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A study on interval-valued generalized fuzzy n-normed linear space

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Abstract

Following the definition of interval-valued fuzzy n-normed linear space given by S. Vijayabalaji et al., in this paper, notion of interval-valued generalized fuzzy n-normed linear space is introduced. The notion of convergent sequence, Cauchy sequence and their relation are studied. Some basic results are established on finite dimensional interval-valued generalized fuzzy n-normed linear space.

Keywords: n-norm, t-norm, fuzzy n-norm, $\mathscr{IV}$-t-norm, interval-valued fuzzy n-normed linear space, interval-valued generalized fuzzy n-normed linear space
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1 Introduction

It was Katsaras [17] who first introduced the idea of fuzzy norm on a linear space. Afterwards, several authors viz. Felbin [12], Cheng and Mordeson [5], Bag and Samanta[2, 3] defined fuzzy normed linear space in different approaches and developed many results of functional analysis in fuzzy setting. Different types of generalized norms (viz. 2-norm[13], n-norm [16], G-norm[18], cone norm [15] etc.) and consequently generalized fuzzy norms (viz. 2-fuzzy norm [26], fuzzy n-norm [28], fuzzy cone norm[1], G-fuzzy norm[4], etc.) have been introduced and established many fundamental results of fuzzy functional analysis.

The concept of interval-valued fuzzy set was introduced by Zadeh [29] in 1975. An interval-valued fuzzy set is characterized by an interval-valued membership function and it is taken as a generalization of fuzzy sets. Li [21], in 2009, introduced distances between two interval-valued fuzzy sets (or numbers) defined on real line $\mathbb{R}$ in three different approaches. Moreover, he noted that each kind of the defined distance is a metric on the corresponding sets. Obviously, the idea of above defined metric space is different from that of George and Veeramani [14]. It should be noted that George and Veeramani applied fuzzy set to express the uncertainty of the distance between two points in a fuzzy metric space and then proposed the concept of the t-norm which generalizes the triangle inequality of general metric space. Motivated by the idea of fuzzy metric space, Shen et al. [23] put forward the concept of continuous interval-valued t-norm ($\mathscr{IV}$-t-norm in short) and defined an interval-valued fuzzy metric space. Following the definition of interval-valued fuzzy metric space, S. Vijayabalaji et al. [27] gave a definition of interval-valued fuzzy n-normed linear space with underlying t-norm as 'min'. In this paper, we introduce the idea of generalized interval-valued fuzzy

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n-normed linear space with underlying general t-norm. Some basic results on convergent sequence, Cauchy sequence, completeness are studied in such spaces. Finally fundamental results on finite dimensional interval-valued generalized fuzzy n-normed linear space are established.

It is worth mentioning the work of P. Debnath et al. [7, 8, 9, 10, 11, 20, 25] which help us to improve the quality of the manuscript.

The organization of the article is as follows. Section 2 consists of some preliminary results related to interval-valued fuzzy n-normed linear space. In Section 3, we define interval-valued generalized fuzzy n-norm over a linear space, give proper example and study convergent sequence, Cauchy sequence and their relation. In Section 4, we establish some fundamental results of functional analysis in finite dimensional interval-valued generalized fuzzy n-normed linear spaces.

2 Preliminaries

In this Section, we collect some preliminary existing notations, basic definitions and results which will be used in this paper. Following is the definition of an n-norm by Gunawan and Mashadi [16].

Definition 2.1. [16] Let $n \in \mathbb{N}$ and X be a real vector space of dimension $d \geq n$. A real-valued function $||.,\cdots,.||$ defined on $\underbrace{X \times \cdots \times X}_{n \text{ times}}$ is called an n-norm on X if it satisfies the following properties:

(i) $||x_1, x_2, \cdots, x_n|| = 0$ iff $x_1, x_2, \cdots, x_n$ are linearly dependent;
(ii) $||x_1, x_2, \cdots, x_n||$ is invariant under any permutation;
(iii) $||x_1, x_2, \cdots, \alpha x_n|| = |\alpha|||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{R}$;
(iv) $||x_1, x_2, \cdots, x_{n-1}, y + z|| \leq ||x_1, x_2, \cdots, x_{n-1}, y|| + ||x_1, x_2, \cdots, x_{n-1}, y||$

for all $x_1, x_2, \cdots, x_n, y, z \in X$. The pair $(X, ||\underbrace{.,\cdots,.}_{n \text{ times}}||)$ is called an n-normed space.

Next we recall the definition of a t-norm and fuzzy n-normed linear space.

Definition 2.2. [19] A t-norm $* : [0,1] \times [0,1] \to [0,1]$ is a function satisfying the following conditions:

(i) $*$ is associative and commutative;
(ii) $\alpha \, * \, 1 = \alpha, \ \forall \alpha \in [0,1]$;
(iii) $\alpha * \gamma \leq \beta * \delta$ whenever $\alpha \leq \beta$ and $\gamma \leq \delta, \ \forall \alpha, \beta, \gamma, \delta \in [0,1]$.

If $*$ is continuous then it is called continuous t-norm.

Definition 2.3. [24] Let X be a linear space over a field $F(\mathbb{R} \ or \ \mathbb{C})$ of dimension $\geq n$ and $*$ be a t-norm. A fuzzy n-norm on X is a fuzzy subset of $X^n \times \mathbb{R}$ if N satisfies the following conditions:

(N1) for all $t \in \mathbb{R}$ with $t \leq 0$, $N(x_1, x_2, \cdots, x_n, t) = 0$;
(N2) for all $t \in \mathbb{R}$ with $t > 0$, $N(x_1, x_2, \cdots, x_n, t) = 1$ iff $x_1, x_2, \cdots, x_n$ are linearly dependent;
(N3) $N(x_1, x_2, \cdots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \cdots, x_n$;
(N4) for all $t \in \mathbb{R}$ with $t > 0$ and $c(\neq 0) \in F$, $N(x_1, x_2, \cdots, cx_n, t) = N(x_1, x_2, \cdots, x_n, \frac{t}{c})$;
(N5) $\forall s, t \in \mathbb{R}, \ N(x_1, x_2, \cdots, x_n + x_n', s + t) \geq N(x_1, x_2, \cdots, x_n, s) * N(x_1, x_2, \cdots, x_n', t)$;
(N6) $\lim_{t \to \infty} N(x_1, x_2, \cdots, x_n, t) = 1$;

for all $x_1, x_2, \cdots, x_n, x_n' \in X$. The pair (X, N) is called a fuzzy n-normed linear space.

