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Existence of multiple solutions for nonlinear fractional Schrödinger-Poisson system involving new fractional operator

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Abstract

In this paper, we prove the existence of multiple solutions in the Bessel Potential space for a new class of nonlinear fractional Schrödinger-Poisson systems involving the distributional Riesz fractional derivative. To reach our goal, we use the symmetric mountain pass theorem under some suitable assumptions on nonlinearity $f(x,u)$ and potential $V(x)$.

Keywords: Fractional Schrödinger-Poisson system, symmetric mountain pass theorem, Palais-Smale condition, distributional Riesz fractional derivative, Bessel potential space.
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1 Introduction

In the last few years, nonlinear systems involving fractional and nonlocal differential operators of elliptic type, have been studied extensively by many scholars, due to numerous applications in many fields of science, such as electrical circuits, optimization, phase transitions, finance, and quantum mechanics. For previous related results, we refer the readers to [4, 6, 9, 16, 22, 23].

Recently, due to the real physical meaning, the fractional Schrödinger-Poisson system has been extensively investigated by many authors. Benci and Fortunato in [3] proposed the following classical Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.1}$$

to describe quantum particles for nonlinear Schrödinger equations interacting with an unknown electrostatic field. It also appears in plasma physics, semiconductor theory, and so on. The nonlinearity f denotes the particles interacting with each other, and the nonlocal term ϕu concerns the interaction with the electric field. We refer the interesting reader to [17, 18] and their references to get more physical background to the system (1.1).
In the last decade, there are many interesting works about the existence of positive solutions, ground states solutions,

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infinitely many solutions, concentration of solutions, and multiplicity of solutions via variational tools and critical point theory, see [2, 8, 14, 26, 28] and the references therein. Che and Chen in [7] studied the following system

$$\begin{cases}(-\Delta)^{\alpha}u+V_{\lambda}(x)u+t\phi u=f(x,u)+g(x)|u|^{r-2}u & \text{in } \mathbb{R}^3,\\ (-\Delta)^{\beta}\phi=u^2 & \text{in } \mathbb{R}^3,\end{cases} \tag{1.2}$$

where $\alpha,\beta\in(0;1]$, $t>0$, $2\beta+2\alpha>3$, $V_{\lambda}(x)$ is allowed to be sign-changing potential, and $(-\Delta)^{\alpha}$ is the fractional Laplacian operator, under some assumption on $f(x,u)$ and $V_{\lambda}(x)$, multiplicity and concentration of solutions are obtained. In [8], a similar system to the (1.2) was studied by Chen, he showed the existence of multiple solutions for the following system

$$\begin{cases}(-\Delta)^{\alpha}u+V(x)u+\phi u=f(x,u)+tg(x)|u|^{r-2}u & \text{in } \mathbb{R}^3,\\ (-\Delta)^{\beta}\phi=u^2 & \text{in } \mathbb{R}^3,\end{cases} \tag{1.3}$$

where $\alpha,\beta\in(0;1]$, $t>0$, $1<r<2$, $2\beta+4\alpha>3$. When $t=0$, system (1.3) reduces to the the following system

$$\begin{cases}(-\Delta)^{\alpha}u+V(x)u+\phi u=f(x,u) & \text{in } \mathbb{R}^3,\\ (-\Delta)^{\beta}\phi=u^2 & \text{in } \mathbb{R}^3.\end{cases} \tag{1.4}$$

In recent years, system like (1.4) has been widely studied by many scholars, for example, Gao et al [13] for ground state solutions when $f(x,u)=f(u)$, Li [15] for non-trivial solution when $V(x)=1$, and Zhang [27] for the existence and multiplicity results.

After the pioneering work of Shieh and Spector [22] concerning the study of a new class of fractional PDEs related to the distributional Riesz fractional gradient, an increasing number of authors have been interested in studying its theoretical structure see e.g [5, 16, 22, 23], and in understanding the applications in the theory of electromagnetic fields, multidimensional processes, and in fractal media see e.g [1, 12, 19] and their works. The latter operator is an intrinsic object of interest for the study of fractional PDEs as stated by Shieh and Spector in [22, 23], they introduced an appropriate functional space to study fractional problems in which the distributional Riesz fractional gradient is present, it also satisfies three basic physical requirements as proved in [24] on fractional gradient analysis.

