does not satisfy $(**)_F$.

Here $H_\infty(\mathbb{D})$ is the space of all bounded analytic functions on the unit disc $\mathbb{D}$.

Proof . Let $G = \{\omega \in \mathbb{C} : Im\ \omega > 0\}$ be the upper halfplane. We prove the theorem in three steps.

Step 1: If $\omega = x + iy \in F$, then $\omega^2 = x^2 - y^2 + 2xyi \in G$. So the map $\psi : F \longrightarrow G$ defined by $\psi(\omega) = \omega^2$ is an isomorphism with the inverse $\psi^{-1} : G \longrightarrow F$ $\psi^{-1}(\omega) = \omega^{\frac{1}{2}}$. Put $u(\omega) = \upsilon(\omega^{\frac{1}{2}})$ for any $\omega \in G$. Therefore the map $T : H_\upsilon(F) \longrightarrow H_u(G)$ defined by

$$(Tf)(\omega) = (f \circ \psi^{-1})(\omega) = f(\omega^{\frac{1}{2}}) \quad \omega \in G, f \in H_\upsilon(F)$$

is an isometric isomorphism since,

$$\|Tf\|_u = \|(Tf)(\omega)\|_u = \sup_{\omega \in G} \mid (Tf)(\omega) \mid u(\omega) = \sup_{\omega \in G} \mid f(\omega^{\frac{1}{2}}) \mid \upsilon(\omega^{\frac{1}{2}}) = \sup_{\omega' \in F} \mid f(\omega') \mid \upsilon(\omega') \mid = \|f\|_\upsilon.$$

Step 2: We show that $u(\omega) = \upsilon(\omega^{\frac{1}{2}})(\omega \in G)$ is a standard weight on G. Since υ is a positve and continuous function on F, so u is a positve and continuous function on G. Choose ω on the imaginary axis, i,e $\omega = ir = re^{i\frac{\pi}{2}}$, so $\omega^{\frac{1}{2}} = r^{\frac{1}{2}} e^{i\frac{\pi}{4}}$. By relation (1.2) we get

$$\lim_{r \to 0^+} u(ir) = \lim_{r \to 0^+} \upsilon(r^{\frac{1}{2}} e^{i\frac{\pi}{4}}) = 0.$$

Above relation implies that u depends only on the imaginary part. Therefore , if $\omega_1, \omega_2 \in G$ and $Im\ \omega_1 \leq Im\ \omega_2$ then by applying (1.2) we have

$$u(\omega_1) = u(Im\ \omega_1 i) = \upsilon((Im\ \omega_1)^{\frac{1}{2}} e^{i\frac{\pi}{4}}) \leq \upsilon((Im\ \omega_2)^{\frac{1}{2}} e^{i\frac{\pi}{4}}) = u(Im\ \omega_2 i) = u(\omega_2).$$

Similarly, $\upsilon(\omega) = u(\omega^2)(\omega \in F)$ is a standard weight on F.

Step 3: Obviously, υ satisfies $(*)_F$ and $(**)_F$ if and only if u satisfies $(*)_G$ and $(**)_G$ respectively, where $(*)_G$ and $(**)_G$ are :

$$\sup_{k \in \mathbb{Z}} \frac{u(2^k i)}{u(2^{k-1} i)} < \infty$$

and

$$\inf_{n \in \mathbb{N}} \sup_{k \in \mathbb{Z}} \frac{u(2^k i)}{u(2^{k-1+n} i)} < 1$$

respectively. Now, since $H_\upsilon(F)$ and $H_u(G)$ are isomorphic, applying Theorem 1.2 of [3] and Theorem 1.3 of [5] complete the proof. □

Benefiting the method of the proof of Theorem 2.2, Theorem 1.2 of [3] and Theorem 1.3 of [5] similarly we obtain:

Theorem 2.3. Let υ be a weight on S satisfying $(*)_S$. Then

(i) $H_\upsilon(S)$ is isomorphic to ℓ_∞ if and only if υ satisfies $(**)_S$.

(ii) $H_{\upsilon_0}(S)$ is isomorphic to c_0 if and only if υ satisfies $(**)_S$.

(iii) $H_\upsilon(S)$ is isomorphic to $H_\infty(\mathbb{D})$ if and only if υ does not satisfy $(**)_S$.