***Remark* 2.4.** $N(x_1, x_2, \cdots, x_n, \cdot)$ is a non-decreasing function of t, which follows from the conditions (N2) and (N5).

Before going to the definition of $\mathscr{IV}$-t-norm, we recall the following notations from [6]. $[I]$ denotes the set of all interval numbers, namely the closed unit interval $[0,1]$ and all closed sub-intervals of $[0,1]$ i.e $[I] = \{\overline{a} : [a^-, a^+]$ such that $0 \leq a^- \leq a^+ \leq 1\}$. If $a^- = a^+$, then the interval number a degenerates into an ordinary real number on I. Conversely, every $a \in I$ induces the interval number $[a, a]$ that we will denote as a if no confusion arises, so that $(I]$ denotes $[I] \setminus \{\overline{0}\}$ and (I) denotes $[I] \setminus \{\overline{0}, \overline{1}\}$.

For $\overline{a}, \overline{b} \in [I]$, $\overline{a} \leq \overline{b}$ if $a^- \leq b^-$ and $a^+ \leq b^+$; $\overline{a} = \overline{b}$ if $a^- = b^-$ and $a^+ = b^+$ and $\overline{a} < \overline{b}$ if $\overline{a} \leq \overline{b}$ but $\overline{a} \neq \overline{b}$. Clearly $([I], \leq)$ is a partial ordered set.

In [23, 22], we see that every $\overline{a}, \overline{b} \in [I]$ satisfies the following operations:

(i) $\overline{a} \wedge \overline{b} = [a^- \wedge b^-, a^+ \wedge b^+]$.
(ii) $\overline{a} \vee \overline{b} = [a^- \vee b^-, a^+ \vee b^+]$.
(iii) $\overline{a}^c = \overline{1} - \overline{a} = [1 - a^+, 1 - a^-]$.
(iv) $\overline{a} - \overline{b} = [a^- - b^+, a^+ - b^-]$ and $\overline{a} + \overline{b} = [a^- + b^-, a^+ + b^+]$.

Definition 2.5. [23] A function $\overline{*} : [I] \times [I] \to [I]$ which satisfies the following properties:

(i) $\overline{*}$ is associative and commutative;
(ii) $\overline{a} \,\overline{*}\, \overline{1} = \overline{a}$ and $\overline{a} \,\overline{*}\, I = [0, a^+]$;
(iii) $\overline{a} \,\overline{*}\, \overline{b} \leq \overline{c} \,\overline{*}\, \overline{d}$ if $\overline{a} \leq \overline{c}$ and $\overline{b} \leq \overline{d}$; for every $\overline{a}, \overline{b}, \overline{c}, \overline{d} \in [I]$, is called an $\mathscr{IV}$-t-norm.

If $\overline{*}$ is continuous, then $\overline{*}$ is called continuous $\mathscr{IV}$-t-norm.

Definition 2.6. [23] A sequence $\{\overline{a}_n\} = \{[a_n^-, a_n^+]\}$ of interval numbers converges to $\overline{a} = [a^-, a^+]$ if $\lim_{n\to\infty} a_n^- = a^-$ and $\lim_{n\to\infty} a_n^+ = a^+$. We write $\lim_{n\to\infty} \overline{a}_n = \overline{a}$.

Following is the definition of an interval-valued fuzzy n-normed linear space by Vijayabalaji et al. [27].

Definition 2.7. [27] Let X be a linear space over a real field F. An interval-valued fuzzy n-norm $\overline{N}$ is a fuzzy subset of $X^n \times \mathbb{R}$ if it satisfies the following properties:

(N1) for all $t \in \mathbb{R}$ with $t \leq 0$, $\overline{N}(x_1, x_2, \cdots, x_n, t) = \overline{0}$;
(N2) for all $t \in \mathbb{R}$ with $t > 0$, $\overline{N}(x_1, x_2, \cdots, x_n, t) = \overline{1}$ iff $x_1, x_2, \cdots, x_n$ are linearly dependent;
(N3) $\overline{N}(x_1, x_2, \cdots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \cdots, x_n$;
(N4) for all $t \in \mathbb{R}$ with $t > 0$, $\overline{N}(x_1, x_2, \cdots, cx_n, t) = \overline{N}(x_1, x_2, \cdots, x_n, \frac{t}{c})$ if $c \neq 0$;
(N5) for all $s, t \in \mathbb{R}$, $\overline{N}(x_1, x_2, \cdots, x_n + x_n', s + t) \geq \min\{\overline{N}(x_1, x_2, \cdots, x_n, s), \overline{N}(x_1, x_2, \cdots, x_n', t)\}$;
(N6) $\overline{N}(x_1, x_2, \cdots, x_n, t)$ is a non-decreasing function of $t \in \mathbb{R}$ and $\lim_{n\to\infty} \overline{N}(x_1, x_2, \cdots, x_n, t) = \overline{1}$

for all $x_1, x_2, \cdots, x_n, x_n' \in X$. Then (X, N) is called an interval-valued fuzzy n-normed linear space (or briefly i-v f-n-NLS).

They consider 'min' instead of general $\mathscr{IV}$-t-norm $\overline{*}$. In this paper, we use the general $\mathscr{IV}$-t-norm $\overline{*}$ in the axiom (N5) of Definition 2.7 and rewrite it in our sense. Further we provide some results on finite dimensional interval-valued generalized fuzzy n-normed linear space. We also define the notion of convergence of sequence, Cauchy sequence and study some well-known results based on completeness and compactness of linear spaces to this setting.

3 Interval-valued generalized fuzzy n-normed linear space

In this Section first we redefine the notion of interval-valued fuzzy n-norm using interval-valued-t-norm $\overline{*}$ and named it as interval-valued generalized fuzzy n-norm.

