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Construction of compact-integral operators on $C^\infty(\mathbb{R}_+)$ and $C^n(\mathbb{R}_+)$ with application in the study of functional integro-differential equations

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Abstract

In this brief note, we present a fixed point theorem in the Fréchet space. Also we study a new family of measures of noncompactness on $C^\infty(\mathbb{R}_+)$ and $C^n(\mathbb{R}_+)$ and we investigate the construction of compact-integral operators on $C^\infty(\mathbb{R}_+)$ and $C^n(\mathbb{R}_+)$. Finally, we provide various examples which illustrate the existence of solutions for a wide variety of functional integral-differential equations.

Keywords: Family of measures of noncompactness, condensing operators, integral-differential equations
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1 Introduction

Many of the operators that arise in the study of integral equations are compact. Some compactness results in this direction are often vital in existencing proving differential, integral and functional integral equations (see [1, 3, 7, 8, 12], for example).

The degree of noncompactness of a set is measured by means of functions called measures of noncompactness. The first measure of noncompactness, the function α, was defined and studied by Kuratowski [17] for purely topological considerations. In 1955, Darbo [11] used this measure to generalize Banachś contraction mapping principle for so-called

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condensing operators. Measures of noncompactness are very useful tools in characterizing compact operators as well as in differential and integral equations (see for instance [1, 4, 5, 6, 9, 10, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22] and the references therein).

Now, we organize this paper as follows. In Section 2, we present a fixed point theorem in the Fréchet space. In Section 3, we study a new family of measures of noncompactness on $C^{\infty}(\mathbb{R}_+)$ and $C^n(\mathbb{R}_+)$. In Section 4, we investigate the construction of compact-integral operators on $C^{\infty}(\mathbb{R}_+)$ and $C^n(\mathbb{R}_+)$. Finally, in Section 5, we provide various examples which show how the previous sections can be useful for proving the existence of solutions for a wide variety of the functional integral-differential equations. A deep approach in this work as a motivating factor for the readers is to see how the operator H, in Theorem 4.6 and Theorem 4.7, is not compact on space $C^{\infty}(\mathbb{R}_+)$, but it can be $\lambda_{k,m}$-condensing on the space $C^{\infty}(\mathbb{R}_+)$ (see Example 5.2) and also how the functional integral-differential equation in Example 5.1 has at least one solution which belongs to the space $C^{\infty}(\mathbb{R}_+)$, but it has no solution in the space $C^n(\mathbb{R}_+)$ for any $n \in \mathbb{N}$.

2 Notation and auxiliary facts

Definition 2.1. Let $\mathcal{M}$ be a class of subsets of a Fréchet space E and $\mathfrak{N}_E$ indicates the subfamily consisting of all relatively compact sets, we say $\mathcal{M}$ is an admissible set if $X \in \mathcal{M}$, then $Conv(X)$, $\overline{X} \in \mathcal{M}$.

Definition 2.2. Suppose that $\mathcal{M}$ be an admissible subset of a Fréchet space E and I be an index set, we say that a family of functions $\{\mu_\alpha\}_{\alpha\in I}$, where $\mu_\alpha : \mathcal{M} \longrightarrow \mathbb{R}_+$, is said to be a family of measures of noncompactness in E if it satisfies the following conditions:

1° The family $ker\{\mu_\alpha\} = \{X \in \mathcal{M} : \mu_\alpha(X) = 0 \ for \ \alpha \in I\}$ is nonempty and $ker\{\mu_\alpha\} \subseteq \mathfrak{N}_E$.

2° $X \subset Y \Longrightarrow \mu_\alpha(X) \leq \mu_\alpha(Y)$ for $\alpha \in I$.

3° $\mu_\alpha(\overline{X}) = \mu_\alpha(X)$ for $\alpha \in I$.

4° $\mu_\alpha(ConvX) = \mu_\alpha(X)$ for $\alpha \in I$.

5° $\mu_\alpha(\lambda X + (1-\lambda)Y) \leq \lambda\mu_\alpha(X) + (1-\lambda)\mu_\alpha(Y)$ for $\lambda \in [0,1]$ and $\alpha \in I$.

6° If $\{X_n\}$ is a sequence of closed sets from $\mathcal{M}$ such that $X_{k+1} \subset X_k$ for $k = 1, 2, \cdots$ and if $\lim_{k\to\infty} \mu_\alpha(X_k) = 0$ for $\alpha \in I$ then $X_\infty = \cap_{k=1}^{\infty} X_k \neq \emptyset$.

We say that a family of measure of noncompactness is regular if it additionally satisfies the following conditions:

7° $\mu_\alpha(X \cup Y) = \max\{\mu_\alpha(X), \mu_\alpha(Y)\}$ for $\alpha \in I$.

8° $\mu_\alpha(X + Y) \leq \mu_\alpha(X) + \mu_\alpha(Y)$ for $\alpha \in I$.

9° $\mu_\alpha(\lambda X) = |\lambda|\mu_\alpha(X)$ for $\lambda \in \mathbb{R}$ and $\alpha \in I$.

10° $ker\{\mu_\alpha\} = \mathfrak{N}_E$.

Definition 2.3. Let $\mathcal{M}$ be an admissible subset of a Fréchet space E. An operator (not necessarily linear) $F : E \longrightarrow E$ is compact if the closure of $F(Y)$ is compact whenever $Y \in \mathcal{M}$.

Theorem 2.4. (Tychonoff fixed point theorem [2]). Let E be a Hausdorff locally convex linear topological space, C be a convex subset of E and $F : C \longrightarrow E$ be a continuous mapping such that

$$F(C) \subseteq A \subseteq C,$$

with A compact. Then F has at least one fixed point.

Theorem 2.5. Let Ω be a nonempty and convex subset of a Fréchet space X, $\mathcal{M}$ an admissible set such that $\Omega \in \mathcal{M}$ and μ_α is a family of regular measures of noncompactness in E and let $F : \Omega \longrightarrow E$ be a continuous operator such that

$$\mu_\alpha(F(X)) \leq \varphi_\alpha(\mu_\alpha(X)), \tag{2.1}$$

and $F(X) \in \mathcal{M}$ for any nonempty subset $X \in \mathcal{M}$ where $\varphi_\alpha : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are given functions such that φ_α are continuous on $\mathbb{R}_+$, $\varphi_\alpha(0) = 0$ and $\varphi_\alpha(t) < t$ for $t > 0$. Assume that $H : \Omega \longrightarrow E$ is a compact and continuous operator. Define $T(x) := F(x) + H(x)$ and assume that $T(x) \in \Omega$ for all $x \in \Omega$. Then T has a fixed point in Ω.