Proof . If $\omega = x + iy \in S$ then $e^\omega \in G$ (note that $Im\ e^\omega = e^x \sin y > 0$). Hence the map $\tau : S \longrightarrow G$ defined by $\tau(\omega) = e^\omega$ is an isomorphism with the inverse $\tau^{-1} : G \longrightarrow S\ \tau^{-1}(\omega) = \log\omega$. Now, if we define $u(\omega) = \upsilon(\log\omega)$ (for any $\omega \in G$) then the map $T : H_\upsilon(S) \longrightarrow H_u(G)$ defined by

$$(Tf)(\omega) = (f \circ \tau^{-1})(\omega) = f(\log\omega) \quad \omega \in G, f \in H_\upsilon(S)$$

is an isometric isomorphism since,

$$\|Tf\|_u = \|(Tf)(\omega)\|_u = \sup_{\omega\in G} \mid (Tf)(\omega) \mid u(\omega) = \sup_{\omega\in G} \mid f(\log\omega) \mid \upsilon(\log\omega) = \sup_{\omega'\in S} \mid f(\omega') \mid \upsilon(\omega') \mid= \|f\|_\upsilon$$

$u(\omega) = \upsilon(\log\omega)$ is a standard weight on G since by relation (1.4) we have

$$\lim_{r\to 0^+} u(ir) = \lim_{r\to 0^+} \upsilon(\log(ir)) = \lim_{r\to 0^+} \upsilon(\log \mid r \mid +i\frac{\pi}{2}) = \lim_{\log|r|\to -\infty} \upsilon(\log \mid r \mid +i\frac{\pi}{2}) = 0.$$

Also , if $\omega_1, \omega_2 \in G$ and $Im\ \omega_1 \le Im\ \omega_2$, then relation (1.4) implies that

$$\begin{aligned} u(\omega_1) &= u(Im\ \omega_1 i) = \upsilon(\log(Im\ \omega_1 i)) = \upsilon(\log \mid Im\ \omega_1 \mid +i\frac{\pi}{2}) \\ &\le \upsilon(\log \mid Im\ \omega_2 \mid +i\frac{\pi}{2}) = \upsilon(\log\omega_2) = u(\omega_2) \end{aligned}$$

Now, the assertion of theorem is fulfilled because υ satisfies $(*)_S$ and $(**)_S$ if and only if u satisfies $(*)_G$ and $(**)_G$ respectively. Indeed independent of the constant factor $\log 2$ multiplied to real part of elemnts of S we have

$$\sup_{k\in\mathbb{Z}} \frac{u(2^k i)}{u(2^{k-1} i)} = \sup_{k\in\mathbb{Z}} \frac{\upsilon(k\log 2 + i\frac{\pi}{2})}{\upsilon((k-1)\log 2 + i\frac{\pi}{2})} = \sup_{k\in\mathbb{Z}} \frac{\upsilon(k + i\frac{\pi}{2})}{\upsilon((k-1) + i\frac{\pi}{2})}$$

and

$$\inf_{n\in\mathbb{N}} \sup_{k\in\mathbb{Z}} \frac{u(2^k i)}{u(2^{k-1+n} i)} = \inf_{n\in\mathbb{N}} \sup_{k\in\mathbb{Z}} \frac{\upsilon(k\log 2 + i\frac{\pi}{2})}{\upsilon((k-1+n)\log 2 + i\frac{\pi}{2})} = \inf_{n\in\mathbb{N}} \sup_{k\in\mathbb{Z}} \frac{\upsilon(k + i\frac{\pi}{2})}{\upsilon((k-1+n) + i\frac{\pi}{2})}.$$

□

2.2 Boundedness of composition operators on $H_\upsilon(F)$ and $H_\upsilon(S)$

In the next theorems, we state results similar to Corollary 1.5 of [2] for composition operators on weighted spaces $H_\upsilon(F)$ and $H_\upsilon(S)$.

