

Fekete-Szegő problem for two new subclasses of bi-univalent functions defined by Bernoulli polynomial

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(Communicated by Mugur Alexandru Acu)

Abstract

This investigation deals with two new subclasses of analytic and bi-univalent functions defined by Bernoulli polynomial. In this paper, coefficient estimation and Fekete-Szegő problems are solved for these newly defined function subclasses. In addition, certain remarks are indicated for the subclasses of bi-starlike and bi-convex functions.

Keywords: Bi-univalent function, coefficient estimates, Fekete-Szegő functional, Bernoulli polynomials
2020 MSC: Primary 30C45; Secondary 11B68

1 Introduction

Let \mathcal{A} denote the class of all analytic functions in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, which are of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in \Delta). \quad (1.1)$$

It is clear that the function f of the form (1.1) satisfy normalization conditions $f(0) = 0$ and $f'(0) = 1$. By \mathcal{S} we show the subclass of \mathcal{A} consisting of all functions, which are univalent in Δ . It is well-known that the familiar Koebe- $\frac{1}{4}$ theorem [9] makes sure that the image of Δ under every function $f \in \mathcal{S}$ contains a disk with radius $\frac{1}{4}$. Thus, every function $f \in \mathcal{S}$ has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, \quad (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w, \quad (|w| < r_0(f), r_0(f) \geq \frac{1}{4}).$$

It is emphasize here that every inverse function f^{-1} need not to be univalent in Δ . If f and f^{-1} are univalent in Δ , then $f \in \mathcal{S}$ is said to be bi-univalent in Δ , and the class of all analytic and bi-univalent functions defined in the unit disk Δ is denoted by Σ . By using series expression of the function f of the form (1.1) one can see that inverse function f^{-1} may be expressed as below:

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots := g(w). \quad (1.2)$$

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It is known that the functions

$$l_1(z) = \frac{z}{1-z} \text{ and } l_2(z) = \frac{1}{2} \log \frac{1+z}{1-z}$$

are in the function class \mathcal{S} . Moreover, the inverses of these functions are, respectively,

$$l_1^{-1}(w) = \frac{w}{1+w} \text{ and } l_2^{-1}(w) = \frac{e^{2w} - 1}{e^{2w} + 1}.$$

Also, the functions $l_1^{-1}(w)$ and $l_2^{-1}(w)$ are in the class \mathcal{S} . So, these functions are in the function class Σ and the function class Σ is a non-empty set. In addition, the familiar Koebe function $k(z) = \frac{z}{(1-z)^2} \notin \Sigma$, since the third coefficient of the function $k^{-1}(z)$ is -5 and it does not satisfy Bieberbach conjecture.

There are a wide literature on some properties of analytic and bi-univalent functions. In the recent years, numerous papers have been published on this topic. For the recent developments in this field the interested readers can refer to the papers [1, 8, 10, 12, 13, 17, 18, 20, 21, 22, 23, 24, 25, 26] and references therein.

If the functions f and $F \in \mathcal{A}$, then f is said to be subordinate to F if there exists a Schwarz function w such that

$$w(0) = 0, |w(z)| < 1 \text{ and } f(z) = F(w(z)) \quad (z \in \Delta).$$

This subordination is shown by

$$f \prec F \quad \text{or} \quad f(z) \prec F(z) \quad (z \in \Delta).$$

If F is univalent function in Δ , then this subordination is equivalent to

$$f(0) = F(0), \quad f(\Delta) \subset F(\Delta).$$

There are comprehensive information about the subordination notion in Monographs written by Miller and Mocanu (see [15]).

In univalent function theory, one of the most attractive problems is known as the Fekete-Szegő problem [11, 14]. This problem is related to coefficients of the functions in the class \mathcal{S} and it is expressed below:

$$|a_3 - \mu a_2^2| \leq 1 + 2 \exp\left(\frac{-2\mu}{1-\mu}\right), \text{ for } 0 \leq \mu < 1.$$

The fundamental inequality $|a_3 - \mu a_2^2| \leq 1$ is obtained as $\mu \rightarrow 1$. The coefficient functional

$$F_\mu(f) = a_3 - \mu a_2^2$$

on the normalized analytic functions f in the open unit disk Δ has a significant impact on univalent function theory. The Fekete-Szegő problem is known as the maximization problem for functional $|F_\mu(f)|$.

Orthogonal polynomials such as Hermite, Laguerre, Jacobi and Bernoulli polynomials are of great importance in applied sciences. In recent years, mathematicians have built a bridge between geometric function theory and orthogonal polynomials. In [2, 3, 4, 6, 7, 19] the authors defined some new subclasses of analytic and univalent functions by using some orthogonal polynomials and they investigated coefficient estimation and Fekete-Szegő problems for the functions belonging to these function classes. In this paper, we define two new subclasses of analytic and bi-univalent functions by using Bernoulli polynomial and investigate initial coefficient estimation and Fekete-Szegő problems for the functions belonging to new classes.