Proof . We define a sequence $\{\Omega_n\}$ by letting $\Omega_0 = \Omega$ and $\Omega_n = Conv(T\Omega_{n-1})$, $n \geq 1$. Then we have $T\Omega_0 = T\Omega \subseteq \Omega = \Omega_0, \Omega_1 = Conv(T\Omega_0) \subseteq \Omega = \Omega_0$, and by continuing this process we obtain

$$\Omega_0 \supseteq \Omega_1 \supseteq \Omega_2 \supseteq \cdots .$$

If there exists an integer $N \geq 0$ such that $\mu_\alpha(\Omega_N) = 0$ for all $\alpha \in I$, then Ω_N is relatively compact. Thus, Tychonoff fixed point theorem implies that T has a fixed point. Next, we assume that $\mu_\alpha(\Omega_n) > 0$ for $n = 1, 2, \ldots$ and for all $\alpha \in I$. Since $\mu_\alpha(\Omega_n)$ is a positive decreasing sequence of real numbers, there is $r_\alpha \geq 0$ such that $\mu_\alpha(\Omega_n) \longrightarrow r_\alpha$ as $n \longrightarrow \infty$. On the other hand, in view of (2.1) and since H is a compact operator and μ_α is a regular measure of noncompactness, we obtain

$$\limsup_{n\longrightarrow\infty} \mu_\alpha(\Omega_{n+1}) \leq \limsup_{n\longrightarrow\infty} \varphi_\alpha(\mu_\alpha(\Omega_n)).$$

This show that $r_\alpha \leq \varphi_\alpha(r_\alpha)$. Consequently $r_\alpha = 0$. Hence we deduce that $\mu_\alpha(\Omega_n) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $\alpha \in I$. Since the sequence (Ω_n) are nested, in view of axiom (6°) of Definition 2.2 we derive that the set $\Omega_\infty = \bigcap_{n=1}^{\infty} \Omega_n$ is nonempty, closed and convex subset of the set Ω. Moreover, the set Ω_∞ is invariant under the operator T and belongs to $Ker\{\mu_\alpha\}$. Now, Tychonoff fixed point theorem implies that T has a fixed point in the set Ω. □

3 Construction of the family of measures of noncompactness on $C^\infty(\mathbb{R}_+)$ and $C^n(\mathbb{R}_+)$

Let $n \in \mathbb{N}$, we denote by $C^n(\mathbb{R}_+)$ the space of all real functions which are n times continuously differentiable on $\mathbb{R}_+$ and $C^\infty(\mathbb{R}_+) = \bigcap_{k\in\mathbb{N}} C^k(\mathbb{R}_+)$ with the family of seminorm

$$|x|_{k,m} = \sup\{|x^{(k')}(t)| : t \in [0, m], 0 \leq k' \leq k\},$$

for $k, m \in \mathbb{N}$. The space $C^\infty(\mathbb{R}_+)$ is a Fréchet space furnished with the distance

$$d(x, y) = \sup\Big\{\frac{1}{2^{km}} \min\{1, |x - y|_{k,m}\} : k, m \in \mathbb{N}\Big\}.$$

A nonempty subset $X \subset C^\infty(\mathbb{R}_+)$ is said to be bounded if

$$|X|_{k,m} := \sup\{|x|_{k,m} : x \in X\} < \infty,$$

for all $k, m \in \mathbb{N}$. Further, let $\mathfrak{M}_{C^\infty}$ be the family of all nonempty and bounded subsets of $C^\infty(\mathbb{R}_+)$. Obviously, $\mathfrak{M}_{C^\infty}$ is an admissible set and $\mathfrak{N}_{C^\infty} \subset \mathfrak{M}_{C^\infty}$.

Let us recall two facts which are crucial in our considerations.

(A) A sequence $\{x_n\}$ is convergent to x in $C^\infty(\mathbb{R}_+)$ if and only if $\{x_n\}$ is convergent to x in $C^k[0, m]$ for all $k, m \in \mathbb{N}$.

(B) A subset $\mathcal{F} \subset C^\infty(\mathbb{R}_+)$ is totally bounded (relatively compact) if and only if $\mathcal{F}^{(k)} = \{f^{(k)} : f \in \mathcal{F}\}$ are bounded and equicontinuous on $[0, m]$ for all $k, m \in \mathbb{N}$.

Now, we recall the definition of quantities which will be used in our further investigations. Let X be a bounded subset of the space $C^\infty(\mathbb{R}_+)$ and $k, m \in \mathbb{N}$. Fix $x \in X$ and $\varepsilon > 0$. Let us denote

$$\omega^{k,m}(x, \varepsilon) = \sup\{|x^{(k)}(t_1) - x^{(k)}(t_2)| : t_1, t_2 \in [0, m], |t_1 - t_2| < \varepsilon\}.$$

Further,

$$\omega^{k,m}(X,\varepsilon) = \sup\{\omega^{k,m}(x,\varepsilon) : x \in X\},$$
$$\omega^{k,m}(X) = \lim_{\varepsilon\to 0} \omega^{k,m}(X,\varepsilon). \tag{3.1}$$

Theorem 3.1. The family of mappings $\{\omega^{k,m}\}$, where $\omega^{k,m} : \mathfrak{M}_{C^\infty} \longrightarrow \mathbb{R}_+$ given by (3.1) defines a regular family of measure of noncompactness on $C^\infty(\mathbb{R}_+)$. Also, $\omega^{k,m_1}(X) \le \omega^{k,m_2}(X)$ for all $X \in \mathfrak{M}_{C^\infty}$, $k \in \mathbb{N}$ and $m_1 \le m_2$.

Proof . The property $ker\{\omega^{k,m}\} = \mathfrak{N}_{C^\infty}$ is a simple consequence of (B). The conditions 2°, 3°, 4° and 5° are obvious. Now, we prove that 6° holds. Suppose that $\{X_n\}$ is a sequence of closed and nonempty sets of $\mathfrak{M}_{C^\infty}$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \cdots$, and $\lim_{n\to\infty} \omega^{k,m}(X_n) = 0$ for all $k, m \in \mathbb{N}$. Now for any $n \in \mathbb{N}$, take $x_n \in X_n$ and set $\mathcal{G} = \overline{\{x_n\}}$.

Claim: $\omega^{k,m}(\mathcal{G}) = 0$ for all $k, m \in \mathbb{N}$.

Let $\varepsilon > 0$ and $k, m \in \mathbb{N}$ be fixed, since $\lim_{n\to\infty} \omega^{k,m}(X_n) = 0$, then there exists $N \in \mathbb{N}$ such that $\omega^{k,m}(X_N) < \varepsilon$. Hence, we can find $\delta_1 > 0$ such that

$$\omega^{k,m}(X_N, \delta_1) < \varepsilon.$$

Thus, we have $\omega^{k,m}(x_n, \delta_1) < \varepsilon$ for all $n \ge N$. Also, we know that the set $\{x_1, x_2, \dots, x_{N-1}\}$ is compact, hence there exists $\delta_2 > 0$ such that $\omega^{k,m}(x_n, \delta_2) < \varepsilon$ for all $1 \le n \le N$. Therefore, we have