Theorem 2.4. Let υ_1 and υ_2 be weights on F such that υ_1 satisifies $(*)_F$. Also suppose $\varphi : F \longrightarrow F$ is a nonconstant holomorphic map. Then the composition operator $C_\varphi : H_{\upsilon_1}(F) \longrightarrow H_{\upsilon_2}(F)$ (defined by $C_\varphi(f) = f \circ \varphi$) is a bounded operator if and only if

$$\sup_{\omega\in F} \frac{\upsilon_2(\omega)}{\upsilon_1(\varphi(\omega))} < \infty.$$

Proof . Consider the following digram

$$\begin{array}{cccc} C_\varphi : & H_{\upsilon_1}(\mathbb{F}) & \longrightarrow & H_{\upsilon_2}(\mathbb{F}) \\ & \downarrow T & & \downarrow T \\ C_{\varphi_1} : & H_{u_1}(G) & \longrightarrow & H_{u_2}(G) \end{array}$$

where $u_1(\omega) = \upsilon_1(\omega^{\frac{1}{2}})$ and $u_2(\omega) = \upsilon_2(\omega^{\frac{1}{2}})$ are standard weights on G corresponding to υ_1 and υ_2 respectively, T is as in the proof of Theorem 2.2 and $\varphi_1 : G \longrightarrow G$ is a holomorphic map defined by $\varphi_1 = \psi \circ \varphi \circ \psi^{-1}$ (here ψ and ψ^{-1}

are as in the proof of Theorem 2.2) . The above diagram is commutative since for any $f \in H_{\upsilon_1}(F)$ and $\omega \in G$

$$\begin{aligned}(T^{-1} \circ C_{\varphi_1} \circ T)(f) &= (T^{-1} \circ C_{\varphi_1})(f(\omega^{\frac{1}{2}}))\\ &= (T^{-1} \circ C_{\varphi_1})((f \circ \psi^{-1})(\omega))\\ &= T^{-1}((f \circ \psi^{-1} \circ \varphi_1)(\omega))\\ &= (f \circ \psi^{-1} \circ \varphi_1 \circ \psi)(\omega)\\ &= (f \circ \psi^{-1} \circ \psi \circ \varphi \circ \psi^{-1})(\omega)\\ &= (f \circ \varphi)(\psi^{-1}(\omega))\\ &= C_\varphi(f).\end{aligned}$$

Therefore, C_φ is bounded if and only if C_{φ_1} is bounded. Also $\varphi_1(\omega) = [\varphi(\omega^{\frac{1}{2}})]^2$ and υ_1 satisfies $(*)_F$ if and only if u_1 satisfies $(*)_G$. Now by Corollary 1.5 of [2] we have C_{φ_1} is bounded if and only if

$$\sup_{\omega \in G} \frac{u_2(\omega)}{u_1(\varphi_1(\omega))} < \infty$$

or equivalently if and only if

$$\sup_{\omega \in G} \frac{\upsilon_2(\omega^{\frac{1}{2}})}{\upsilon_1(\varphi(\omega^{\frac{1}{2}}))} < \infty.$$

□

Let υ_1 and υ_2 be weights on S. Repeating the proof of the Theorem 2.4 with $H_{\upsilon_1}(S)$ and $H_{\upsilon_2}(S)$ instead of $H_{\upsilon_1}(F)$ and $H_{\upsilon_2}(F)$, $\varphi_1 = \tau \circ \varphi \circ \tau^{-1}$ (here τ and τ^{-1} are as in the proof of Theorem 2.3) and considering relations $u_1(\omega) = \upsilon_1(\log \omega)$ and $u_2(\omega) = \upsilon_2(\log \omega)$ we have

Theorem 2.5. Let υ_1 and υ_2 be weights on S such that υ_1 satisifies $(*)_S$. Also suppose $\varphi : S \longrightarrow S$ is a nonconstant holomorphic map. Then the composition operator $C_\varphi : H_{\upsilon_1}(S) \longrightarrow H_{\upsilon_2}(S)$ (defined by $C_\varphi(f) = f \circ \varphi$) is a bounded operator if and only if

$$\sup_{\omega \in S} \frac{\upsilon_2(\omega)}{\upsilon_1(\varphi(\omega))} < \infty.$$

2.3 An especial subspace of $H_\upsilon(F)$

In this step we consider an especial subspace of $H_\upsilon(F)$, $H_\upsilon^{2\pi}(F)$ which is defined as follow.

$$H_\upsilon^{2\pi}(F) = \{f \in H_\upsilon(F) : \ f(\omega) = f((\omega^2 + 2\pi)^{\frac{1}{2}}) \text{ for all } \omega \in F\}.$$

Evidently, for any $\omega \in F$,$(\omega^2 + 2\pi)^{\frac{1}{2}}$ is again in F. In figure 3 we have visualized $(\omega^2 + 2\pi)^{\frac{1}{2}}$ whenever $Re\ \omega$ and $Im\ \omega$ are in the interval $[1, 100]$.