Definition 3.1. Let $n \in \mathbb{N}$ and X be a linear space of dimension $\geq n$ over a real field F and $\overline{*}$ be $\mathscr{IV}$-t-norm. An interval valued fuzzy set $\overline{N}$ of $X^n \times \mathbb{R}$ is called an interval-valued generalized fuzzy n-norm(in short i-v G-fuzzy n-norm) if the following conditions hold:

$(G\overline{N}1)$ for all $t \in \mathbb{R}$ with $t \leq 0$, $\overline{N}(x_1, x_2, \cdots, x_n, t) = \overline{0}$;
$(G\overline{N}2)$ for all $t \in \mathbb{R}$ with $t > 0$, $\overline{N}(x_1, x_2, \cdots, x_n, t) = \overline{1}$ iff $x_1, x_2, \cdots, x_n$ are linearly dependent;
$(G\overline{N}3)$ $\overline{N}(x_1, x_2, \cdots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \cdots, x_n$;
$(G\overline{N}4)$ for all $t \in \mathbb{R}$ with $t > 0$, $\overline{N}(x_1, x_2, \cdots, cx_n, t) = \overline{N}(x_1, x_2, \cdots, x_n, \frac{t}{c})$ if $c \neq 0$;

$(G\overline{N}5)$ for all $s,t \in \mathbb{R}$, $\overline{N}(x_1,x_2,\cdots,x_n+x_n',s+t) \geq \overline{N}(x_1,x_2,\cdots,x_n,s) \,\overline{*}\, \overline{N}(x_1,x_2,\cdots,x_n',t)$;
$(G\overline{N}6)$ $\lim\limits_{t\to\infty} \overline{N}(x_1,x_2,\cdots,x_n,t) = \overline{1}$

for all $x_1,x_2,\cdots,x_n,x_n' \in X$. Then the triplet $(X,\overline{N},\overline{*})$ is called an interval-valued generalized fuzzy n-normed linear space (in short i-v g-fuzzy n-nls).

***Remark* 3.2.** $\overline{N}(x_1,x_2,\cdots,x_n,t)$ is a non-decreasing function of $t \in \mathbb{R}$. It follows from $(G\overline{N}2)$ and $(G\overline{N}5)$. For, if $s > t$ then

$$\begin{aligned}\overline{N}(x_1,x_2,\cdots,x_{n-1},x,s) &= N(x_1,x_2,\cdots,x_{n-1},x+\theta,t+(s-t))\\ &\geq \overline{N}(x_1,x_2,\cdots,x_{n-1},x,t) \,\overline{*}\, \overline{N}(x_1,x_2,\cdots,x_{n-1},\theta,(s-t))\\ &= \overline{N}(x_1,x_2,\cdots,x_{n-1},x,t) \,\overline{*}\, \overline{1}\\ &= \overline{N}(x_1,x_2,\cdots,x_{n-1},x,t)\end{aligned}$$

Example 3.3. Let $(X,||\underbrace{.,\cdots,.}_{n \text{ times}}||)$ be an n-normed linear space. Define a function $\overline{N}$ by

$$\overline{N}(x_1,x_2,\cdots,x_n,t) = \begin{cases}[\frac{t}{t+||x_1,x_2,\cdots,x_n||},1] & \text{if } t>0\\ \overline{0} & \text{if } t\leq 0\end{cases}$$

for all $x_1,x_2,\cdots,x_n \in X$. We choose $\overline{*}$ defined as $\overline{a}\,\overline{*}\,\overline{b} = [a^-\wedge b^-, a^+\wedge b^+]$ where $\overline{a}=[a^-,a^+]$ and $\overline{b}=[b^-,b^+]$. Now we show that $\overline{N}$ is a i-v g-fuzzy n-norm on X.

(i) If $t \in \mathbb{R}$ with $t \leq 0$, from definition we have, $\overline{N}(x_1,x_2,\cdots,x_n,t) = \overline{0}$. So, $(G\overline{N}1)$ holds.

(ii) If $x_1,x_2,\cdots,x_n \in X$ are linearly dependent then $||x_1,x_2,\cdots,x_n|| = 0$. Then $\overline{N}(x_1,x_2,\cdots,x_n,t) = [1^-,1^+] = \overline{1}$, for all $t > 0$.
Conversely suppose that, $\overline{N}(x_1,x_2,\cdots,x_n,t) = \overline{1}$ $\forall t > 0$, then

$$\begin{aligned}&[\frac{t}{t+||x_1,x_2,\cdots,x_n||},1] = [1^-,1^+] \quad \forall t>0\\ \Longrightarrow\ &\frac{t}{t+||x_1,x_2,\cdots,x_n||} = 1 \quad \forall t>0\\ \Longrightarrow\ &||x_1,x_2,\cdots,x_n|| = 0\\ \Longrightarrow\ &x_1,x_2,\cdots,x_n \text{ are linearly dependent in } X.\end{aligned}$$

Thus $(G\overline{N}2)$ holds.

(iii) As $||x_1,x_2,\cdots,x_n||$ is invariant under any permutation of $\{x_1,x_2,\cdots,x_n\}$ thus $\overline{N}(x_1,x_2,\cdots,x_n,t)$ for all $t > 0$ is also so. Hence $(G\overline{N}3)$ holds.

(iv) For $t > 0$ and $c \neq 0$, if $\overline{N}(x_1,x_2,\cdots,cx_n,t) = \overline{0}$ then

$$t \leq ||x_1,x_2,\cdots,cx_n|| = |c|||x_1,x_2,\cdots,x_n|| \quad i.e. \ \frac{t}{|c|} \leq ||x_1,x_2,\cdots,x_n||.$$

Again if $\overline{N}(x_1,x_2,\cdots,cx_n,t) \neq \overline{0}$, then

$$\begin{aligned}\overline{N}(x_1,x_2,\cdots,cx_n,t) =&[\frac{t}{t+||x_1,x_2,\cdots,cx_n||},1]\\ =&[\frac{\frac{t}{|c|}}{\frac{t}{|c|}+||x_1,x_2,\cdots,x_n||},1]\\ =&\overline{N}(x_1,x_2,\cdots,x_n,\frac{t}{|c|})\end{aligned}$$

So for $t > 0$ and $c \neq 0$, $\overline{N}(x_1,x_2,\cdots,cx_n,t) = \overline{N}(x_1,x_2,\cdots,x_n,\frac{t}{|c|})$. Thus $(G\overline{N}4)$ holds.