In the present paper, we build upon all the works just described, by using the distributional Riesz fractional derivative instead of the usual fractional Laplacian, we study the following new class of fractional Schrödinger-Poisson system

$$\begin{cases}-div^{\alpha}(\nabla^{\alpha}u)+V(x)u+\phi u=f(x,u)+tg(x)|u|^{r-2}u & \text{in } \mathbb{R}^3,\\ -div^{\beta}(\nabla^{\beta}\phi)=u^2 & \text{in } \mathbb{R}^3,\end{cases} \tag{1.5}$$

where $\alpha,\beta\in(0;1]$, $t>0$ is a parameter, $r\in(1,2)$, $2\beta+4\alpha>3$, and $-div^{\alpha}(\nabla^{\alpha})$ is the distributional Riesz fractional derivative, and we give its consistency with the usual fractional Laplacian in this work. The starting point of research pursued in [22] for the development of a general theory for fractional PDEs involving this operator, is the distributional Riesz fractional gradient ∇^{α} of order $\alpha\in(0,1)$ (fractional gradient for short). For $1<p<\infty$, if $u\in L^p\left(\mathbb{R}^N\right)$ such that $I_{1-\alpha}*u$ is well defined, ∇^{α} can be characterized as (see [16, 22])

$$(\nabla^{\alpha}u)_j=\frac{\partial^{\alpha}u}{\partial x_j^{\alpha}}=\frac{\partial}{\partial x_j}I_{1-\alpha}*u,\ 0<\alpha<1,\ j=1,...,N,$$

where $\dfrac{\partial}{\partial x_j}$, is defined for every $w\in C_0^{\infty}(\mathbb{R}^N)$ in the following sense

$$\langle\frac{\partial^{\alpha}u}{\partial x_j^{\alpha}},w\rangle=-\langle I_{1-\alpha}*u,\frac{\partial w}{\partial x_j}\rangle=-\int_{\mathbb{R}^N}(I_{1-\alpha}*u)\frac{\partial w}{\partial x_j}dx,$$

where I_{α} denotes the Riesz potential of order α, $0<\alpha<1$:

$$\left(I_{\alpha}*u\right)(x)=\gamma\left(N,\alpha\right)\int\limits_{\mathbb{R}^N}\frac{u\left(y\right)}{\left|x-y\right|^{N-\alpha}}dy,\quad with\quad \gamma(N,\alpha):=\pi^{-\frac{N}{2}}\frac{\Gamma(\frac{N-\alpha}{2})}{2^{\alpha}\Gamma(\frac{\alpha}{2})}.$$

Thus, the fractional gradient ∇^{α} and the fractional divergence (div^{α}) can be written in finite integral form for smooth function u and vector w ([10, 16, 23], respectively by

$$\begin{aligned}\nabla^{\alpha}u(x) &:= \gamma(N,\alpha)\lim_{\epsilon\to 0}\int_{\mathbb{R}^N}\frac{zu(x+z)}{|z|^{N+\alpha+1}}\chi_{\epsilon}(0,z)dz\\ &= \gamma(N,\alpha)\int_{\mathbb{R}^N}[u(x)-u(y)]\frac{1}{|x-y|^{N+\alpha}}\frac{x-y}{|x-y|}dy,\end{aligned}$$

$$\begin{aligned}div^{\alpha}w(x) &:= \gamma(N,\alpha)\lim_{\epsilon\to 0}\int_{\mathbb{R}^N}\frac{z.w(x+z)}{|z|^{N+\alpha+1}}\chi_{\epsilon}(0,z)dz\\ &= \gamma(N,\alpha)\int_{\mathbb{R}^N}[w(x)-w(y)].\frac{1}{|x-y|^{N+\alpha}}\frac{x-y}{|x-y|}dy,\end{aligned}$$

where $\chi_{\epsilon}(0,z)$, is the characteristic function of the set $\{(0,z) : |z| > \epsilon\}$ for $\epsilon > 0$. It was observed in [22] that for $u \in C_0^{\infty}(\mathbb{R}^N)$, the composition of fractional divergence div^{α} and fractional gradient ∇^{α} it coincides with the fractional Laplacian as follows:

$$\begin{aligned}(-\Delta)^{\alpha}u &= -\sum_{j=1}^{N}\frac{\partial^{\alpha}}{\partial x_j^{\alpha}}\frac{\partial^{\alpha}}{\partial x_j^{\alpha}}u\\ &= -div^{\alpha}(\nabla^{\alpha}u), \qquad (1.6)\end{aligned}$$

where the well known fractional Laplacian can be given ([11]), for $\alpha \in (0,1)$ by

$$(-\Delta)^{\alpha}u(x) = \frac{1}{2}\gamma^2(N,\alpha)\int_{\mathbb{R}^N}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2\alpha}}dy.$$