$$\omega^{k,m}(x_n, \delta) < \varepsilon.$$

If we define $\delta < \min\{\delta_1, \delta_2\}$, then we obtain

$$\omega^{k,m}(\mathcal{G}, \delta) < \varepsilon,$$

and $\omega^{k,m}(\mathcal{G}) = 0$ for all $k, m \in \mathbb{N}$. This claim shows that there exist a subsequence $\{x_{n_j}\}$ and $x_0 \in C^\infty(\mathbb{R}_+)$ such that $x_{n_j} \to x_0$. Since $x_n \in X_n$, $X_{n+1} \subset X_n$ and X_n is closed for all $n \in \mathbb{N}$, we get

$$x_0 \in \bigcap_{n=1}^{\infty} X_n = X_\infty,$$

which completes the proof of 6°. It remains to prove 7°, 8° and 9°. Suppose that $X, Y \in \mathfrak{M}_{C^\infty}$. Since for all $\varepsilon > 0$, $\lambda > 0$ and $k, m > 0$, we have

$$\omega^{k,m}(X \cup Y, \varepsilon) \le \max\{\omega^{k,m}(X,\varepsilon), \omega^{k,m}(Y,\varepsilon)\},$$
$$\omega^{k,m}(X + Y, \varepsilon) \le \omega^{k,m}(X,\varepsilon) + \omega^{k,m}(Y,\varepsilon),$$
$$\omega^{k,m}(\lambda X, \varepsilon) \le \lambda\omega^{k,m}(X,\varepsilon),$$

then the hypotheses 7°, 8° and 9° are satisfied. Moreover, if $m_1 \le m_2$, then for all $X \in \mathfrak{M}_{C^\infty}$ and $\varepsilon > 0$ we have

$$\{|x^{(k)}(t_1) - x^{(k)}(t_2)| : t_1, t_2 \in [0, m_1], |t_1 - t_2| < \varepsilon\}$$
$$\subset \{|x^{(k)}(t_1) - x^{(k)}(t_2)| : t_1, t_2 \in [0, m_2], |t_1 - t_2| < \varepsilon\},$$

and we obtain $\omega^{k,m_1}(X) \le \omega^{k,m_2}(X)$. □

On the other hand, the space $C^n(\mathbb{R}_+)$ is a Fréchet space furnished with the family of semi-norms

$$|x|_{k,m} = \sup\{|x^{(k)}(t)| : t \in [0, m]\},$$

for $m \in \mathbb{N}$ and $0 \le k \le n$ and the distance

$$d(x, y) = \sup\Big\{\frac{1}{2^{km}} \min\{1, |x - y|_{k,m}\} : m \in \mathbb{N}, 0 \le k \le n\Big\}.$$

A nonempty subset $X \subset C^n(\mathbb{R}_+)$ is said to be bounded if

$$\sup\{|x|_{k,m} : x \in X\} < \infty,$$

for all $k, m \in \mathbb{N}$. Further, let $\mathfrak{M}_{C^n}$ be the family of all nonempty and bounded subsets of $C^n(\mathbb{R}_+)$. Obviously, $\mathfrak{M}_{C^n}$ is an admissible set and $\mathfrak{N}_{C^n} \subset \mathfrak{M}_{C^n}$. Let us recall two facts which are crucial in our considerations.

(A) A sequence $\{x_l\}$ is convergent to x in $C^n(\mathbb{R}_+)$ if and only if $\{x_l\}$ is convergent to x in $C^k[0,m]$ for all $m \in \mathbb{N}$ and $0 \le k \le n$.

(B) A subset $\mathcal{F} \subset C^n(\mathbb{R}_+)$ is totally bounded (relatively compact) if and only if $\mathcal{F}^{(k)} = \{f^{(k)} : f \in \mathcal{F}\}$ are bounded and equicontinuous on $[0,m]$ for all $m \in \mathbb{N}$ and $0 \le k \le n$.

Let X be a bounded subset of the space $C^n(\mathbb{R}_+)$ and $k, m \in \mathbb{N}$. Fix $x \in X$ and $\varepsilon > 0$. Then, the condition (3.1) also holds on $C^n(\mathbb{R}_+)$.

Theorem 3.2. The family of mappings $\{\omega^{k,m}\}$, where $\omega^{k,m} : \mathfrak{M}_{C^n} \longrightarrow \mathbb{R}_+$ given by (3.1) defines a regular family of measure of noncompactness on $C^n(\mathbb{R}_+)$. Also, $\omega^{k,m_1}(X) \le \omega^{k,m_2}(X)$ for all $X \in \mathfrak{M}_{C^n}$, $0 \le k \le n$ and $m_1 \le m_2$.

4 Compact-integral operators on $C^\infty(\mathbb{R}_+)$ and $C^n(\mathbb{R}_+)$

In this section, we obtain main results about the compactness and continuity of Volterra and Fredholm integral operators.

Theorem 4.1. Suppose that the following conditions hold true.

(i) $T : C^\infty(\mathbb{R}_+) \longrightarrow C(\mathbb{R}_+)$ be a continuous operator and there exists a continuous function $a : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that
$$|Tx(t)| \le a(t),$$
for all $t \in \mathbb{R}_+$ and $x \in C^\infty(\mathbb{R}_+)$.

(ii) $k : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a continuous function and has a continuous derivative of order n with respect to the first argument such that $n \in \mathbb{N}$. Also, functions $s \longrightarrow a(s)k(t,s)$ and $s \longrightarrow a(s)\frac{\partial^i k}{\partial t^i}(t,s)$ are integrable over $\mathbb{R}_+$ for any fixed $t \in \mathbb{R}_+$ and $i \in \mathbb{N}$.

Then $H : C^\infty(\mathbb{R}_+) \longrightarrow C^\infty(\mathbb{R}_+)$ defined by
$$Hx(t) = \int_0^\infty k(t,s)Tx(s)ds, \tag{4.1}$$
is a compact and continuous operator and
$$|Hx|_{n,m} \le \alpha_{n,m},$$
where $\alpha_{n,m} = \sup\{\int_0^\infty \frac{\partial^i k}{\partial t^i}(t,s)a(s)ds : t \in [0,m], 0 \le i \le n\}$.

Remark 4.2. Note that according to the hypotheses (ii) of Theorem 4.1, we have
$$\lim_{T \longrightarrow \infty} \int_T^\infty \frac{\partial^n k}{\partial t^n}(t,s)a(s)ds = 0,$$
for any fixed $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$. Since $[0,m]$ is compact interval, so we get
$$\lim_{T \longrightarrow \infty} \sup\{\int_T^\infty \frac{\partial^i k}{\partial t^i}(t,s)a(s)ds : t \in [0,m], 0 \le i \le n\} = 0,$$
for all $m, n \in \mathbb{N}$.