(v) We consider only the cases when both $s > 0$ and $t > 0$. Other cases are obvious.
Case I: $s > ||x_1, x_2, \cdots, x_n||$ and $t > ||x_1, x_2, \cdots, x'_n||$. Then $\overline{N}(x_1, x_2, \cdots, x_n, s) = [\frac{s}{s+||x_1,x_2,\cdots,x_n||}, 1]$ and $\overline{N}(x_1, x_2, \cdots, x'_n, t) = [\frac{t}{t+||x_1,x_2,\cdots,x'_n||}, 1]$. Now, $\overline{N}(x_1, x_2, \cdots, x_n, s) \mathbin{\overline{*}} \overline{N}(x_1, x_2, \cdots, x'_n, t) = [\frac{s}{s+||x_1,x_2,\cdots,x_n||} \wedge \frac{t}{t+||x_1,x_2,\cdots,x'_n||}, 1]$. Suppose that $\overline{N}(x_1, x_2, \cdots, x_n, s) \leq \overline{N}(x_1, x_2, \cdots, x_n, t)$ i.e $\frac{s}{s+||x_1,x_2,\cdots,x_n||} \leq \frac{t}{t+||x_1,x_2,\cdots,x'_n||}$, then we have $t||x_1, x_2, \cdots, x_n|| \geq s||x_1, x_2, \cdots, x'_n||$. Using this inequality, we obtain

$$\begin{aligned}
&\overline{N}(x_1, x_2, \cdots, x_n + x'_n, s+t) - \overline{N}(x_1, x_2, \cdots, x_n, s) \\
&= \frac{s+t}{(s+t) + ||x_1, x_2, \cdots, x_n + x'_n||} - \frac{s}{s + ||x_1, x_2, \cdots, x_n||} \\
&= \frac{s||x_1, x_2, \cdots, x_n|| + t||x_1, x_2, \cdots, x'_n|| - s||x_1, x_2, \cdots, x_n + x'_n||}{\{(s+t) + ||x_1, x_2, \cdots, x_n + x'_n||\}\{s + ||x_1, x_2, \cdots, x_n||\}} \\
&\geq \frac{t||x_1, x_2, \cdots, x_n|| - s||x_1, x_2, \cdots, x'_n||}{\{(s+t) + ||x_1, x_2, \cdots, x_n + x'_n||\}\{s + ||x_1, x_2, \cdots, x_n||\}} \geq 0
\end{aligned}$$

i.e $\overline{N}(x_1, x_2, \cdots, x_n + x'_n, s+t) \geq \overline{N}(x_1, x_2, \cdots, x_n, s)$. Similarly, if $\overline{N}(x_1, x_2, \cdots, x'_n, t) \geq \overline{N}(x_1, x_2, \cdots, x_n, s)$, then $\overline{N}(x_1, x_2, \cdots, x_n + x'_n, s+t) \geq \overline{N}(x_1, x_2, \cdots, x'_n, t)$. Hence $(G\overline{N}5)$ holds.

(vi) Since $\lim_{t\to\infty} \frac{t}{t+||x_1,x_2,\cdots,x'_n||} = 1$, so $[\lim_{t\to\infty} \frac{t}{t+||x_1,x_2,\cdots,x_n||},\ 1] = [\ 1^-,\ 1^+]$ and hence $\lim_{t\to\infty} \overline{N}(x_1, x_2, \cdots, x_n, t) = \overline{1}$. Therefore $(G\overline{N}6)$ holds.

Hence $(X, \overline{N}, \overline{*})$ is an i-v g-fuzzy n-nls.

Definition 3.4. Let $(X, \overline{N}, \overline{*})$ be an i-v g-fuzzy n-nls and $\{x_n\}$ be a sequence in X. Then $\{x_r\}$ is said to be convergent and converges to $x \in X$ if for each $y_1, y_2, \cdots, y_{n-1} \in X$, $\overline{\delta} \in (I)$, $t > 0$, there exists $m \in \mathbb{N}$ such that

$$\overline{N}(y_1, y_2, \cdots, y_{n-1}, x_r - x, t) > \overline{1} - \overline{\delta} \quad \forall n \geq m$$

where $\overline{\delta} = [\delta^-, \delta^+],\ 0 < \delta^- < \delta^+ < 1$.

Proposition 3.5. In an i-v g-fuzzy n-nls $(X, \overline{N}, \overline{*})$, a sequence $\{x_r\}$ converges to $x \in X$ if and only if for each $y_1, y_2, \cdots, y_{n-1} \in X$, $\overline{N}(y_1, y_2, \cdots, y_{n-1}, x_r - x, t) \to \overline{1}$ as $r \to \infty$.

Proof . Let us choose $t_0 > 0$ and suppose $\{x_r\}$ converges to $x \in X$. Then for each $y_1, y_2, \cdots, y_{n-1} \in X$, $\overline{\delta} \in (I)$, there exists $m \in \mathbb{N}$ such that

$$\begin{aligned}
&\overline{N}(y_1, y_2, \cdots, y_{n-1}, x_r - x, t_0) > \overline{1} - \overline{\delta}, \quad \forall r \geq m \\
&i.e.\ \ \overline{1} - \overline{N}(y_1, y_2, \cdots, y_{n-1}, x_r - x, t_0) < \overline{\delta} \quad \forall r \geq m.
\end{aligned}$$

Since $\overline{\delta} \in (I)$ is chosen arbitrarily, taking limit as $n \to \infty$, we get

$$\lim_{r\to\infty} \overline{N}(y_1, y_2, \cdots, y_{n-1}, x_r - x, t_0) = \overline{1}.$$

Conversely, suppose that, for each $t > 0$ and for each $y_1, y_2, \cdots, y_{n-1} \in X$, $\overline{N}(y_1, y_2, \cdots, y_{n-1}, x_r - x, t) \to \overline{1}$ as $r \to \infty$. So for every $\overline{\delta} \in (I)$, there exists $m \in \mathbb{N}$ such that

$$\overline{N}(y_1, y_2, \cdots, y_{n-1}, x_r - x, t_0) > \overline{1} - \overline{\delta}, \quad \forall r \geq m.$$

This completes the proof. □

Proposition 3.6. In an i-v g-fuzzy n-nls with the continuity of the underlying $\mathscr{I}\mathscr{V}$-t-norm $\overline{*}$ at $(\overline{1}, \overline{1})$, every convergent sequence has unique limit.