Furthermore, for $u, w \in C_0^{\infty}(\mathbb{R}^N)$ equation (1.6) means, that the following holds

$$\int_{\mathbb{R}^N}\nabla^{\alpha}u.\nabla^{\alpha}wdx = \int_{\mathbb{R}^N}(-\Delta)^{\alpha}u.wdx = \int_{\mathbb{R}^N}(-\Delta)^{\frac{\alpha}{2}}u.(-\Delta)^{\frac{\alpha}{2}}wdx,$$

which is a key result for the weak formulation of PDEs involving fractional operator. We refer to [10, 16, 22, 23, 24] for more information about this fractional operator.

In this works, we assume that the functions f, g and V satisfy the following conditions:
(H_1) : $f \in C(\mathbb{R}^3 \times \mathbb{R};\mathbb{R})$ for every $x \in \mathbb{R}^3$ and $u \in \mathbb{R}$, there exists constant $K_1 > 0$, and $p \in (4; 2_{\alpha}^*)$ such that

$$|f(x,u)| \leq K_1(|u| + |u|^{p-1}),$$

where $2_{\alpha}^* = \frac{6}{3-2\alpha}$ the fractional critical Sobolev exponent,
(H_2) : $f(x,-u) = -f(x,u)$, $x \in \mathbb{R}^3$, $u \in \mathbb{R}$,
(H_3) : There exist $\mu > 4$ and $\lambda > 0$ such that

$$0 < \mu F(x,u) \leq uf(x,u)$$

holds for $|u| \geq \lambda$ and $\inf_{x\in\mathbb{R}^3,|u|=\lambda}F(x,u) > 0$, where $F(x,u) = \int_0^u f(x,s)ds$,
(H_4) : $g : \mathbb{R}^3 \to \mathbb{R}^+$ and $g(x) \in L^{\frac{2}{2-r}}(\mathbb{R}^3)$,
(V) : $V \in C(\mathbb{R}^3,\mathbb{R}), V_0 := \inf_{x\in\mathbb{R}^3} V(x) > 0$, where V_0 is a constant and

$$\lim_{|x|\to+\infty}V(x) = +\infty.$$

We next fix the following notations. For any $p \in [1,\infty)$, $L^p\left(\mathbb{R}^N\right)$ denotes the Lesbesgue space with the norm $\|u\|_{L^p} = (\int_{\mathbb{R}^N}|u|^p dx)^{\frac{1}{p}}$. $L^r\left(\mathbb{R}^N\right)$ the weighted Lesbesgue space for $1 < r < 2$ with the norm $\|u\|_{L^r} = (\int_{\mathbb{R}^N}g(x)\,|u|^r dx)^{\frac{1}{r}}$.

Under the above hypothesis, our main result can be stated as follows.

Theorem 1.1. Assume f satisfies (H_1)-(H_4) and (V). Then, we can find $t_0 > 0$, such that problem (1.5) has multiple solutions for every $t < t_0$.

To our knowledge, this paper is the seconde contribution to studying this class of fractional Schrödinger-Poisson systems in the Bessel potential space. The rest of this paper is organized as follows, in the next section, we introduce some work space and key results that will be used in this paper. In section 3, we use the symmetric mountain pass theorem to prove Theorem 1.1. In section 4, we give a discussion about our research results.

2 Preliminaries and variational settings

In this section, we first recall some necessary variational settings for system (1.5), and the complete introduction on fractional Sobolev space $W^{\alpha,2}(\mathbb{R}^N)$ and Bessel potential space$L^{\alpha,2}(\mathbb{R}^N)$ can be found respectively, in [11, 22].

For $\alpha \in (0,1)$ and $u \in C_0^\infty(\mathbb{R}^N)$, the vector space of fractional differentiable functions $S^{\alpha,2}(\mathbb{R}^N)$ is defined as the closure of $C_0^\infty(\mathbb{R}^N)$ with the norm

$$\|u\|^2_{S^{\alpha,2}} = \|u\|^2_{L^2} + \|\nabla^\alpha u\|^2_{L^2}. \tag{2.1}$$