Bernoulli polynomials are defined the following generating functions (see [16]):

$$F(x, z) = \frac{ze^{xz}}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} z^k, \quad |z| < 2\pi, \quad (1.3)$$

where $B_k(x)$ is the k -th Bernoulli polynomial in variable x .

2 The Class $\mathcal{Y}_{\Sigma}^{\mu}(\lambda, \delta)$

In this section we introduce a new function class $\mathcal{Y}_{\Sigma}^{\mu}(\lambda, \delta)$ and investigate initial coefficient bounds estimations and the Fekete-Szegő inequality for this class.

Definition 2.1. Let $\lambda \geq 1, \mu \geq 0$ and $\delta \geq 0$. If the function $f(z) \in \Sigma$ of the form (1.1) satisfies the following conditions, then it is called in the class $\mathcal{Y}_{\Sigma}^{\mu}(\lambda, \delta)$:

$$(1 - \lambda) \left(\frac{f(z)}{z} \right)^{\mu} + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} + \xi \delta z f''(z) \prec \frac{ze^{xz}}{e^z - 1} = F(x, z), \quad (2.1)$$

$$(1 - \lambda) \left(\frac{g(w)}{w} \right)^{\mu} + \lambda g'(w) \left(\frac{g(w)}{w} \right)^{\mu-1} + \xi \delta w g''(w) \prec \frac{we^{xw}}{e^w - 1} = F(x, w), \quad (2.2)$$

where $\xi = \frac{2\lambda + \mu}{2\lambda + 1}$ and the function g is of the form (1.2).

Remark 2.2. Taking $\lambda = 1$ and $\delta = \mu = 0$ in Definition 2.1 we obtain the class $\mathcal{Y}_{\Sigma}^0(1, 0)$ of bi-starlike functions and it satisfies the following subordinations:

$$\frac{zf'(z)}{f(z)} \prec \frac{ze^{xz}}{e^z - 1} = F(x, z), \quad (2.3)$$

$$\frac{wg'(w)}{g(w)} \prec \frac{we^{xw}}{e^w - 1} = F(x, w). \quad (2.4)$$

Theorem 2.3. Suppose that $\lambda \geq 1, \mu \geq 0$ and $\delta \geq 0$. If $f \in \mathcal{Y}_{\Sigma}^{\mu}(\lambda, \delta)$, then

$$|a_2| \leq \frac{|B_1(x)| \sqrt{2|B_1(x)|}}{\sqrt{|B_1^2(x)(\mu + 2\lambda)(\mu + 1 + \frac{12\delta}{2\lambda + 1}) - B_2(x)(\lambda + \mu + 2\xi\delta)^2|}}, \quad (2.5)$$

$$|a_3| \leq \frac{B_1^2(x)}{(\lambda + \mu + 2\xi\delta)^2} + \frac{|B_1(x)|}{\left| (\mu + 2\lambda)(1 + \frac{6\delta}{12\lambda + 1}) \right|} \quad (2.6)$$

and for $\eta \in \mathbb{R}$

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|B_1(x)|}{(\mu + 2\lambda)(1 + \frac{6\delta}{12\lambda + 1})}, & |1 - \eta| \leq T(x, \lambda, \mu, \delta) \\ \frac{2|B_1(x)|^3 |1 - \eta|}{|B_1^2(x)(\mu + 2\lambda)(\mu + 1 + \frac{12\delta}{2\lambda + 1}) - B_2(x)(\lambda + \mu + 2\xi\delta)^2|}, & |1 - \eta| \geq T(x, \lambda, \mu, \delta) \end{cases}, \quad (2.7)$$

where $T(x, \lambda, \mu, \delta) = \frac{|B_1^2(x)(\mu + 2\lambda)(\mu + 1 + \frac{12\delta}{2\lambda + 1}) - B_2(x)(\lambda + \mu + 2\xi\delta)^2|}{2B_1^2(x)(\mu + 2\lambda)(1 + \frac{6\delta}{12\lambda + 1})}$.

Proof . Let $f(z) \in \mathcal{Y}_{\Sigma}^{\mu}(\lambda, \delta)$, $\lambda \geq 1, \mu \geq 0$ and $\delta \geq 0$. By Definition 2.1, there are two Schwarz functions $p, r : \Delta \rightarrow \Delta$,

$$p(z) = p_1 z + p_2 z^2 + p_3 z^3 + \dots, \quad (2.8)$$

$$r(w) = r_1 w + r_2 w^2 + r_3 w^3 + \dots \quad (2.9)$$

such that

$$(1 - \lambda) \left(\frac{f(z)}{z} \right)^{\mu} + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} + \xi \delta z f''(z) = F(x, p(z)) \quad (2.10)$$