Proof . Let $(X, \overline{N}, \overline{*})$ be an i-v g-fuzzy n-nls and $\{x_n\}$ be a sequence in X. If possible suppose that $\{x_r\}$ converges to two distinct points x and y in X. Since $x \neq y$, there exists a set of linearly independent set of vectors $\{y_1, y_2, \cdots, y_{n-1}, x - y\}$ in X. Then for $y_1, y_2, \cdots, y_{n-1} \in X$ and $t_0 > 0$,

$$\lim_{r\to\infty} \overline{N}(y_1, y_2, \cdots, y_{n-1}, x_r - x, \frac{t_0}{2}) = \overline{1} \ \text{ and } \ \lim_{r\to\infty} \overline{N}(y_1, y_2, \cdots, y_{n-1}, x_r - y, \frac{t_0}{2}) = \overline{1}.$$

Now,

$$\overline{N}(y_1, y_2, \cdots, y_{n-1}, x-y, t_0) \geq \overline{N}(y_1, y_2, \cdots, y_{n-1}, x_r - x, \frac{t_0}{2}) \,\overline{*}\, \overline{N}(y_1, y_2, \cdots, y_{n-1}, x_r - y, \frac{t_0}{2})$$
$$\to \overline{1} \,\overline{*}\, \overline{1} = \overline{1} \;\; as \;\; r \to \infty \;\; (\text{since } \overline{*} \text{ is continuous } at \ (\overline{1}, \overline{1}))$$

Therefore,

$$\overline{N}(y_1, y_2, \cdots, y_{n-1}, x-y, t_0) = \overline{1}.$$

Since $t_0 > 0$ chosen arbitrarily, $\overline{N}(y_1, y_2, \cdots, y_{n-1}, x-y, t) = \overline{1}$, for all $t > 0$. This implies that $\{y_1, y_2, \cdots, y_{n-1}, x-y\}$ is a linearly dependent set in X, which is a contradiction to our assumption. □

Definition 3.7. A sequence $\{x_r\}$ in an i-v g-fuzzy n-nls $(X, N, \overline{*})$ is said to be a Cauchy sequence if for each $y_1, y_2, \cdots, y_{n-1} \in X$, $\overline{\delta} \in (I)$, $t > 0$, there exists $m \in \mathbb{N}$ such that

$$\overline{N}(y_1, y_2, \cdots, y_{n-1}, x_r - x_s, t) > \overline{1} - \overline{\delta} \quad \forall r, s \geq m$$

where $\overline{\delta} = [\delta^-, \delta^+],\ 0 < \delta^- < \delta^+ < 1$.

Proposition 3.8. In an i-v g-fuzzy n-nls $(X, \overline{N}, \overline{*})$, a sequence $\{x_r\}$ is a Cauchy sequence iff for each $y_1, y_2, \cdots, y_{n-1} \in X$, $\overline{N}(y_1, y_2, \cdots, y_{n-1}, x_r - x_s, t) \to \overline{1}$ as $r, s \to \infty$.

Proof . The proof is same as the proof of the Proposition 3.5. □

Proposition 3.9. In an i-v g-fuzzy n-nls $(X, \overline{N}, \overline{*})$, every convergent sequence is Cauchy.

Proof . Let $\{x_r\}$ be a sequence in an i-v g-fuzzy n-nls $(X, \overline{N}, \overline{*})$ converging to $x \in X$. Let us choose $\overline{\delta} \in (I)$. For $\overline{\delta} \in (I)$, we can choose $\overline{\lambda} \in (I)$ such that $(\overline{1} - \overline{\lambda}) \,\overline{*}\, (\overline{1} - \overline{\lambda}) > (\overline{1} - \overline{\delta})$. Then for each $y_1, y_2, \cdots, y_{n-1} \in X$, $t > 0$ and $\overline{\lambda} \in (I)$, there exists $m \in \mathbb{N}$ such that

$$\overline{N}(y_1, y_2, \cdots, y_{n-1}, x_r - x, \frac{t}{2}) > \overline{1} - \overline{\lambda}, \quad \forall r \geq m.$$

Now,

$$\begin{aligned}\overline{N}(y_1, y_2, \cdots, y_{n-1}, x_r - x_s, t) =& \overline{N}(y_1, y_2, \cdots, y_{n-1}, x_r - x + x - x_s, \frac{t}{2} + \frac{t}{2}) \\ \geq& \overline{N}(y_1, y_2, \cdots, y_{n-1}, x_r - x, \frac{t}{2}) \,\overline{*}\, \overline{N}(y_1, y_2, \cdots, y_{n-1}, x_s - x, \frac{t}{2}) \\ >& (\overline{1} - \overline{\lambda}) \,\overline{*}\, (\overline{1} - \overline{\lambda}) \quad \forall r, s \geq m \\ >& (\overline{1} - \overline{\delta}) \quad \forall r, s \geq m.\end{aligned}$$

This shows that $\{x_r\}$ is a Cauchy sequence in X. □

***Remark* 3.10.** The converse of the above Proposition 3.9 is not necessarily true. For justification, we consider the Example 3.3. We first show that, a sequence $\{x_r\}$ is a convergent (or Cauchy) sequence in $(X, N, \overline{*})$ iff $\{x_r\}$ is convergent (or Cauchy) in $(X, ||\underbrace{., \cdots, .}_{n \text{ times}}||)$.

$$\{x_n\} \text{ is a convergent sequence in } (X, N, \overline{*}) \text{ which converges to } x \in X$$
$$\text{if and only if for each } y_1, y_2, \cdots, y_{n-1} \in X,\ t > 0,\ \lim_{r \to \infty} \overline{N}(y_1, y_2, \cdots, y_{n-1}, x_r - x, t_0) = \overline{1}$$
$$\text{if and only if } \lim_{r \to \infty} \frac{t}{t + ||y_1, y_2, \cdots, y_{n-1}, x_r - x||} = 1^- \quad \forall t > 0$$
$$\text{if and only if } \lim_{r \to \infty} ||y_1, y_2, \cdots, y_{n-1}, x_r - x|| = 0$$
$$\text{if and only if } \{x_r\} \text{ is convergent in } (X, ||\underbrace{., \cdots, .}_{n \text{ times}}||).$$

We follow same line of proof for Cauchy sequence. Therefore, if $||., \cdots, .||$ be an incomplete norm on a linear space X, the i-v g-fuzzy n-nls as defined in the Example 3.3 is an incomplete i-v g-fuzzy n-nls.

4 Finite dimensional interval-valued generalized fuzzy n-normed linear space

This Section consists some results on interval-valued generalized fuzzy n-norm defined over a finite dimensional linear space. We start with the following lemma.