By Theorem 1.7 in [22], it is exactly the Bessel potential space $L^{\alpha,2}(\mathbb{R}^N)$ defined for $\alpha \in \mathbb{R}_+$, by

$$L^{\alpha,2}(\mathbb{R}^N) = \{u : u = G_\alpha * f \quad for\ some \quad f \in L^2(\mathbb{R}^N)\},$$

where the Bessel potential G_α is defined by (see [21, 22])

$$G_\alpha(x) := \frac{1}{(4\pi)^{\frac{\alpha}{2}}\Gamma(\frac{\alpha}{2})}\int_0^{+\infty} e^{\frac{-\pi|x|^2}{t}} e^{\frac{-t}{4\pi}} t^{\frac{\alpha-N}{2}-1}dt.$$

The norm of this Bessel potential Space is $\|u\|_{L^{\alpha,2}} = \|f\|_{L^2}$ if $G_\alpha * f$. Now, we summarize the key properties of this Bessel potential space (see p.7 in [22]).

Theorem 2.1.
1. If $\alpha \in (0,1)$, then $H^\alpha(\mathbb{R}^N) = W^{\alpha,2}(\mathbb{R}^N) = L^{\alpha,2}(\mathbb{R}^N) = S^{\alpha,2}(\mathbb{R}^N)$, with the norm given by (2.1).
2. If $\alpha \geq 0$ and $2 \leq q \leq 2^*_\alpha$, then $L^{\alpha,2}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$, and the embedding is locally compact if $2 \leq q < 2^*_\alpha$,

Remark 2.2. (i) Though the work space in this paper involves $\|\nabla^\alpha u\|_{L^2}$, we will not separate the Bessel potential space $L^{\alpha,2}(\mathbb{R}^N)$ from the fractional Sobolev space $H^\alpha(\mathbb{R}^N)$ despite the fact that $L^{\alpha,2}(\mathbb{R}^N)$ is topologically compatible with $H^\alpha(\mathbb{R}^N)$.

(ii) As stated in [22], the most appropriate functional framework to deal with the system (1.5) is the Bessel potential space $L^{\alpha,2}(\mathbb{R}^N)$.

The homogeneous Sobolev space $D^{\alpha,2}(\mathbb{R}^N)$ for $\alpha \in (0,1)$, is defined by

$$D^{\alpha,2}(\mathbb{R}^N) = \left\{u \in L^{2^*_\alpha}(\mathbb{R}^N) : \nabla^\alpha u \in L^2(\mathbb{R}^N\right\},$$

which is the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm and the inner product, respectively

$$\|u\|_{D^{\alpha,2}} = (\int_{\mathbb{R}^N} |\nabla^\alpha u|^2 dx)^{\frac{1}{2}}.$$

$$\langle u, w\rangle_{D^{\alpha,2}} = \int_{\mathbb{R}^N} (\nabla^\alpha u.\nabla^\alpha w)dx.$$

This definition coincides with any standard definition of the homogeneous fractional Sobolev space $D^{\alpha,2}(\mathbb{R}^N)$. The solvability of the linear fractional PDEs with variable coefficients is established by the following theorem.

Theorem 2.3. ([22]) Let $\Omega \subset \mathbb{R}^N$ is an arbitrary bounded open set. Assume that $v \in L^{\alpha,2}(\mathbb{R}^N)$ and $h \in L^2(\Omega)$, such that $I_{1-\alpha} * v$ is well defined and $A : \mathbb{R}^N \longrightarrow \mathbb{R}^{N\times N}$ with coefficients bounded and measurable such that

$$\lambda |y|^2 \leq A(x)y.y \quad and \quad A(x)y.y \leq \Lambda |y|^2$$

For all $x, y \in \mathbb{R}^N$ and some $\lambda, \Lambda > 0$. Then, there exists a unique $u \in L^{\alpha,2}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} A(x)\nabla^\alpha u.\nabla^\alpha w dx = \int_{\mathbb{R}^N} hw dx$$

for every $w \in L^{\alpha,2}(\mathbb{R}^N)$ and $u = v$ in $\mathbb{R}^N \backslash \Omega$. In this work A is the identity.

From now on, we restrict the work space in dimension $N = 3$.