and

$$(1 - \lambda) \left(\frac{g(w)}{w} \right)^{\mu} + \lambda g'(w) \left(\frac{g(w)}{w} \right)^{\mu-1} + \xi \delta w g''(w) = F(x, r(w)), \quad (2.11)$$

where $z, w \in \Delta$. It is well-known the definition of Schwarz function that $|p_i| \leq 1$ and $|r_i| \leq 1$ for $\forall i \in \mathbb{N}$. A basic calculation yields that right hand sides of the equations (2.10) and (2.11) are, respectively,

$$F(x, p(z)) = B_0(x) + [B_1(x)p_1]z + [B_1(x)p_2 + \frac{B_2(x)}{2!}p_1^2]z^2 + \left[B_1(x)p_3 + B_2(x)p_1p_2 + \frac{B_3(x)}{3!}p_1^3 \right]z^3 + \dots \quad (2.12)$$

and

$$F(x, r(w)) = B_0(x) + [B_1(x)r_1]w + [B_1(x)r_2 + \frac{B_2(x)}{2!}r_1^2]w^2 + \left[B_1(x)r_3 + B_2(x)r_1r_2 + \frac{B_3(x)}{3!}r_1^3 \right] w^3 + \dots, \quad (2.13)$$

where $B_0(x) = 1$. In addition, left hand sides of the equations (2.10) and (2.11) are, respectively,

$$\begin{aligned} (1 - \lambda) \left(\frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} + \xi \delta z f''(z) = \\ 1 + (\lambda + \mu + 2\delta\xi)a_2z + (\mu + 2\lambda) \left[\frac{\mu-1}{2}a_2^2 + \left(1 + \frac{6\delta}{2\lambda+1} \right) a_3 \right] z^2 + \dots \end{aligned} \quad (2.14)$$

and

$$\begin{aligned} (1 - \lambda) \left(\frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left(\frac{g(w)}{w} \right)^{\mu-1} + \xi \delta w g''(w) = \\ 1 - (\lambda + \mu + 2\delta\xi)a_2w + (\mu + 2\lambda) \left[\left(\frac{\mu+3}{2} + \frac{12\delta}{2\lambda+1} \right) a_2^2 - \left(1 + \frac{6\delta}{2\lambda+1} \right) a_3 \right] w^2 + \dots \end{aligned} \quad (2.15)$$

Here, by comparing the coefficients of the equations (2.12) and (2.14) we obtain

$$(\lambda + \mu + 2\delta\xi)a_2 = B_1(x)p_1 \quad (2.16)$$

$$(\mu + 2\lambda) \left[\frac{\mu-1}{2}a_2^2 + \left(1 + \frac{6\delta}{2\lambda+1} \right) a_3 \right] = B_1(x)p_2 + \frac{B_2(x)}{2!}p_1^2. \quad (2.17)$$

Also, by similar point of view from the equations (2.13) and (2.15) we have

$$-(\lambda + \mu + 2\delta\xi)a_2 = B_1(x)r_1 \quad (2.18)$$

$$(\mu + 2\lambda) \left[\frac{\mu+3}{2} + \frac{12\delta}{2\lambda+1} a_2^2 - \left(1 + \frac{6\delta}{2\lambda+1} \right) a_3 \right] = B_1(x)r_2 + \frac{B_2(x)}{2!}r_1^2. \quad (2.19)$$

Now, from the equations (2.16) and (2.18), it follows that

$$p_1 = -r_1, \quad (2.20)$$

$$2(\lambda + \mu + 2\delta\xi)^2 a_2^2 = B_1^2(x)(p_1^2 + r_1^2) \quad (2.21)$$

and

$$a_2^2 = \frac{B_1^2(x)(p_1^2 + r_1^2)}{2(\lambda + \mu + 2\delta\xi)^2}. \quad (2.22)$$

Summation of the expressions (2.17) and (2.19) imply that

$$(\mu + 2\lambda) \left(\mu + 1 + \frac{12\delta}{2\lambda+1} \right) a_2^2 = B_1(x)(p_2 + r_2) + \frac{B_2(x)}{2}(p_1^2 + r_1^2). \quad (2.23)$$

Using equation (2.22) in (2.23) one can easily see that

$$a_2^2 = \frac{B_1^3(x)(p_2 + r_2)}{B_1^2(x)(\mu + 2\lambda) \left(\mu + 1 + \frac{12\delta}{2\lambda+1} \right) - B_2(x)(\lambda + \mu + 2\delta\xi)^2}. \quad (2.24)$$

Since $|p_i| \leq 1, |r_i| \leq 1$ for $\forall i \in \mathbb{N}$, by using triangle inequality in (2.24) we can write that

$$|a_2| \leq \frac{|B_1(x)| \sqrt{2|B_1(x)|}}{\sqrt{\left| B_1^2(x)(\mu + 2\lambda) \left(\mu + 1 + \frac{12\delta}{2\lambda+1} \right) - B_2(x)(\lambda + \mu + 2\delta\xi)^2 \right|}}. \quad (2.25)$$