Lemma 4.1. Let $(X, \overline{N}, \overline{*})$ be an i-v g-fuzzy n-nls and $\overline{*}$ be continuous at $(\overline{1}, \overline{1})$. If $\{x_1, x_2, \cdots, x_k\}$ is a linearly independent set of vectors in X, then there exist $c > 0$ and $\overline{\delta} \in (I)$ such that for each set of scalars $\{\alpha_1, \cdots, \alpha_k\}$, there exists $y_1, y_2, \cdots, y_{n-1} \in X$ such that

$$\overline{N}(y_1, y_2, \cdots, y_{n-1}, \sum_{i=1}^{k} \alpha_i x_i, c\sum_{i=1}^{k} |\alpha_i|) < \overline{1} - \overline{\delta}. \tag{4.1}$$

Proof . Let $s = \sum_{i=1}^{k} |\alpha_i|$. If $s = 0$ then $\alpha_i = 0, \ \forall i = 1, 2, \cdots, k$. Then $\overline{N}(y_1, y_2, \cdots, y_{n-1}, \theta, c\sum_{i=1}^{k} |\alpha_i|) = \overline{0}$ and the result holds for any $c > 0$ and $\overline{\delta} \in (I)$. Next suppose that $s \neq 0$. Then (4.1) is equivalent to

$$\overline{N}(y_1, y_2, \cdots, y_{n-1}, \sum_{i=1}^{k} \beta_i x_i, c) < \overline{1} - \overline{\delta} \tag{4.2}$$

where $\sum_{i=1}^{k} |\beta_i| = 1$. If possible suppose that (4.2) does not hold. Then for any $c > 0$ and $\overline{\delta} \in (I)$, there exists a set of scalars $\{\beta_1, \beta_2, \cdots, \beta_k\}$ with $\sum_{i=1}^{k} |\beta_i| = 1$ such that for any $y_1, y_2, \cdots, y_{n-1} \in X$,

$$\overline{N}(y_1, y_2, \cdots, y_{n-1}, \sum_{i=1}^{k} \beta_i x_i, c) \geq \overline{1} - \overline{\delta}. \tag{4.3}$$

In particular, for each positive integer m, if we choose $c = \frac{1}{m}$ and $\overline{\delta_m} = [\delta_m^-, \delta_m^+] = [\frac{1}{m+2}, \frac{1}{m+1}]$, then there exists a set of scalars $\{\beta_1^{(m)}, \beta_2^{(m)}, \cdots, \beta_k^{(m)}\}$ with $\sum_{i=1}^{k} |\beta_i^{(m)}| = 1$ such that for any $y_1, y_2, \cdots, y_{n-1} \in X$,

$$\overline{N}(y_1, y_2, \cdots, y_{n-1}, z_m, c) \geq \overline{1} - \overline{\delta_m} \tag{4.4}$$

where $z_m = \sum_{i=1}^{k} \beta_i^{(m)} x_i, \ m = 1, 2, \cdots$. Since $\sum_{i=1}^{k} |\beta_i^{(m)}| = 1, \ m = 1, 2, \cdots$, we have $0 \leq |\beta_i^{(m)}| \leq 1$ for $i = 1, 2, \cdots, k$. So for each fixed i, the sequence $\{\beta_i^{(m)}\}$ is bounded and hence $\{\beta_i^{(m)}\}$ has a convergent subsequence. Let β_1 denotes the limit of that subsequence and let $\{z_{1,m}\}$ denotes the corresponding subsequence of $\{z_m\}$. By the same argument $\{z_{1,m}\}$ has a subsequence $\{z_{2,m}\}$ for which the corresponding subsequence of scalars $\{\beta_2^{(m)}\}$ converges to β_2. Proceeding in this way, after n steps we obtain a subsequence $\{z_{k,m}\}$ where $z_{k,m} = \sum_{i=1}^{k} \gamma_i^{(m)} x_i$ with $\sum_{i=1}^{k} |\gamma_i^{(m)}| = 1$ and $\{\gamma_i^{(m)}\}$ converges to β_i as $m \to \infty$, for each $i = 1, 2, \cdots, k$. Let $y = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k$. Now, for all $t > 0$ and for all $y_1, y_2, \cdots, y_{n-1} \in X$,

$$\begin{aligned} \overline{N}(y_1, y_2, \cdots, y_{n-1}, z_{k,m} - y, t) =& \overline{N}(y_1, y_2, \cdots, y_{n-1}, \underset{i=1}{k} (\gamma_i^{(m)} - \beta_i) x_i, \frac{kt}{k}) \\ \geq& \overline{N}(y_1, y_2, \cdots, y_{n-1}, (\gamma_1^{(m)} - \beta_1) x_1, \frac{t}{k}) \ \overline{*} \ \cdots \ \overline{*} \ \overline{N}(y_1, y_2, \cdots, y_{n-1}, (\gamma_k^{(m)} - \beta_k) x_k, \frac{t}{k}) \\ =& \overline{N}(y_1, y_2, \cdots, y_{n-1}, x_1, \frac{t}{k|\gamma_1^{(m)} - \beta_1|}) \ \overline{*} \ \cdots \ \overline{*} \ \overline{N}(y_1, y_2, \cdots, y_{n-1}, x_k, \frac{t}{k|\gamma_k^{(m)} - \beta_k|}) \end{aligned}$$

which implies that

$$\lim_{m \to \infty} \overline{N}(y_1, y_2, \cdots, y_{n-1}, z_{k,m} - y, t) \geq \ \overline{1} \ \overline{*} \ \cdots \ \overline{*} \ \overline{1} \ \ \forall t > 0 \text{ (by the continuity of } \overline{*} \text{ at } (\overline{1}, \overline{1})).$$

Thus,

$$\lim_{m \to \infty} \overline{N}(y_1, y_2, \cdots, y_{n-1}, z_{k,m} - y, t) = \overline{1} \ \ \forall t > 0.$$

Hence $\{z_{k,m}\}$ converges to y. Now, for $s > 0$, we choose m such that $\frac{1}{m} < s$. Then we have,

$$\begin{aligned} \overline{N}(y_1, y_2, \cdots, y_{n-1}, z_{k,m}, s) &= \overline{N}(y_1, y_2, \cdots, y_{n-1}, z_{k,m} + \theta, s + \frac{1}{m} - \frac{1}{m}) \\ &\geq \overline{N}(y_1, y_2, \cdots, y_{n-1}, z_{k,m}, \frac{1}{m}) \ \overline{*} \ \overline{N}(y_1, y_2, \cdots, y_{n-1}, \theta, s - \frac{1}{m}) \\ &\geq (\overline{1} - \overline{\delta_m}) \ \overline{*} \ \overline{1} \qquad \text{(since } \{y_1, y_2, \cdots, y_{n-1}, \theta\} \text{ is a linearly dependent set)} \\ &= (\overline{1} - \overline{\delta_m}). \end{aligned}$$