Lemma 2.4. ([11]) For any $\alpha \in (0, 1)$, $D^{\alpha,2}(\mathbb{R}^3)$ is continuously embedded in $L^{2^*_\alpha}(\mathbb{R}^3)$, i.e there exists $K_\alpha > 0$ such that :

$$\left(\int_{\mathbb{R}^3} |u|^{2^*_\alpha} dx\right)^{\frac{2}{2^*_\alpha}} \leq K_\alpha \int_{\mathbb{R}^3} |\nabla^\alpha u|^2 dx, \ u \in D^{\alpha,2}\left(\mathbb{R}^3\right).$$

Next, let us consider the variational setting of (1.5). For convenience, we use the letters K_i, $i = 1, 2...$ repeatedly to denote various constant which may change from line to line. If $2\beta + 4\alpha \geq 3$, then $L^{\alpha,2}(\mathbb{R}^3)$ is continuously embedded in $L^{\frac{12}{3+2\beta}}(\mathbb{R}^3)$. For $u \in L^{\alpha,2}(\mathbb{R}^3)$, the linear operator $\mathcal{L}_u : D^{\beta,2}(\mathbb{R}^3) \to \mathbb{R}$ is defined as:

$$\mathcal{L}_u(w) = \int_{\mathbb{R}^3} u^2 w dx.$$

By Hölder inequality and Lemma 2.4, we obtain

$$|\mathcal{L}_u(w)| \leq \|u\|^2_{L^{\frac{12}{3+2\beta}}} \|w\|_{L^{2^*_\beta}} \tag{2.2}$$

$$\leq K \|u\|^2_{L^{\alpha,2}} \|w\|_{D^{\beta,2}}. \tag{2.3}$$

Hence, according to the Lax-Milgram theorem, there exists a unique $\phi_u^\beta \in D^{\beta,2}\left(\mathbb{R}^3\right)$ such that

$$\int_{\mathbb{R}^3} \nabla^\beta \phi_u^\beta.\nabla^\beta w dx = \int_{\mathbb{R}^3} u^2 w dx \quad \forall w \in D^{\beta,2}\left(\mathbb{R}^3\right). \tag{2.4}$$

i.e. ϕ_u^β is a weak solution of $-div^\beta\left(\nabla^\beta \phi_u^\beta\right) = u^2$. Moreover,

$$\left\|\phi_u^\beta\right\|_{D^{\beta,2}} \leq K \|u\|^2_{L^{\alpha,2}}. \tag{2.5}$$

Since $2\beta + 4\alpha \geq 3$ and $\beta \in (0, 1]$, then $\frac{12}{3+2\beta} \in (2, 2^*_\alpha)$. From Lemma 2.4, (2.2) and (2.4) we derive

$$\left\|\phi_u^\beta\right\|^2_{D^{\beta,2}} = \int_{\mathbb{R}^3} \left|\nabla^\beta \phi_u^\beta\right|^2 dx = \int_{\mathbb{R}^3} u^2 \phi_u^\beta dx,$$

and

$$\left\|\phi_u^\beta\right\|^2_{D^{\beta,2}} \leq \|u\|^2_{L^{\frac{12}{3+2\beta}}} \left\|\phi_u^\beta\right\|_{L^{2^*_\beta}} \leq K \|u\|^2_{L^{\frac{12}{3+2\beta}}} \left\|\phi_u^\beta\right\|_{D^{\beta,2}}. \tag{2.6}$$

Then,

$$\left\|\phi_u^\beta\right\|_{D^{\beta,2}} \leq K \|u\|^2_{L^{\frac{12}{3+2\beta}}}. \tag{2.7}$$

For $x \in \mathbb{R}^3$, we have

$$\phi_u^\beta(x) = c_\beta \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2\beta}} dy, \tag{2.8}$$

which is called β-Riesz potential (see [21]), where

$$c_\beta = \pi^{-\frac{3}{2}} 2^{-2\beta} \frac{\Gamma\left(\frac{3-2\beta}{2}\right)}{\Gamma(\beta)}.$$

Substituting ϕ_u^β in (1.5), it leads to the following fractional Schrödinger equation

$$-div^\alpha(\nabla^\alpha u) + V(x)u + \phi_u^\beta u = f(x,u) + tg(x)|u|^{r-2}u, \quad x \in \mathbb{R}^3 . \tag{2.9}$$

Now, we introduce our working space

$$E = \left\{ (u \in L^{\alpha,2}\left(\mathbb{R}^3\right) : \int_{\mathbb{R}^3} |\nabla^\alpha u|^2 + V(x)u^2)dx < \infty \right\},$$

which is a Hilbert space equipped with the norm and the inner product respectively,

$$\|u\|_E^2 = \int_{\mathbb{R}^3} (|\nabla^\alpha u|^2 + V(x)|u|^2)dx,$$

$$\langle u, w\rangle_E = \int_{\mathbb{R}^3} (\nabla^\alpha u.\nabla^\alpha w + V(x)uw)dx.$$