Proof . In view of the imposed assumptions, the function $Hx(t)$ is continuous on $\mathbb{R}_+$ for any $x \in C^\infty(\mathbb{R}_+)$. Also, for any $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$ we have
$$\frac{d^n(Hx)}{dt^n}(t) = \int_0^\infty \frac{\partial^n k}{\partial t^n}(t,s)Tx(s)ds,$$
and Hx has continuous derivative of order $n \in \mathbb{N}$. Now, we show that the map H is continuous. To do this, let us fix $\varepsilon > 0$, $m \in \mathbb{N}$ and take arbitrary $x, y \in C^\infty(\mathbb{R}_+)$ with $d(x,y) < \varepsilon$. Then, for $t \in [0,m]$, we have
$$\Big|Hx(t) - Hy(t)\Big| = \Big|\int_0^\infty k(t,s)\Big[Tx(s) - Ty(s)\Big]ds\Big|.$$

This result together condition (ii) imply that there exists $b>0$ such that

$$\sup\Big\{\int_b^\infty a(s)|k(t,s)|ds : t\in[0,m]\Big\}<\varepsilon,$$

and we obtain

$$\begin{aligned}\Big|Hx(t)-Hy(t)\Big| &\le \Big|\int_0^b k(t,s)\Big[Tx(s)-Ty(s)\Big]ds\Big|+2\Big|\int_b^\infty k(t,s)a(s)ds\Big|\\ &\le bK^0_{b,m}\omega^b_{\{r_j\}}(T,\varepsilon)+2\varepsilon,\end{aligned}$$

and similarly

$$\Big|\frac{d^n(Hx)}{dt^n}(t)-\frac{d^n(Hy)}{dt^n}(t)\Big| \le bK^n_{b,m}\omega^b_{\{r_j\}}(T,\varepsilon)+2\varepsilon,$$

for all $n\in\mathbb{N}$, where

$$\begin{aligned}&r_j=\sup\{|x^{(j)}(t)| : t\in[0,m]\}+2^{jm}\varepsilon,\quad j\in\mathbb{N},\\ &K^0_{b,m}=\sup\{|k(t,s)| : t\in[0,m], s\in[0,b]\},\\ &K^n_{b,m}=\sup\{|\frac{\partial^n k}{\partial t^n}(t,s)| : t\in[0,m], s\in[0,b]\},\\ &\omega^b_{\{r_j\}}(T,\varepsilon)=\sup\{|Tx(s)-Ty(s)| : s\in[0,b],\ x,y\in[-r_j,r_j],\ d(x,y)\le\varepsilon\}.\end{aligned}$$

By using the continuity of T on the compact set $\prod_{j\in\mathbb{N}}[-r_j,r_j]$ (Tychonoff's theorem implies that $\prod_{j\in\mathbb{N}}[-r_j,r_j]$ is a compact space), we have $\omega^b_{\{r_j\}}(T,\varepsilon)\longrightarrow 0$ as $\varepsilon\longrightarrow 0$. Moreover, in view of assumption (ii), we can choose b in such a way that the last term of the above estimate is sufficiently small. Thus, H is a continuous operator on $C^\infty(\mathbb{R}_+)$. Now, let X be a nonempty and bounded subset of $C^\infty(\mathbb{R}_+)$, $n,m\in\mathbb{N}$ and assume that $\varepsilon>0$ is an arbitrary constant. Then for $x\in X$ and $t_1,t_2\in[0,m]$, with $|t_2-t_1|\le\varepsilon$, we have

$$\Big|\frac{d^n(Hx)}{dt^n}(t_2)-\frac{d^n(Hx)}{dt^n}(t_1)\Big|=\Big|\int_0^\infty\frac{\partial^n k}{\partial t^n}(t_2,s)Tx(s)ds-\int_0^\infty\frac{\partial^n k}{\partial t^n}(t_1,s)Tx(s)ds\Big|.$$

Combining this result with condition (ii) imply that there exists $b>0$ such that

$$\sup\Big\{\int_b^\infty a(s)|k(t,s)|ds : t\in[0,m]\Big\}<\varepsilon,$$

and we obtain

$$\begin{aligned}\Big|\frac{d^n(Hx)}{dt^n}(t_2)-\frac{d^n(Hx)}{dt^n}(t_1)\Big| &\le\Big|\int_0^b[\frac{\partial^n k}{\partial t^n}(t_2,s)-\frac{\partial^n k}{\partial t^n}(t_1,s)]a(s)ds\Big|+|\int_b^\infty\frac{\partial^n k}{\partial t^n}(t_1,s)a(s)ds|+|\int_b^\infty\frac{\partial^n k}{\partial t^n}(t_2,s)a(s)ds|\\ &\le\int_0^b a(s)ds\,\omega^{n,m}(k,\varepsilon)+2\varepsilon,\end{aligned}\tag{4.2}$$

where

$$\omega^{n,m}(k,\varepsilon)=\sup\{|\frac{\partial^n k}{\partial t^n}(t_2,s)-\frac{\partial^n k}{\partial t^n}(t_1,s)| : t_1,t_2\in[0,m], s\in[0,b], |t_2-t_1|\le\varepsilon\}.$$

Since x was an arbitrary element of X in (4.2), we obtain

$$\omega^{n,m}(H(X),\varepsilon)\le\int_0^b a(s)ds\,\omega^{n,m}(k,\varepsilon)+2\varepsilon.$$

On the other hand, by using the uniform continuity of k on the compact set $[0,m]\times[0,b]$, we have $\omega^{n,m}(k,\varepsilon)\longrightarrow 0$ as $\varepsilon\longrightarrow 0$. Therefore, we obtain

$$\omega^{n,m}(H(X))=0.$$

Also, the hypothesis (ii) ensures that $|Hx|_{k,m}\le\alpha_{k,m}$ for all $k,m\in\mathbb{N}$. □

Corollary 4.3. Let $h:\mathbb{R}_+\times\mathbb{R}^\omega\longrightarrow\mathbb{R}$ be continuous and there exists a continuous function $a:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ such that $|h(t,(x_j)_{j=1}^\infty)|\le a(t)$, for all $t\in\mathbb{R}_+$ and $(x_j)_{j=1}^\infty\in(\mathbb{R})^\omega$. Let the hypotheses (ii) of Theorem 4.1 is satisfied. Then $H:C^\infty(\mathbb{R}_+)\longrightarrow C^\infty(\mathbb{R}_+)$ defined by

$$Hx(t)=\int_0^\infty k(t,s)h(s,x(s),x'(s),x''(s),\ldots)ds,$$

is compact and continuous operator and $|Hx|_{k,m}\le\alpha_{k,m}$, where $\alpha_{k,m}$ satisfies in Theorem 4.1.

Proof . It is enough to defined $Tx(t)=h(t,x(t),x'(t),x''(t),\ldots)$ in Theorem 4.1. □

Theorem 4.4. Let $T:C^n(\mathbb{R}_+)\longrightarrow C(\mathbb{R}_+)$ be continuous and hypotheses (i) and (ii) of Theorem 4.1 are satisfied. Then $H:C^n(\mathbb{R}_+)\longrightarrow C^n(\mathbb{R}_+)$ defined by (4.1) is compact and continuous operator and $|Hx|_{k,m}\le\alpha_{k,m}$, where $\alpha_{k,m}$ satisfies in Theorem 4.1.