Example 2.6. If we define $f : F \longrightarrow \mathbb{C}$ by $f(\omega) = e^{i\omega^2}$, then

$$f((\omega^2 + 2\pi)^{\frac{1}{2}}) = e^{i((\omega^2+2\pi)^{\frac{1}{2}})^2} = e^{i(\omega^2+2\pi)} = e^{i\omega^2} = f(\omega).$$

Also,

$$\begin{aligned}\|f\|_\upsilon &= \sup_{\omega \in F} \mid e^{i\omega^2} \mid \upsilon(\omega)\\ &= \sup_{\omega \in F} \mid e^{i((Re\ \omega)^2 - (Im\ \omega^2))} \mid \mid e^{-2Re\ \omega\ Im\ \omega} \mid \ \upsilon(\omega)\\ &= \sup_{\omega \in F} \mid e^{-2Re\ \omega\ Im\ \omega} \mid \ \upsilon(\omega)\\ &\le \sup_{\omega \in F} \upsilon(\omega).\end{aligned}$$

Now, we consider two cases.

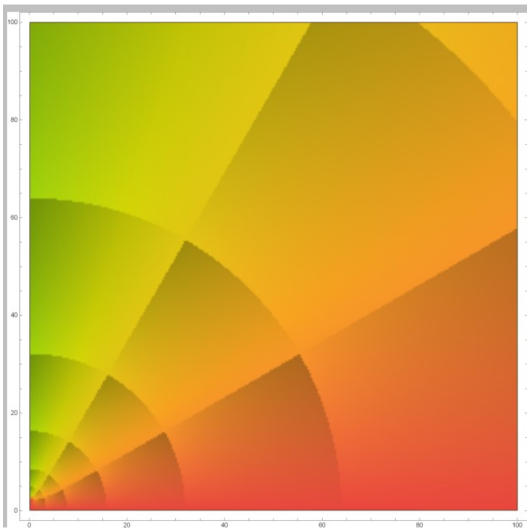

Figure 3: Visualization of $(\omega^2 + 2\pi)^{\frac{1}{2}}$ in F

Case 1: Let υ be any bounded weight on F (for instance the weight υ_2 in Example 1.2). Then $f(\omega) = e^{i\omega^2} \in H_{\upsilon}^{2\pi}(F)$.

Case 2: In particular, put $\upsilon(\omega) = (Im\ \omega^2)^{\frac{1}{2}} = \upsilon_1(\omega)(\upsilon_1$ is as in Example 1.2). Then

$$\|f\|_{\upsilon} = \sup_{\omega \in F} e^{-2Re\ \omega\ Im\ \omega} \sqrt{2Re\ \omega\ Im\ \omega} < \infty,$$

since the function $g(x) = \frac{\sqrt{x}}{e^x}(x > 0)$ attains its supremum$(\frac{\sqrt{2}}{2\sqrt{e}})$ at the point $x = \frac{1}{2}$ and obviously $\lim_{x\to+\infty} g(x) = 0$. Again $f(\omega) = e^{i\omega^2} \in H_{\upsilon}^{2\pi}(F)$ where $\upsilon(\omega) = (Im\ \omega^2)^{\frac{1}{2}}$.

Remark 2.7. Note that in Example 2.6 we have shown that $H_{\upsilon}^{2\pi}(F)$ is not equivalent to the trivial space $\{0\}$ not only for weights which satisfy both $(*)_F$ and $(**)_F$ but also for weights which satisfy only $(*)_F$.

Let u be a standard weight on G. Put

$$H_u^{2\pi}(G) = \{f \in H_u(G) :\ f(\omega) = f(\omega + 2\pi) \text{ for all } \ \omega \in G\}.$$

It is well-known that (see Proposition 2.1 of [1]) there is a smallest integer b_u such that $\sup_{\omega \in G} e^{b_u Im\ \omega} \upsilon(\omega) < \infty$ and for each $f \in H_u^{2\pi}(G)$ there exist $\gamma_k \in \mathbb{C}$ such that $f(\omega) = \Sigma_{k=b_u}^{\infty} \gamma_k e^{ik\omega}$. We have:

Lemma 2.8. $H_u^{2\pi}(G)$ and $H_{\upsilon}^{2\pi}(F)$ are isometrically isomorphic.