On the other hand, if we subtract (2.17) from (2.19), then we get

$$a_3 = a_2^2 + \frac{B_1(x)(p_2 - r_2)}{2(\mu + 2\lambda) \left(1 + \frac{6\delta}{12\lambda+1}\right)}. \quad (2.26)$$

If we replace (2.22) in (2.26), then we can write that

$$a_3 = \frac{B_1^2(x)(p_1^2 + r_1^2)}{2(\lambda + \mu + 2\delta\xi)^2} + \frac{B_1(x)(p_2 - r_2)}{2(\mu + 2\lambda) \left(1 + \frac{6\delta}{12\lambda+1}\right)}. \quad (2.27)$$

Now, using triangle inequality in the last equality we deduce

$$|a_3| \leq \frac{B_1^2(x)}{(\lambda + \mu + 2\delta\xi)^2} + \frac{|B_1(x)|}{\left|(\mu + 2\lambda) \left(1 + \frac{6\delta}{12\lambda+1}\right)\right|} \quad (2.28)$$

Finally, taking into account the equations (2.24) and (2.26), we can write that

$$\begin{aligned} a_3 - \eta a_2^2 &= \frac{(1-\eta)B_1^3(x)(p_2 + r_2)}{B_1^2(x)(\mu+2\lambda)(\mu+1+\frac{12\delta}{2\lambda+1}) - B_2(x)(\lambda+\mu+2\delta\xi)^2} + \frac{B_1(x)(p_2 - r_2)}{2(\mu+2\lambda) \left(1 + \frac{6\delta}{12\lambda+1}\right)} \\ &= \left[\frac{(1-\eta)B_1^3(x)}{B_1^2(x)(\mu+2\lambda)(\mu+1+\frac{12\delta}{2\lambda+1}) - B_2(x)(\lambda+\mu+2\delta\xi)^2} + \frac{B_1(x)}{2(\mu+2\lambda) \left(1 + \frac{6\delta}{12\lambda+1}\right)} \right] p_2 \\ &\quad + \left[\frac{(1-\eta)B_1^3(x)}{B_1^2(x)(\mu+2\lambda)(\mu+1+\frac{12\delta}{2\lambda+1}) - B_2(x)(\lambda+\mu+2\delta\xi)^2} - \frac{B_1(x)}{2(\mu+2\lambda) \left(1 + \frac{6\delta}{12\lambda+1}\right)} \right] r_2 \\ &= B_1(x) \left\{ \left[t(\eta) + \frac{1}{2(\mu+2\lambda) \left(1 + \frac{6\delta}{12\lambda+1}\right)} \right] p_2 + \left[t(\eta) - \frac{1}{2(\mu+2\lambda) \left(1 + \frac{6\delta}{12\lambda+1}\right)} \right] r_2 \right\} \end{aligned}$$

for $\eta \in \mathbb{R}$, where $t(\eta) = \frac{(1-\eta)B_1^2(x)}{B_1^2(x)(\mu+2\lambda)(\mu+1+\frac{12\delta}{2\lambda+1}) - B_2(x)(\lambda+\mu+2\delta\xi)^2}$. A straightforward calculation implies here that

$$\begin{aligned} |a_3 - \eta a_2^2| &\leq |B_1(x)| \left\{ \left| t(\eta) - \frac{1}{2(\mu+2\lambda) \left(1 + \frac{6\delta}{12\lambda+1}\right)} \right| + \left| t(\eta) + \frac{1}{2(\mu+2\lambda) \left(1 + \frac{6\delta}{12\lambda+1}\right)} \right| \right\} \\ &= \begin{cases} \frac{|B_1(x)|}{(\mu+2\lambda) \left(1 + \frac{6\delta}{12\lambda+1}\right)}, & |1-\eta| \leq T(x, \lambda, \mu, \delta) \\ \frac{2|B_1(x)|^3 |1-\eta|}{|B_1^2(x)(\mu+2\lambda)(\mu+1+\frac{12\delta}{2\lambda+1}) - B_2(x)(\lambda+\mu+2\delta\xi)^2|}, & |1-\eta| \geq T(x, \lambda, \mu, \delta) \end{cases} \end{aligned}$$