From above we have,

$$N^-(y_1, y_2, \cdots, y_{n-1}, z_{k,m}, s) \geq (1 - \frac{1}{m+1}) \text{ and } N^+(y_1, y_2, \cdots, y_{n-1}, z_{k,m}, s) \geq (1 - \frac{1}{m+2}).$$

Therefore, $\lim_{m\to\infty} N^-(y_1, y_2, \cdots, y_{n-1}, z_{k,m}, s) = 1$ and $\lim_{m\to\infty} N^+(y_1, y_2, \cdots, y_{n-1}, z_{k,m}, s) = 1$. This implies

$$\lim_{m\to\infty} \overline{N}(y_1, y_2, \cdots, y_{n-1}, z_{k,m}, s) = \overline{1} \quad \forall s > 0$$

and hence $\underset{m\to\infty}{lim}\, z_{k,m} = \theta$. Since limit of a convergent sequence is unique, we conclude that $y = \theta$. Hence $y = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k = \theta$ which implies $\beta_1 = \beta_2 = \cdots = \beta_k = 0$, since $\{x_1, x_2, \cdots, x_k\}$ ia a linearly independent set of vectors. This is a contradiction to the fact that $\sum_{i=1}^{k} |\beta_i| = 1$. Therefore (4.2) holds and hence the lemma is proved. □

Definition 4.2. An i-v g-fuzzy n-nls $(X, \overline{N}, \overline{*})$ is said to be compact if every sequence in X has a convergent subsequence which converges to some point in X.

Definition 4.3. Let $(X, \overline{N}, \overline{*})$ be an i-v g-fuzzy n-nls and $F \subseteq X$. F is said to be bounded if for each $y_1, y_2, \cdots, y_{n-1} \in X$ and $\overline{\delta} \in (I)$, there exists $t_0 > 0$ such that

$$\overline{N}(y_1, y_2, \cdots, y_{n-1}, x, t_0) > \overline{1} - \overline{\delta} \qquad \forall\ x \in F.$$

Theorem 4.4. Let $(X, \overline{N}, \overline{*})$ be a finite dimensional i-v g-fuzzy n-nls and $\overline{*}$ be continuous at $(\overline{1}, \overline{1})$. A subset A of X is compact iff A is closed and bounded.

Proof . Suppose that A is compact. Choose $x \in \overline{A}$. Then there exists a sequence $\{x_r\} \subset A$ such that $\underset{r\to\infty}{lim}\, x_r = x$. Since A is compact, there exist a subsequence $\{x_{r_k}\}$ of $\{x_r\}$ converges to a point in A. Again $\{x_r\}$ converges to x and so $\{x_{r_k}\}$ also converges to x. Thus, $x \in A$ i.e A is closed. If possible suppose that A is not bounded. Then there exists $y_1, y_2, \cdots, y_{n-1} \in X$ and $\overline{\delta_0} \in (I)$ such that for each positive integer r, there exists $x_r \in A$ for which

$$\overline{N}(y_1, y_2, \cdots, y_{n-1}, x_r, r) \leq \overline{1} - \overline{\delta_0}. \tag{4.5}$$

Since A is compact, there exist a subsequence $\{x_{r_l}\}$ of $\{x_r\}$ converging to some element $x(say) \in A$. So for $y_1, y_2, \cdots, y_{n-1} \in X$,

$$\underset{l\to\infty}{lim} \overline{N}(y_1, y_2, \cdots, y_{n-1}, x_{r_l} - x, t) = \overline{1} \quad \forall t > 0. \tag{4.6}$$

From (4.5), we can write

$$\overline{N}(y_1, y_2, \cdots, y_{n-1}, x_{r_l}, r_l) \leq \overline{1} - \overline{\delta_0}.$$

Thus,

$$\begin{aligned}
\overline{1} - \overline{\delta_0} &\geq \overline{N}(y_1, y_2, \cdots, y_{n-1}, x_{r_l}, r_l) \\
&= \overline{N}(y_1, y_2, \cdots, y_{n-1}, x_{r_l} - x + x, r_l - t + t) \\
&\geq \overline{N}(y_1, y_2, \cdots, y_{n-1}, x_{r_l} - x, t) \,\overline{*}\, \overline{N}(y_1, y_2, \cdots, y_{n-1}, x, r_l - t) \\
&\geq \overline{1} \,\overline{*}\, \overline{1} \;\; as\ l \to \infty \\
&= \overline{1}
\end{aligned}$$

This implies $\overline{1} - \overline{\delta_0} \geq \overline{1}$ i.e $\overline{\delta_0} \leq \overline{0}$ - a contradiction. Hence A is bounded.

For the converse part, suppose that A is closed and bounded and $dimX = k$. Let $B = \{e_1, e_2, \cdots, e_k\}$ be a basis for X and $\{x_r\}$ be a sequence in X. Since B is a basis for X, there exists a set of suitable scalars $\beta_1^{(r)}, \beta_2^{(r)}, \cdots, \beta_k^{(r)}$ such that $x_r = \beta_1^{(r)} e_1 + \beta_2^{(r)} e_2 + \cdots + \beta_k^{(r)} e_k,\ r = 1, 2, \cdots$. Then by Lemma 4.1, there exists $c > 0$ and $\overline{\delta} \in (I)$ such that for the scalars $\beta_1^{(r)}, \beta_2^{(r)}, \cdots, \beta_k^{(r)}$, there exists $y_1, y_2, \cdots, y_{n-1} \in X$ such that

$$\overline{N}(y_1, y_2, \cdots, y_{n-1}, \sum_{i=1}^{k} \beta_i x_i, c \sum_{i=1}^{k} |\beta_i|) < \overline{1} - \overline{\delta},\ r = 1, 2, \cdots. \tag{4.7}$$

Since $\{x_r\}$ is bounded, $\overline{\delta} \in (I)$ and $y_1, y_2, \cdots, y_{n-1} \in X$, there exists $t > 0$ such that

$$\overline{N}(y_1, y_2, \cdots, y_{n-1}, \sum_{i=1}^{k} \beta_i x_i, t) > \overline{1} - \overline{\delta},\ r = 1, 2, \cdots. \tag{4.8}$$