Proof . The proof is similar to the proof of Theorem 4.1. □

Corollary 4.5. Let $h:\mathbb{R}_+\times\mathbb{R}^{n+1}\longrightarrow\mathbb{R}$ be continuous and there exists a continuous function $a:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ such that $|h(t,(x_j)_{j=1}^{n+1})|\le a(t)$, for all $t\in\mathbb{R}_+$ and $(x_j)_{j=1}^{n+1}\in\mathbb{R}^{n+1}$. Let the hypotheses (ii) of Theorem 4.1 is satisfied. Then $H:C^\infty(\mathbb{R}_+)\longrightarrow C^\infty(\mathbb{R}_+)$ defined by

$$Hx(t)=\int_0^\infty k(t,s)h(s,x(s),x'(s),x''(s),\ldots,x^{(n)}(s))ds,$$

is a compact and continuous operator and $|Hx|_{k,m}\le\alpha_{k,m}$, where $\alpha_{k,m}$ satisfies in Theorem 4.1.

Proof . It is enough to defined $Tx(t)=h(t,x(t),x'(t),x''(t),\ldots,x^{(n)}(t))$. □

Theorem 4.6. Let $\beta:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ be continuous such that $\beta(t)\le t$ for all $t\in\mathbb{R}_+$ and assume that $g:\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^{n+1}\longrightarrow\mathbb{R}$ is continuous and has a continuous derivative of order n with respect to the first argument such that

$$\begin{cases}\sup\{|g(t,t,x_0,x_1,\ldots,x_n)|:t\in\mathbb{R}_+,x_i\in\mathbb{R}\}=0,\\ \sup\{|\frac{\partial^k g}{\partial t^k}(t,t,x_0,x_1,\ldots,x_n)|:t\in\mathbb{R}_+,x_i\in\mathbb{R}\}=0,\qquad 1\le k\le n\end{cases}$$

Also, there exist nondecreasing and continuous functions $\psi:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ and $M_i:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ $(0\le i\le n)$ such that

$$\begin{cases}\sup\{\left|\int_0^t\frac{\partial^k g}{\partial t^k}\big(t,s,x_0(\beta(s)),x_1(\beta(s)),\ldots,x_n(\beta(s))\big)ds\right|:\ t\in[0,m],\ |x_i|_{k,m}\le r\}\le M_k(m)\psi(r),\\ \sup\{\left|\int_0^t g\big(t,s,x_0(\beta(s)),x_1(\beta(s)),\ldots,x_n(\beta(s))\big)ds\right|:\ t\in[0,m],\ |x_i|_{k,m}\le r\}\le M_0(m)\psi(r),\end{cases}$$

for any $r\in\mathbb{R}_+$ and $m\in\mathbb{N}$. Then $H:C^n(\mathbb{R}_+)\longrightarrow C^n(\mathbb{R}_+)$ defined by

$$Hx(t)=\int_0^t g(t,s,x(\beta(s)),x'(\beta(s)),x''(\beta(s)),\ldots,x^{(n)}(\beta(s)))ds,$$

is a compact and continuous operator and $|Hx|_{k,m}\le M_k(m)\psi(|x|_{n,m})$.

Proof . In view of the imposed assumptions, the operator $Hx(t)$ is continuous on $\mathbb{R}_+$ for any $x\in C^n(\mathbb{R}_+)$. Also, for any $t\in\mathbb{R}_+$ and $1\le k\le n$ we have

$$\begin{aligned}\frac{d^k(Hx)}{dt^k}(t)&=\frac{\partial^{k-1}g}{\partial t^{k-1}}\big(t,t,x(\beta(t)),x'(\beta(t)),x''(\beta(t)),\ldots,x^{(n)}(\beta(t))\big)\\&\quad+\int_0^t\frac{\partial^k g}{\partial t^k}\big(t,s,x(\beta(s)),x'(\beta(s)),x''(\beta(s)),\ldots,x^{(n)}(\beta(s))\big)ds,\end{aligned}$$

and Hx has continuous derivative of order $1 \leq k \leq n$. Now, we show that the map H is continuous. (For this we only need to H is a continuous operator of $C^n[0,m]$ into itself). For this, take $x \in C^n(\mathbb{R}_+)$, $m \in \mathbb{N}$ and $\varepsilon > 0$ arbitrarily, and consider $y \in C^n(\mathbb{R}_+)$ with $\max_{0\leq k\leq n} |x-y|_{k,m} < \varepsilon$ and $t \in [0,m]$. Then we have

$$\begin{aligned} |Hx(t) - Hy(t)| &\leq \int_0^t |g(t,s,x(s),x'(s),\ldots,x^{(n)}(s)) - g(t,s,y(s),y'(s),\ldots,y^{(n)}(s))|ds \\ &\leq m\zeta_m(\varepsilon), \end{aligned}$$

where

$$a = \max_{0\leq k\leq n} |x-y|_{k,m} + \varepsilon,$$

$$\zeta_m(\varepsilon) = \sup\{\big|g(t,s,x_0,x_1,\ldots,x_n) - g(t,s,y_0,y_1,\ldots,y_n)\big| : t,s \in [0,m], x_i,y_i \in [-a,a], |x_i - y_i| \leq \varepsilon\}.$$

By similar argument, we have

$$|\frac{d^k(Hx)}{dt^k}(t) - \frac{d^k(Hy)}{dt^k}(t)| \leq m\theta_m(\varepsilon),$$

where

$$\theta_m(\varepsilon) = \sup\{\big|\frac{\partial^k g}{\partial t^k}(t,s,x_0,\ldots,x_n) - \frac{\partial^k g}{\partial t^k}(t,s,y_0,\ldots,y_n)\big| : t,s \in [0,m], x_i,y_i \in [-a,a], |x_i - y_i| \leq \varepsilon, 0 \leq k \leq n\}.$$

Thus, we get

$$|Hx - Hy|_{k,m} \leq m\theta_m(\varepsilon),$$

and since g is continuous on $[0,m]\times[0,m]\times[-a,a]^{n+1}$, we have $\theta(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$ for all $m \in \mathbb{N}$ and $k = 0,1,\ldots,n$. Thus H is a continuous operator from $C^n(\mathbb{R}_+)$ into $C^n(\mathbb{R}_+)$. Now, let X be a nonempty and bounded subset of $C^n(\mathbb{R}_+)$, $m \in \mathbb{N}$, $0 \leq k \leq n$ and assume that $\varepsilon > 0$ is an arbitrary constant. Then for $x \in X$ and $t_1, t_2 \in [0,m]$, with $|t_2 - t_1| \leq \varepsilon$ we have