The proof is thus completed. \square

Remark 2.4. Taking $\lambda = 1$ and $\delta = \mu = 0$ in Theorem 2.3 we obtain some bounds for the class $\mathcal{J}_{\Sigma}^0(1, 0)$ of bi-starlike functions as below:

$$|a_2| \leq \frac{B_1(x)\sqrt{2|B_1(x)|}}{\sqrt{|2B_1^2(x) - B_2(x)|}}, \quad (2.29)$$

$$|a_3| \leq B_1^2(x) + \frac{|B_1(x)|}{2} \quad (2.30)$$

and for some $\eta \in \mathbb{R}$

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|B_1(x)|}{2}, & |1-\eta| \leq \frac{|2B_1^2(x) - B_2(x)|}{4B_1^2(x)} \\ \frac{2|B_1(x)|^3 |1-\eta|}{|2B_1^2(x) - B_2(x)|}, & |1-\eta| \geq \frac{|2B_1^2(x) - B_2(x)|}{4B_1^2(x)}. \end{cases} \quad (2.31)$$

3 The Class $\mathcal{Y}_\Sigma(\zeta, \gamma)$

In this section we define a new function class $\mathcal{Y}_\Sigma(\zeta, \gamma)$ and determine some bounds for initial coefficients and the Fekete-Szegő functional.

Definition 3.1. Let $\gamma > 0$ and $\zeta > 0$. If the function $f(z) \in \Sigma$ of the form (1.1) satisfies the following conditions, then it is called in the class $\mathcal{Y}_\Sigma(\zeta, \gamma)$:

$$1 + \frac{1}{\gamma} \left(\frac{zf'(z) + z^2f''(z)}{(1-\zeta)z + \zeta zf'(z)} - 1 \right) \prec \frac{ze^{xz}}{e^z - 1}, \quad (3.1)$$

$$1 + \frac{1}{\gamma} \left(\frac{wg'(w) + w^2g''(w)}{(1-\zeta)w + \zeta wg'(w)} - 1 \right) \prec \frac{we^{xw}}{e^w - 1}, \quad (3.2)$$

where the function g is of the form (1.2).

Remark 3.2. Taking $\zeta = \gamma = 1$ in Definition 3.1 we obtain the class $\mathcal{Y}_\Sigma(1, 1)$ of bi-convex functions and it satisfies the following subordinations:

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{ze^{xz}}{e^z - 1} = F(x, z), \quad (3.3)$$

$$1 + \frac{wg''(w)}{g'(w)} \prec \frac{we^{xw}}{e^w - 1} = F(x, w). \quad (3.4)$$

Theorem 3.3. Suppose that $\zeta \neq 2$, $0 \leq \zeta < 3$, and $\gamma > 0$. If $f \in \mathcal{Y}_\Sigma(\zeta, \gamma)$, then

$$|a_2| \leq \frac{\gamma |B_1(x)| \sqrt{|B_1(x)|}}{\sqrt{|(4\zeta^2 - 11\zeta + 9)B_1^2(x)\gamma - B_2(x)2(2-\zeta)^2|}}, \quad (3.5)$$

$$|a_3| \leq \frac{B_1^2(x)\gamma^2}{4(2-\zeta)^2} + \frac{|B_1(x)|\gamma}{|3(3-\zeta)|} \quad (3.6)$$

and for $\eta \in \mathbb{R}$

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{\gamma^2 |B_1(x)|}{3(3-\zeta)}, & |1-\eta| \leq \frac{|[(4\zeta^2 - 11\zeta + 9)B_1^2(x)\gamma^2 - 2B_2(x)(2-\zeta)^2]|}{3\gamma^2 B_1^2(x)(3-\zeta)} \\ \frac{\gamma^2 |B_1(x)|^3 |1-\eta|}{|[(4\zeta^2 - 11\zeta + 9)B_1^2(x)\gamma^2 - 2B_2(x)(2-\zeta)^2]|}, & |1-\eta| \geq \frac{|[(4\zeta^2 - 11\zeta + 9)B_1^2(x)\gamma^2 - 2B_2(x)(2-\zeta)^2]|}{3\gamma^2 B_1^2(x)(3-\zeta)} \end{cases}. \quad (3.7)$$

Proof . Let $\zeta \neq 2$, $0 \leq \zeta < 3$, $\gamma > 0$ and $f(z) \in \mathcal{Y}_\Sigma(\zeta, \gamma)$. By Definition 3.1, there are two Schwarz functions $u, v : \Delta \rightarrow \Delta$,

$$u(z) = u_1z + u_2z^2 + u_3z^3 + \dots, \quad (3.8)$$

$$v(w) = v_1w + v_2w^2 + v_3w^3 \dots \quad (3.9)$$

such that

$$1 + \frac{1}{\gamma} \left(\frac{zf'(z) + z^2f''(z)}{(1-\zeta)z + \lambda zf'(z)} - 1 \right) = F(x, u(z)) \quad (3.10)$$

and

$$1 + \frac{1}{\gamma} \left(\frac{wg'(w) + w^2g''(w)}{(1-\zeta)w + \zeta wg'(w)} - 1 \right) = F(x, v(w)), \quad (3.11)$$

where $z, w \in \Delta$. It is well-known the definition of Schwarz function that $|u_i| \leq 1$ and $|v_i| \leq 1$ for $\forall i \in \mathbb{N}$. A basic calculation yields that right hand sides of the equations (3.10) and (3.11) are, respectively,