(4.7) and (4.8) together implies

$$\overline{N}(y_1, y_2, \cdots, y_{n-1}, \sum_{i=1}^{k} \beta_i x_i, c \sum_{i=1}^{k} |\beta_i|) < \overline{N}(y_1, y_2, \cdots, y_{n-1}, \sum_{i=1}^{k} \beta_i x_i, t),\ r = 1, 2, \cdots$$

$$i.e\ \ c \sum_{i=1}^{k} |\beta_i| \leq t,\ r = 1, 2, \cdots \quad (\text{since } \overline{N}(., \cdots, ., t) \text{ is a nondecreasing function w.r.t } t).$$

This implies that $\sum_{i=1}^{k} |\beta_i| \leq \frac{t}{c},\ r = 1, 2, \cdots$. Hence, we have $|\beta_i| \leq \frac{t}{c}$, for each $i = 1, 2, \cdots, k$ and $r = 1, 2, \cdots$. Thus for each $i = 1, 2, \cdots, k$, $\{\beta_i^{(r)}\}$ is a bounded sequence of scalars in $\mathbb{R}$. Hence $\{\beta_i^{(r)}\}$ has a converging subsequence. Then we follow the line of proof of Lemma 4.1 to show that there exist a subsequence of $\{x_r\}$ which converges to some element in A. This proves that A is compact. □

Theorem 4.5. Let $\overline{N}$ be an i-v g-fuzzy norm over a finite dimensional linear space X and the underlying $\mathscr{I}\mathscr{V}$-t-norm $\overline{*}$ be continuous at $(\overline{1}, \overline{1})$. Then $(X, \overline{N}, \overline{*})$ is complete.

Proof . Let $dimX = k$ and $\{e_1, e_2, \cdots, e_k\}$ be a basis for X. Let $\{x_r\}$ be a Cauchy sequence in $(X, \overline{N}, \overline{*})$. Then there exists a set of suitable scalars $\beta_1^{(r)}, \beta_2^{(r)}, \cdots, \beta_k^{(r)}$ such that $x_r = \beta_1^{(r)} e_1 + \beta_2^{(r)} e_2 + \cdots + \beta_k^{(r)} e_k,\ r = 1, 2, \cdots$. Since $\{x_r\}$ is a Cauchy sequence, for each $y_1, y_2, \cdots, y_{n-1} \in X$,

$$\lim_{r,s \to \infty} \overline{N}(y_1, y_2, \cdots, y_{n-1}, x_r - x_s, t) = \overline{1},\ \forall t > 0. \tag{4.9}$$

Now by Lemma 4.1, there exists $c > 0$ and $\overline{\delta} \in (I)$ such that

$$\overline{N}(y_1, y_2, \cdots, y_{n-1}, \sum_{i=1}^{k} e_i(\beta_i^{(r)} - \beta_i^{(s)}), c \sum_{i=1}^{k} |\beta_i^{(r)} - \beta_i^{(s)}|) < \overline{1} - \overline{\delta}. \tag{4.10}$$

If $\sum_{i=1}^{k} |\beta_i^{(r)} - \beta_i^{(s)}| = 0$ then $\beta_i^{(r)} = \beta_i^{(s)}$ for all $i = 1, 2, \cdots, k$, which implies $\{x_r\}$ is a constant sequence and hence the theorem is done. So suppose that $\sum_{i=1}^{k} |\beta_i^{(r)} - \beta_i^{(s)}| \neq 0$. Again for $\overline{\delta} \in (I)$, the relation (4.9) implies there exists $n_0 \in \mathbb{N}$ such that

$$\overline{N}(y_1, y_2, \cdots, y_{n-1}, \sum_{i=1}^{k} e_i(\beta_i^{(r)} - \beta_i^{(s)}), t) > \overline{1} - \overline{\delta} \quad \forall r, s \geq n_0. \tag{4.11}$$

Therefore from (4.10) and (4.11), for all $r, s \geq n_0$, we have

$$\overline{N}(y_1, y_2, \cdots, y_{n-1}, \sum_{i=1}^{k} e_i(\beta_i^{(r)} - \beta_i^{(s)}), c \sum_{i=1}^{k} |\beta_i^{(r)} - \beta_i^{(s)}|) < \overline{N}(y_1, y_2, \cdots, y_{n-1}, \sum_{i=1}^{k} e_i(\beta_i^{(r)} - \beta_i^{(s)}), t)$$

which implies that

$$c \sum_{i=1}^{k} |\beta_i^{(r)} - \beta_i^{(s)}| < t \quad \forall r, s \geq n_0 \ \ (\text{since } \overline{N}(., \cdots, ., t) \text{ is a nondecreasing function w.r.t } t).$$

Since $t > 0$ is arbitrary, $\lim_{r,s \to \infty} \sum_{i=1}^{k} |\beta_i^{(r)} - \beta_i^{(s)}| = 0$ i.e $\lim_{r,s \to \infty} |\beta_i^{(r)} - \beta_i^{(s)}| = 0,\ i = 1, 2, \cdots, k$. Hence $\{\beta_i^{(r)}\}$ is a Cauchy sequence of scalars for each $i = 1, 2, \cdots, k$ and thus each sequence $\{\beta_i^{(r)}\}$ converges. Let, $\lim_{r \to \infty} \beta_i^{(r)} = \beta_i$ for $i = 1, 2, \cdots, k$. Define $x = \sum_{i=1}^{k} \beta_i e_i$. Then clearly $x \in X$. By similar line of proof of the Lemma 4.1, we can conclude that

$$\lim_{r \to \infty} \overline{N}(y_1, y_2, \cdots, y_{n-1}, x_r - x, t) = \overline{1} \ \ \forall t > 0.$$

This completes the proof. □

Conclusion:

This article contains the definition of interval-valued generalized fuzzy n-normed linear space which generalize the notion of interval-valued fuzzy n-normed linear space. Here we use the general $\mathscr{IV}$-t-norm $\overline{*}$ instead of $\mathscr{IV}$-t-norm 'min'. We study the notion of convergent sequence, Cauchy sequence and establish some well-known results of fuzzy functional analysis including completeness and compactness on finite dimensional i-v g-fuzzy n-nls. Our results are more interesting and important than existing results as we consider general $\mathscr{IV}$-t-norm $\overline{*}$ instead of particular $\mathscr{IV}$-t-norm 'min' (which is used in existing results) and consequently results are also more general than existing results. We think it will enrich the contents of fuzzy mathematics and in future there is a wide scope of research with underlying general $\mathscr{IV}$-t-norm $\overline{*}$ setting to develop the results of functional analysis in this new fuzzy setting.

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