$$\begin{aligned} \Big|\frac{d^k(Hx)}{dt^k}(t_2) - \frac{d^k(Hx)}{dt^k}(t_1)\Big| \leq& \Big|\int_0^{t_2} \frac{\partial^k g}{\partial t^k}(t_2,s,x(s),x'(s),x''(s),\ldots,x^{(n)}(s))ds \\ &- \int_0^{t_1} \frac{\partial^k g}{\partial t^k}(t_1,s,x(s),x'(s),x''(s),\ldots,x^{(n)}(s))ds\Big| \\ \leq& |\int_{t_1}^{t_2} \frac{\partial^k g}{\partial t^k}(t_2,s,x(s),x'(s),x''(s),\ldots,x^{(n)}(s))ds| \\ &+ \int_0^{t_1} |\frac{\partial^k g}{\partial t^k}(t_2,s,x(s),x'(s),x''(s),\ldots,x^{(n)}(s)) \\ &- \frac{\partial^k g}{\partial t^k}(t_1,s,x(s),x'(s),x''(s),\ldots,x^{(n)}(s))|ds \\ \leq& \varepsilon U_{k,m} + m\omega^{k,m}(g,\varepsilon), \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} U_{k,m} &= \sup\{|\frac{d^k g}{dt^k}(t,s,x_0,x_1,\ldots,x_n)| : t,s \in [0,m], |x_i| \leq |X|_{n,m}, 0 \leq k \leq n\}, \\ \omega^{k,m}(g,\varepsilon) &= \sup\{|\frac{d^k g}{dt^k}(t_2,s,x_0,x_1,\ldots,x_n) - \frac{d^k g}{dt^k}(t_1,s,x_0,x_1,\ldots,x_n)| : t_1,t_2 \in [0,m], \\ &\qquad s \in [0,m], |t_2 - t_1| \leq \varepsilon, |x_i| \leq |X|_{n,m}\}. \end{aligned}$$

Since x was an arbitrary element of X in (4.3), we obtain

$$\omega^{k,m}(H(X),\varepsilon) \leq \varepsilon U_{k,m} + m\omega^{k,m}(g,\varepsilon).$$

On the other hand, by using the uniform continuity of g on the compact set $[0,m]\times[0,m]\times[-a,a]^{n+1}$, we have $\omega^{k,m}(g,\varepsilon)\longrightarrow 0$ as $\varepsilon\longrightarrow 0$. Therefore, we obtain

$$\omega^{k,m}(H(X))=0,$$

for all $m\in\mathbb{N}$ and $0\le k\le n$. Also, the hypothesis (ii) ensures that $|Hx|_{k,m}\le M_k\psi(|x|_{n,m})$ for all $m\in\mathbb{N}$ and $0\le k\le n$. $\square$

Theorem 4.7. Assume that $g:\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuous and has a continuous derivative of order n with respect to the first argument such that

$$\begin{cases}\sup\{|g(t,t,x)|:t\in\mathbb{R}_+,x\in\mathbb{R}\}=0,\\ \sup\{|\frac{\partial^k g}{\partial t^k}(t,t,x)|:t\in\mathbb{R}_+,x\in\mathbb{R}\}=0, \qquad 1\le k\le n,\end{cases}$$

and there exist nondecreasing and continuous functions $\psi:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ and $M_i:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ $(0\le i\le n)$ such that

$$\begin{cases}\sup\{\left|\int_0^t\frac{\partial^k g}{\partial t^k}\big(t,s,x(s)\big)ds\right|:\ t\in[0,m],\ |x|_{k,m}\le r\}\le M_k(m)\psi(r),\\ \sup\{\left|\int_0^t g\big(t,s,x(s)\big)ds\right|:\ t\in[0,m],\ |x|_{k,m}\le r\}\le M_0(m)\psi(r),\end{cases}$$

for any $r\in\mathbb{R}_+$ and $m\in\mathbb{N}$. Also, $T:C^n(\mathbb{R}_+)\longrightarrow C(\mathbb{R}_+)$ is a continuous operator and there exists a positive increasing sequence $\{\beta_m\}$ such that for any $x\in C^n(\mathbb{R}_+)$ and $m\in\mathbb{N}$ we have

$$|Tx|_{k,m}\le\beta_m|x|_{k,m}.$$

Then $H:C^n(\mathbb{R}_+)\longrightarrow C^n(\mathbb{R}_+)$ defined by

$$Hx(t)=\int_0^t g(t,s,Tx(s))ds,$$

is a compact and continuous operator and $|Hx|_{k,m}\le M_k(m)\psi(\beta_m|x|_{k,m})$.

Proof . In view of the imposed assumptions, the operator $Hx(t)$ is continuous on $\mathbb{R}_+$ for any $x\in C^n(\mathbb{R}_+)$. Also, for any $t\in\mathbb{R}_+$ and $1\le k\le n$ we have

$$\begin{aligned}\frac{d^k(Hx)}{dt^k}(t) &= \frac{\partial^{k-1}g}{\partial t^{k-1}}\big(t,t,Tx(t)\big)\\ &+\int_0^t\frac{\partial^k g}{\partial t^k}\big(t,s,Tx(s)\big)ds,\end{aligned}$$

and Hx has continuous derivative of order $1\le k\le n$. Now, we show that the map H is continuous. (For this we only need to H is a continuous operator of $C^n[0,m]$ into itself). For this, take $x\in C^n(\mathbb{R}_+)$, $m\in\mathbb{N}$ and $\varepsilon>0$ arbitrarily, and consider $y\in C^n(\mathbb{R}_+)$ with $\max_{0\le k\le n}|x-y|_{k,m}<\varepsilon$ and $t\in[0,m]$. Then we have

$$\begin{aligned}|Hx(t)-Hy(t)| &\le \int_0^t|g\big(t,s,Tx(s)\big)-g\big(t,s,Ty(s)\big)|ds\\ &\le m\zeta_m(\varepsilon),\end{aligned}$$

where

$$a=\max_{0\le k\le n}|x-y|_{k,m}+\varepsilon,$$

$$\zeta_m(\varepsilon)=\sup\{\big|g(t,s,x)-g(t,s,y)\big|:t,s\in[0,m],x,y\in[-a,a],|x-y|\le\varepsilon\}.$$

By similar argument, we have

$$|\frac{d^k(Hx)}{dt^k}(t)-\frac{d^k(Hy)}{dt^k}(t)|\leq(m+1)\theta_m(\varepsilon),$$

where

$$\theta_m(\varepsilon)=\sup\{\big|\frac{\partial^k g}{\partial t^k}(t,s,x)-\frac{\partial^k g}{\partial t^k}(t,s,y)\big|:t,s\in[0,m],x_i,y_i\in[-a,a],|x_i-y_i|\leq\varepsilon,0\leq k\leq n\}.$$