$$F(x, u(z)) = B_0(x) + [B_1(x)u_1]z + [B_1(x)u_2 + \frac{B_2(x)}{2!}u_1^2]z^2 + \left[B_1(x)u_3 + B_2(x)u_1u_2 + \frac{B_3(x)}{3!}u_1^3 \right]z^3 + \dots \quad (3.12)$$

and

$$F(x, v(w)) = B_0(x) + [B_1(x)v_1]w + [B_1(x)v_2 + \frac{B_2(x)}{2!}v_1^2]w^2 + \left[B_1(x)v_3 + B_2(x)v_1v_2 + \frac{B_3(x)}{3!}v_1^3 \right] w^3 + \dots, \quad (3.13)$$

where $B_0(x) = 1$. In addition, left hand sides of the equations (3.10) and (3.11) are, respectively,

$$1 + \frac{1}{\gamma} \left(\frac{zf'(z) + z^2f''(z)}{(1-\zeta)z + \zeta zf'(z)} - 1 \right) = 1 + \frac{2(2-\zeta)}{\gamma} a_2 z + \frac{3(3-\zeta)a_3 - 4\zeta(2-\zeta)a_2^2}{\gamma} z^2 + \dots \quad (3.14)$$

and

$$1 + \frac{1}{\gamma} \left(\frac{wg'(w) + w^2g''(w)}{(1-\zeta)w + \zeta wg'(w)} - 1 \right) = 1 - \frac{2(2-\zeta)}{\gamma} a_2 w + \frac{3(3-\zeta)(2a_2^2 - a_3) - 4\zeta(2-\zeta)a_2^2}{\gamma} w^2 + \dots \quad (3.15)$$

Here, by comparing the coefficients of the equations (3.12) and (3.14) we obtain

$$\frac{2(2-\zeta)}{\gamma} a_2 = B_1(x)u_1 \quad (3.16)$$

and

$$\frac{3(3-\zeta)a_3 - 4\zeta(2-\zeta)a_2^2}{\gamma} = B_1(x)u_2 + \frac{B_2(x)}{2!}u_1^2. \quad (3.17)$$

Also, by similar point of view from the equations (3.13) and (3.15) we have

$$-\frac{2(2-\zeta)}{\gamma} a_2 = B_1(x)v_1 \quad (3.18)$$

and

$$\frac{3(3-\zeta)(2a_2^2 - a_3) - 4\zeta(2-\zeta)a_2^2}{\gamma} = B_1(x)v_2 + \frac{B_2(x)}{2!}v_1^2. \quad (3.19)$$

Now, from the equations (3.16) and (3.18), it follows that

$$u_1 = -v_1, \quad (3.20)$$

$$8(2-\zeta)^2 a_2^2 = B_1^2(x)(u_1^2 + v_1^2)\gamma^2 \quad (3.21)$$

and

$$a_2^2 = \frac{B_1^2(x)(u_1^2 + v_1^2)\gamma^2}{8(2-\zeta)^2}. \quad (3.22)$$

Summation of the expressions 3.17 and 3.19 imply that

$$\frac{(8\zeta^2 - 22\zeta + 18)}{\gamma} a_2^2 = B_1(x)(u_2 + v_2) + \frac{B_2(x)4(2-\zeta)^2 a_2^2}{B_1^2(x)\gamma^2}. \quad (3.23)$$

Using equation (3.22) in (3.23) one can easily see that

$$a_2^2 = \frac{B_1^3(x)(u_2 + v_2)\gamma^2}{2[(4\zeta^2 - 11\zeta + 9)B_1^2(x)\gamma - B_2(x)2(2-\zeta)^2]}. \quad (3.24)$$

Since $|u_i| \leq 1, |v_i| \leq 1$ for $\forall i \in \mathbb{N}$, by using triangle inequality in (3.24) we can write that

$$|a_2| \leq \frac{\gamma |B_1(x)| \sqrt{2|B_1(x)|}}{\sqrt{2|(4\zeta^2 - 11\zeta + 9)B_1^2(x)\gamma - B_2(x)2(2-\zeta)^2|}}. \quad (3.25)$$

On the other hand, if we subtract (3.17) from (3.19), then we get

$$a_3 = a_2^2 + \frac{B_1(x)(u_2 - v_2)\gamma}{6(3-\zeta)}. \quad (3.26)$$

If we replace (3.22) in (3.26), then we can write that

$$a_3 = \frac{B_1^2(x)(u_1^2 + v_1^2)\gamma^2}{8(2-\zeta)^2} + \frac{B_1(x)(u_2 - v_2)\gamma}{6(3-\zeta)}. \quad (3.27)$$