Proof . If we use notations of the proof of Theorem 2.2 then the map $T^{-1} : H_u(G) \longrightarrow H_{\upsilon}(F)$ defined by $T^{-1}(f) = (f \circ \psi)(\omega) = f(\omega^2)$ $(f \in H_u(G))$ is obviously an isometric isomorphism. Choose $f \in H_u^{2\pi}(G) \subset H_u(G)$ so for any $\omega \in F$

$$\begin{aligned} T^{-1}(f)((\omega^2 + 2\pi)^{\frac{1}{2}}) &= f(\omega^2 + 2\pi) \\ &= f(\omega^2) \\ &= T^{-1}(f)(\omega). \end{aligned}$$

Therefore the restriction of T^{-1} on $H_u^{2\pi}(G)$ is an isometric isomorphism from $H_u^{2\pi}(G)$ onto $H_{\upsilon}^{2\pi}(F)$. □

As an immediate consequence of Lemma 2.8 we obtain

Corollary 2.9. $H_{\upsilon}^{2\pi}(F) = \{f \in H_{\upsilon}(F) : \text{there are } \gamma_k \in \mathbb{C} \text{ such that } f(\omega) = \Sigma_{k=b_{\upsilon}}^{\infty} \gamma_k e^{ik\omega^2}\ (\omega \in F)\}$, where $k = b_{\upsilon}$ is the smallest integer such that $\sup_{\omega \in F} e^{-b_{\upsilon} 2Re\ \omega\ Im\ \omega} \upsilon(\omega) < \infty$.

Proof . Let υ be a standard weight on F with $H_\upsilon(F) \neq \{0\}$ and u be the corresponding standard weight to υ on G, that is $u(\omega^2) = \upsilon(\omega)$ for any $\omega \in F$. Therefore, for any $f \in H_u^{2\pi}(G)$, $T^{-1}(f) \in H_\upsilon^{2\pi}(F)(T^{-1}$ is as in the proof of Lemma 2.8) and

$$T^{-1}(f)(\omega) = f(\omega^2) = \Sigma_{k=b_u}^{\infty} \gamma_k e^{ik\omega^2} \quad (\omega \in F)$$

where b_u is such that $\sup_{\omega^2 \in G} e^{-b_u Im\ \omega^2} u(\omega^2) < \infty$ or equivalently $\sup_{\omega \in F} e^{-b_u 2Re\ \omega\ \omega Im\ \omega} \upsilon(\omega) < \infty$. Let $b_\upsilon = b_u$ we are done. $\square$

Remark 2.10. Let b_υ be the smallest integer such that $\sup_{\omega \in F} e^{-b_\upsilon 2Re\ \omega\ Im\ \omega} \upsilon(\omega) < \infty$. Since υ is increasing on the angle bisector of quarter plane, so $b_\upsilon \geq 0$. Furthermore $b_\upsilon = 0$ if and only if υ is bounded. For instance $b_{\upsilon_1} = 1$ and $b_{\upsilon_2} = 0$ where υ_1 and υ_2 are as in Example 1.2.

Here we recall the following theorem from [1]. For definition and structure of the space $(\Sigma_{n=0}^{\infty} \oplus H_n)_0$ we refer the reader to [7].

Theorem 2.11. Let u be a standard weight on G. Either $H_u^{2\pi}(G)$ is isomorphic to ℓ_∞ and $H_{u_0}^{2\pi}(G)$ is isomorphic to c_0, or $H_u^{2\pi}(G)$ is isomorphic to $H_\infty(\mathbb{D})$ and $H_{u_0}^{2\pi}(G)$ is isomorphic to $(\Sigma_{n=0}^{\infty} \oplus H_n)_0$.

Proof . See Theorem 3.1 of [1]. $\square$

Now applying Lemma 2.8 and Theorem 2.11, we obtain the following dichotomy for isomorphism classification of the space $H_\upsilon^{2\pi}(F)$.

Theorem 2.12. Let υ be a standard weight on F. Either $H_\upsilon^{2\pi}(F)$ is isomorphic to ℓ_∞, or $H_\upsilon^{2\pi}(F)$ is isomorphic to $H_\infty(\mathbb{D})$.

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