Thus, we get

$$|Hx-Hy|_{k,m}\quad\leq\quad(m+1)\theta_m(\varepsilon),$$

and since g and T are continuous on $[0,m]\times[0,m]\times[-a,a]$ and $C^n[0,m]$, we have $\theta(\varepsilon)\longrightarrow 0$ as $\varepsilon\longrightarrow 0$ for all $m\in\mathbb{N}$ and $k=0,1,\dots,n$. Thus H is a continuous operator from $C^n(\mathbb{R}_+)$ into $C^n(\mathbb{R}_+)$. Now, let X be a nonempty and bounded subset of $C^n(\mathbb{R}_+)$, $m\in\mathbb{N}$, $0\leq k\leq n$ and assume that $\varepsilon>0$ is an arbitrary constant. Then for $x\in X$ and $t_1,t_2\in[0,m]$, with $|t_2-t_1|\leq\varepsilon$ we have

$$\begin{aligned}\Big|\frac{d^k(Hx)}{dt^k}(t_2)-\frac{d^k(Hx)}{dt^k}(t_1)\Big|\leq&\Big|\int_0^{t_2}\frac{\partial^k g}{\partial t^k}\big(t_2,s,Tx(s)\big)ds-\int_0^{t_1}\frac{\partial^k g}{\partial t^k}\big(t_1,s,Tx(s)\big)ds\Big|\\ \leq&|\int_{t_1}^{t_2}\frac{\partial^k g}{\partial t^k}\big(t_2,s,Tx(s)\big)ds|+\int_0^{t_1}|\frac{\partial^k g}{\partial t^k}\big(t_2,s,Tx(s)\big)-\frac{\partial^k g}{\partial t^k}\big(t_1,s,Tx(s)\big)|ds\\ \leq&\varepsilon U_{k,m}+m\omega^{k,m}(g,\varepsilon),\end{aligned}\tag{4.4}$$

where

$$\begin{aligned}U_{k,m}&=\sup\{|\frac{d^k g}{dt^k}(t,s,x)|:t,s\in[0,m],|x|\leq|T(X)|_{n,m},0\leq k\leq n\},\\ \omega^{k,m}(g,\varepsilon)&=\sup\{|\frac{d^k g}{dt^k}(t_2,s,x)-\frac{d^k g}{dt^k}(t_1,s,x)|:t_1,t_2,s\in[0,m],|t_2-t_1|\leq\varepsilon,|x|\leq|T(X)|_{n,m}\}.\end{aligned}$$

Since x was an arbitrary element of X in (4.4), we obtain

$$\omega^{k,m}(H(X),\varepsilon)\quad\leq\quad\varepsilon U_{k,m}+m\omega^{k,m}(g,\varepsilon).$$

On the other hand, by using the uniform continuity of g on the compact set $[0,m]\times[0,m]\times[-a,a]$, we have $\omega^{k,m}(g,\varepsilon)\longrightarrow 0$ as $\varepsilon\longrightarrow 0$. Therefore, we obtain

$$\omega^{k,m}(H(X))=0,$$

for all $m\in\mathbb{N}$ and $0\leq k\leq n$. Also, the hypothesis (ii) ensures that $|Hx|_{k,m}\leq M_k\psi(|x|_{n,m})$ for all $m\in\mathbb{N}$ and $0\leq k\leq n$. □

Remark 4.8. The operator H, in Theorem 4.6 and Theorem 4.7, is not compact in the space $C^\infty(\mathbb{R}_+)$, but it can be $\lambda_{k,m}$-condensing in the space $C^\infty(\mathbb{R}_+)$ (see example 5.2).

5 Application and Examples

In this section, as an application of the above section, we prove the existence of solutions for some integral equations in $C^\infty(\mathbb{R}_+)$ and $C^n(\mathbb{R}_+)$.

Example 5.1. Consider the following functional integral-differential equation

$$x(t)=\frac{1}{2}x(\frac{t}{2})+\sum_{n=0}^{\infty}\int_0^\infty\frac{e^{-(1+t)s}(-s)^n x^{(n)}(s)}{1+(x^{(n)}(s))^2}ds.\tag{5.1}$$

Now, we define

$$Fx(t) := \frac{1}{2}x(\frac{t}{2}) \qquad \text{and} \qquad Hx(t) := \int_0^\infty e^{-(1+t)s}\sum_{n=0}^{\infty}\frac{(-s)^n x^{(n)}(s)}{1+(x^{(n)}(s))^2}ds.$$

If we define

$$k(t,s) = e^{-(1+t)s}, \qquad h(t,(x_j)_{j=1}^\infty) = \sum_{n=0}^{\infty}\frac{(-t)^n x_n}{1+(x_n)^2} \qquad \text{and} \qquad a(t) = \frac{1}{1+s},$$

then, by Corollary 4.3, H is a compact operator. Also, F is a $\lambda_{k,m}$-condensing if $\lambda_{k,m} = \frac{1}{2^{k+1}}$. On the other hand, to find Ω such that $T(x) = F(x) + H(x) \in \Omega$ for all $x \in \Omega$. For this, it is enough to define

$$\Omega = \{x \in C^\infty(\mathbb{R}_+) : |x|_{k,m} \leq r_{k,m}\},$$

such that $r_{k,m} > \frac{2^{k+1}}{2^{k+1}-1}$. Therefore, by Theorem 2.5, T has a fixed point and the functional differential-integral equation (5.1) has at least one solution which belongs to the space $C^\infty(\mathbb{R}_+)$, but it has no solution in the space $C^n(\mathbb{R}_+)$ for any $n \in \mathbb{N}$.

Example 5.2. Consider the following functional integro-differential equation

$$y(t) = \int_0^{\frac{t}{2}}(\frac{t}{2} - s)e^{-\frac{t}{2}}y''(s)ds + \int_0^\infty e^{-v}\cos(vt)\ln(2+\cos vy(v))dv. \tag{5.2}$$

Now, we define

$$Fx(t) := \int_0^{\frac{t}{2}}(\frac{t}{2} - s)e^{-\frac{t}{2}}y''(s)ds \quad \text{and} \quad Hx(t) := \int_0^\infty e^{-v}\cos(vt)\ln(2+\cos vy(v))dv.$$

If we define

$$k(t,s) = e^{-s}\cos(st), \qquad Tx(t) = \ln(2+\cos tx(t)) \qquad \text{and} \qquad a(t) = 1,$$

then, by Theorem 4.1, H is a compact operator. On the other hand,

$$\frac{d^n(Fx)}{dt^n}(t) = \frac{1}{2^n}\Big(\int_0^{\frac{t}{2}}((-1)^n(\frac{t}{2}-s)e^{-\frac{t}{2}} + (-1)^{n+1}ne^{-\frac{t}{2}})y''(s)ds + \sum_{i=2}^{i=n}\binom{n}{i}(-1)^i e^{-\frac{t}{2}}x^{(n)}(\frac{t}{2})\Big).$$

Thus, F is a $\lambda_{k,m}$-condensing if $\lambda_{k,m} = 0$ for $k = 0,1$ and $\lambda_{k,m} < 1 - \frac{1+n}{2^n}$. To find Ω such that $L(x) = F(x) + H(x) \in \Omega$ for all $x \in \Omega$. For this, it is enough to define

$$\Omega = \{x \in C^\infty(\mathbb{R}_+) : |x|_{k,m} \leq r_{k,m}\},$$

such that $r_{k,m} > \frac{2^n \ln 3}{\frac{7}{8}+\frac{n}{2}}$. Therefore, by Theorem 2.5, L has a fixed point and the functional differential-integral equation (5.2) has at least one solution which belongs to the space $C^\infty(\mathbb{R}_+)$. Also, for each $n \geq 2$, it has a solution in space $C^n(\mathbb{R}_+)$.