Now, using triangle inequality in the last equality we deduce

$$|a_3| \leq \frac{B_1^2(x)\gamma^2}{4(2-\zeta)^2} + \frac{|B_1(x)|\gamma}{|3(3-\zeta)|}. \quad (3.28)$$

Finally, taking into account the equations (3.24) and (3.26), we can write that

$$\begin{aligned} a_3 - \eta a_2^2 &= \frac{(1-\eta)B_1^3(x)(u_2 + v_2)\gamma^2}{2[(4\zeta^2 - 11\zeta + 9)B_1^2(x)\gamma - B_2(x)2(2-\zeta)^2]} + \frac{B_1(x)(u_2 - v_2)\gamma}{6(3-\zeta)} \\ &= B_1(x)\gamma \left[\frac{(1-\eta)B_1^2(x)\gamma}{2[(4\zeta^2 - 11\zeta + 9)B_1^2(x)\gamma - B_2(x)2(2-\zeta)^2]} + \frac{1}{6(3-\zeta)} \right] u_2 \\ &\quad + B_1(x)\gamma \left[\frac{(1-\eta)B_1^2(x)\gamma}{2[(4\zeta^2 - 11\zeta + 9)B_1^2(x)\gamma - B_2(x)2(2-\zeta)^2]} - \frac{1}{6(3-\zeta)} \right] v_2 \\ &= B_1(x)\gamma \left\{ \left[\kappa(\eta) + \frac{1}{6(3-\zeta)} \right] u_2 + \left[\kappa(\eta) - \frac{1}{6(3-\zeta)} \right] v_2 \right\} \end{aligned}$$

for $\eta \in \mathbb{R}$, where $\kappa(\eta) = \frac{(1-\eta)B_1^2(x)\gamma}{2[(4\zeta^2 - 11\zeta + 9)B_1^2(x)\gamma - B_2(x)2(2-\zeta)^2]}$. A straightforward calculation implies here that

$$\begin{aligned} |a_3 - \eta a_2^2| &\leq |B_1(x)|\gamma \left\{ \left| \kappa(\eta) + \frac{1}{6(3-\zeta)} \right| + \left| \kappa(\eta) - \frac{1}{6(3-\zeta)} \right| \right\} \\ &\leq \begin{cases} \frac{\gamma^2 |B_1(x)|}{3(3-\zeta)}, & |1-\eta| \leq \frac{|[(4\zeta^2 - 11\zeta + 9)B_1^2(x)\gamma^2 - 2B_2(x)(2-\zeta)^2]|}{3\gamma^2 B_1^2(x)(3-\zeta)}, \\ \frac{\gamma^2 |B_1(x)|^3 |1-\eta|}{|[(4\zeta^2 - 11\zeta + 9)B_1^2(x)\gamma^2 - 2B_2(x)(2-\zeta)^2]|}, & |1-\eta| \geq \frac{|[(4\zeta^2 - 11\zeta + 9)B_1^2(x)\gamma^2 - 2B_2(x)(2-\zeta)^2]|}{3\gamma^2 B_1^2(x)(3-\zeta)}. \end{cases} \end{aligned}$$

□

Remark 3.4. Taking $\gamma = \zeta = 1$ in Theorem 3.3 we obtain some bounds for the class $\mathcal{Y}_{\Sigma}(1, 1)$ of bi-convex functions as below:

$$|a_2| \leq \frac{B_1(x)\sqrt{|B_1(x)|}}{\sqrt{2|B_1^2(x) - B_2(x)|}}, \quad (3.29)$$

$$|a_3| \leq \frac{B_1^2(x)}{4} + \frac{|B_1(x)|}{6} \quad (3.30)$$

and

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|B_1(x)|}{6}, & |1-\eta| \leq \frac{|B_1^2(x) - B_2(x)|}{3B_1^2(x)} \\ \frac{|B_1(x)|^3 |1-\eta|}{2|B_1^2(x) - B_2(x)|}, & |1-\eta| \geq \frac{|B_1^2(x) - B_2(x)|}{3B_1^2(x)} \end{cases}. \quad (3.31)$$

4 Conclusion

In the present investigation two new subclasses of analytic and bi-univalent functions are introduced by using Bernoulli polynomial. Also, some coefficient bounds are estimated for certain coefficients of functions belonging to these subclasses defined. In addition, the Fekete-Szegő problem are handled for the mentioned function subclasses. Finally, a few remarks are indicated for the certain function subclasses which are related to bi-starlike and bi-convex functions.