Example 5.3. Consider the following functional integro-differential equation

$$\big(x(t) - \frac{1}{t+2}x(\ln t + 2)\big)''' = \Big(\cos t\ln(|x(t)|+1) + \frac{x''(t)\arctan(t^2+1)}{(x''(t))^2+1} + \sqrt{|\int_0^t x(s)ds| + 1} - 1\Big), \tag{5.3}$$

with the initial condition of the form

$$x(0) = x'(0) = x''(0) = 0. \tag{5.4}$$

The differential equation (5.3) with the initial condition (5.4) has at least one solution in the space $C^3(\mathbb{R}_+)$ if and only if a nonlinear differential-integral equation

$$x(t) = \frac{1}{t+2}x(\ln t + 2) + \frac{1}{2}\int_0^t (t-s)^2\Big(\cos s \ln(|x(s)|+1) + \frac{x''(s)\arctan(s^2)}{(x''(s))^2+1} + \sqrt{|\int_0^s x(v)dv| + 1} - 1\Big)ds,$$

has at least one solution in the space $C^3(\mathbb{R}_+)$. Now, we define

$$\begin{aligned}
Fx(t) &:= \frac{1}{t+2}x(\ln t + 2),\\
H_1x(t) &:= \frac{1}{2}\int_0^t (t-s)^2\big(\cos s\ln(|x(s)|+1) + \frac{x''(s)\arctan(s^2)}{(x''(s))^2+1}\big)ds\\
H_2x(t) &= \frac{1}{2}\int_0^t (t-s)^2(\sqrt{|\int_0^s x(v)dv| + 1} - 1)ds.
\end{aligned}$$

If we define

$$g(t,s,x_0,x_1,x_2) = \frac{(t-s)^2}{2}(\cos s\ln(|x_0|+1) + \frac{x_2\arctan(s^2)}{(x_2)^2+1}), \qquad \text{and} \qquad \beta(t) = t.$$

Then, by Theorem 4.6, H_1 is a compact operator. Also, if we define

$$g(t,s,x) = \frac{(t-s)^2}{2}(\sqrt{|x|+1} - 1), \text{ and } Tx(t) = \int_0^t x(s)ds,$$

then, by Theorem 4.7, H_2 is a compact operator. Thus, $H_1 + H_2$ is a compact operator. To find Ω such that $L(x) = F(x) + H_1(x) + H_2(x) \in \Omega$ for all $x \in \Omega$, it is enough to define

$$\Omega = \{x \in C^3(\mathbb{R}_+) : |x|_{k,m} \le r_{k,m}\},$$

such that $\frac{m^3}{6}(\ln(r_{k,m}+1) + \frac{\pi}{6} + \sqrt{r_{k,m}+1} - 1) \le r_{k,m}$ for $k = 0,1,2,3$ and $m \in \mathbb{N}$. Therefore, by Theorem 2.5, L has a fixed point and the functional differential-integral equation (5.3) has at least one solution which belongs to the space $C^3(\mathbb{R}_+)$.

References

[1] R.P. Agarwal, M. Benchohra, and D. Seba, *On the application of measure of noncompactness to the existence of solutions for fractional differential equations*, Results Math. **55** (2009), 221–230.

[2] R. Agarwal, M. Meehan, and D. O'Regan, *Fixed Point Theory and Applications*, Cambridge University Press, 2004.

[3] A. Aghajani, J. Banaś, and Y. Jalilian, *Existence of solutions for a class of nonlinear Volterra singular integralm equations*, Comput. Math. Appl. **62** (2011), 1215–1227.

[4] A. Aghajani, R. Allahyari, and M. Mursaleen, *A generalization of Darbo's theorem with application to the solvability of systems of integral equations*, J. Comput. Appl. Math. **260** (2014), 68–77.

[5] A. Aghajani, M. Mursaleen, and A. Shole Haghighi, *Fixed point theorems for Meir-Keeler condensing operators via measure of noncompactness*, Acta Math. Sci. **35B** (2015), no. 3, 552–566.

[6] R. Arab, *The existence of fixed points via the measure of non-compactness and its application to functional integral equations*, Mediterr. J. Math. **13** (2016), 759–773.

[7] A. Ayad, *Spline approximation for first order Fredholm delay integro-differential equations*, Int. J. Comput. Math. **70** (1999), no. 3, 467–476.

[8] J. Banaś, *Measures of noncompactness in the study of solutions of nonlinear differential and integral equations*, Cent. Eur. J. Math. **10** (2012), no. 6, 2003–2011.

[9] M. Benchohra, Z. Bouteffal, J. Henderson, and S. Litimein, *Measure of noncompactness and fractional integro-differential equations with state-dependent nonlocal conditions in Fréchet spaces*, Aims Math. **5** (2020), no. 1, 15–25.

[10] F. Bloom, *Asymptotic bounds for solutions to a system of damped integro-differential equations of electromagnetic theory*, J. Math. Anal. Appl. **73** (1980), no. 2, 524–542.

[11] G. Darbo, *Punti uniti in trasformazioni a codominio non compatto*, Rend. Sem. Mat. Uni. Padova. **24** (1955), 84–92.

[12] S. Djebali and K. Hammache, *Fixed point theorems for nonexpansive maps in Banach spaces*, Nonlinear Anal. **73** (2010), 3440–3449.

[13] J. Garcia-Falset, *Existence of fixed points and measures of weak noncompactness*, Nonlinear Anal. **71** (2009), 2625–2633.

[14] D. Guo, *Existence of solutions for n^{th}-order integro-differential equations in Banach spaces*, Comput. Math. Appl. **41** (2001), 597–606.

[15] S. Hadi Bonab, V. Parvaneh, and J. Rezaei Roshan, *The solvability of an iterative system of functional integral equations with self-composition of arbitrary order*, submitted.

[16] H. Hosseinzadeh, H. Isik, S. Hadi Bonab, and R. George, *Coupled measure of noncompactness and functional integral equations*, Open Math. **20** (2022), 38–49.

[17] K. Kuratowski, *Sur les espaces complets*, Fund. Math. **15** (1930), 301–309.

[18] B. Mohammadi, V. Parvaneh, and H. Hosseinzadeh, *Some Wardowski-Mizogochi-Takahashi-Type generalizations of the multi-valued version of Darbo's fixed point theorem with applications*, Int. J. Nonlinear Anal. Appl. 2023, 10.22075/ijnaa.2023.23057.2470

[19] M. Mursaleen, V. Rakocevic, *A survey on measures of non-compactness with some applications in infinite systems of differential equations*, Aequ. Math. **96** (2022), 489-–514.

[20] V. Parvaneh, S. Banaei, J.R. Roshan, and M. Mursaleen, *On tripled fixed point theorems via measure of noncompactness with applications to a system of fractional integral equations*, Filomat **35** (2021), no. 14, 4897–4915.

[21] M. Rabbani, R. Arab, B. Hazarika, and N. Aghazadeh, *Existence of solution of functional integral equations by measure of noncompactness and sinc interpolation to find solution*, Fixed Point Theory 23 (2022), no. 1, 331–334.

[22] J.R. Roshan, *Existence of solutions for a class of system of functional integral equation via measure of noncompactness*, J. Comput. Appl. Math. **313** (2017), 129–141.