References

- [1] İ. Aktaş and N. Yılmaz, *Initial coefficients estimate and Fekete-Szegő problems for two new subclasses of bi-univalent functions*, Konuralp J. Math. **10** (2022), no. 1, 138–148.
- [2] T. Al-Hawary, A. Amourah, and B.A. Frasin, *Fekete-Szegő inequality for bi-univalent functions by means of Horadam polynomials*, Bol. Soc. Mat. Mex. **27** (2021), 1–12.
- [3] A. Amourah, B.A. Frasin, and T. Abdeljawad, *Fekete-Szegő inequality for analytic and bi-univalent functions subordinate to Gegenbauer polynomials*, J. Funct. Spaces **2021** (2021), 5574673.
- [4] A. Amourah, B.A. Frasin, M. Ahmad, and F. Yousef, *Exploiting the Pascal distribution series and Gegenbauer polynomials to construct and study a new subclass of analytic bi-univalent functions*, Symmetry **14** (2022), no. 1, 147.
- [5] D.A. Brannan and T.S. Taha, *On some classes of bi-univalent function*, Stud.Univ. Babeş-Bolyai Math. **31** (1986), 70–77.
- [6] M. Buyankara and M. Çağlar, *On Fekete-Szegő problem for a new subclass of bi-univalent functions defined by Bernoulli polynomials*, Acta Univ. Apulensis Math. Inform. **71** (2022) 137–145.
- [7] M. Buyankara, M. Çağlar and LI. Cotîrlă, *New subclasses of bi-univalent functions with respect to the symmetric points defined by Bernoulli polynomials*, Axioms **11** (2022), 652.
- [8] L.I. Cotîrlă, *New classes of analytic and bi-univalent functions*, AIMS Math. **6** (2021), 10642–10651.
- [9] P.L. Duren, *Univalent Functions*, Springer Science and Business Media, 2001.
- [10] J. Dziok, *A general solution of the Fekete-Szegő problem*, Bound. Value Probl. **2013** (2013), no. 1, 1–13.
- [11] M. Fekete and G. Szegő *Eine bemerkung über ungerade schlichte funktionen*, J. Lond. Math. Soc. **1** (1933), no. 2, 85–89.
- [12] B.A. Frasin and M.K. Aouf, *New subclasses of bi-univalent functions*, Appl. Math. Lett. **24** (2011), 1569–1573.
- [13] H.Ö. Güney, G. Murugusundaramoorthy, and J. Sokoł, *Subclasses of bi-univalent functions related to shell-like curves connected with Fibonacci numbers*, Acta Univ. Sapientiae Math. **10** (2018), no. 1, 70–84.
- [14] M. Lewin, *On a coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc. **18** (1967), 63–68.
- [15] S.S. Miller and P.T. Mocanu, *Differential Subordinations*, Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, 2000.
- [16] P. Natalini and A. Bernardini, *A generalization of the Bernoulli polynomials*, J. Appl. Math. **2003** (2003), no. 3, 155–163.
- [17] H. Orhan, İ. Aktaş, and H. Arıkan, *On new subclasses of bi-univalent functions associated with the (p, q) -Lucas polynomials and bi-Bazilevič type functions of order $\rho + \xi$* , Turk. J. Math. **47** (2023), no. 1, 98–109.
- [18] GI. Oros and LI. Cotîrlă, *Coefficient estimates and the Fekete-Szegő problem for new classes of m -fold symmetric bi-univalent functions*, Mathematics **10** (2022), 129.
- [19] H.M. Srivastava, Ş. Altınkaya, and S. Yalçın, *Certain Subclasses of bi-univalent functions associated with the Horadam polynomials*, Iran. J. Sci. Technol. Trans. A Sci. **43** (2019), 1873–1879.
- [20] H.M. Srivastava, S. Gaboury, and F. Ghanim, *Coefficient estimates for some general subclasses of analytic and bi-univalent functions*, Afr. Mat. **28** (2017), 693–706.
- [21] H.M. Srivastava, A.K. Mishra, and P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett. **23** (2010), 1188–1192.
- [22] H.M. Srivastava, G. Murugusundaramoorthy, and K. Vijaya, *Coefficient estimates for some families of bi-Bazilevič functions of the Ma-Minda type involving the Hohlov operator*, J. Class. Anal. **2** (2013), no. 2, 167–181.
- [23] A.K. Wanas and L.I. Cotîrlă, *Initial coefficient estimates and Fekete-Szegő inequalities for new families of bi-univalent functions governed by $(p - q)$ Wanas operator*, Symmetry **13** (2021), no. 11, 2118.
- [24] A.K. Wanas and L.I. Cotîrlă, *New applications of Gegenbauer polynomials on a new family of bi-Bazilevič functions*

-
- governed by the q -Srivastava-Attiya Operator*, Mathematics 10 (2022), no. 8, 1309.
- [25] N. Yılmaz and İ. Aktaş, *On some new subclasses of bi-univalent functions defined by generalized bivariate Fibonacci polynomial*, Afr. Mat. **33** (2022), no. 2, 59.
- [26] P. Zaprawa, *On the Fekete-Szegő problem for classes of bi-univalent functions*, Bull. Belg. Math. Soc. Simon Stevin **21** (2014), no. 1, 169–178.