Assume that (V) hold, by Lemma 2.3 in [25], E is compactly embedded in $L^p\left(\mathbb{R}^3\right)$ for $p \in [2, 2_\alpha^*)$, and continuously embedded in $L^p\left(\mathbb{R}^3\right)$ for $p \in [2, 2_\alpha^*]$. We define the energy functional $J : E \to \mathbb{R}$ associated to (1.5) by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla^\alpha u|^2 + V(x)u^2\right)dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^\beta u^2 dx - \int_{\mathbb{R}^3} F(x,u)\,dx - \frac{t}{r}\int_{\mathbb{R}^3} g(x)|u|^r dx.$$

Hence, J is well defined in E and $J \in C^1(E, \mathbb{R})$. Moreover its derivative is

$$< J'(u), w >= \int_{\mathbb{R}^3} (\nabla^\alpha u.\nabla^\alpha w + V(x)uw + \phi_u^\beta uw - f(x,u)w - tg(x)|u|^{r-2}uw)dx, \; w \in E. \tag{2.10}$$

Definition 2.5. 1. If $u \in E$ is a weak solution of (2.9), then the pair $(u, \phi) \in E \times D^{\beta,2}\left(\mathbb{R}^3\right)$ is a weak solution of (1.5).
2. $u \in E$ is a weak solution of (2.9) if

$$\int_{\mathbb{R}^3} \left(\nabla^\alpha u.\nabla^\alpha w + V(x)uw + \phi_u^\beta uw - f(x,u)\,w - tg(x)|u|^{r-2}uw\right)dx = 0.$$

Definition 2.6. The functional J satisfies the Palais-Smale condition at level $c \in \mathbb{R}$, denoted by $(\mathrm{PS})_c$, if any sequence $\{u_n\} \subset E$ satisfying

$$J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0 \;\text{ as }\; n \to \infty.$$

has a strongly convergent subsequence.

We choose $\{e\}_i$ an orthonormal basis of E and define $X_i = \mathbb{R}e_i$,

$$Y_k = \oplus_{i=1}^k X_i \quad Z_k = \overline{\oplus_{i=k}^\infty X_i} \quad k \in \mathbb{Z}.$$

Evidently, we have $E = Y_k \oplus Z_k$. To prove our result, we need the following symmetric mountain-pass theorem.

Theorem 2.7. (Symmetric mountain-pass theorem, see [20]) Assume that $E = Y_k \oplus Z_k$ be a Banach space where Y is finite dimensional, let $J \in C^1(E, \mathbb{R})$ be even, satisfies the $(\mathrm{PS})_c$ condition and $J(0) = 0$, if
(i) there exist constants $\rho, \delta > 0$ satisfying $J\left|_{\partial B_\rho \bigcap Z}\right. = \inf_{u \in Z, \|u\| = \rho} J(u) \geq \delta$,
(ii) for every finite dimensional subspace $\tilde{E} \subset E$, there is a constant $K = K(\tilde{E}) > 0$ such that $\max_{u \in \tilde{E}, \|u\| \geq K} J(u) < 0$,

then, J has an unbounded sequence of critical points.

3 Proof of main result

Lemma 3.1. J satisfies $(\mathrm{PS})_c$ condition for every $c \in \mathbb{R}$ on E, If (H_1)-(H_4) and (V) hold.

Proof . Let $\{u_n\}$ be a $(\mathrm{PS})_c$ sequence in E, we will prove that $\{u_n\}$ bounded in E using arguing by contradiction. By (H_3) and (H_4) for sufficiently large $n \in \mathbb{N}$, we have

$$\begin{aligned} c + \|u_n\|_E \geq & J(u_n) - \frac{1}{\mu}\langle J'(u_n), (u_n)\rangle \\ = & \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|_E^2 + \left(\frac{1}{4} - \frac{1}{\mu}\right)\int_{\mathbb{R}^3} \phi_{u_n}^\beta u_n^2 dx + \int_{\mathbb{R}^3}\left(\frac{u_n f(x,u_n)}{\mu} - F(x,u_n)\right)dx + (\frac{1}{\mu} - \frac{1}{r})t\int_{\mathbb{R}^3} g(x)|u_n|^r dx \\ \geq & \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|_E^2 + (\frac{1}{\mu} - \frac{1}{r})t\|g\|_{L^{\frac{2}{2-r}}} C^r \|u_n\|_E^r, \end{aligned} \tag{3.1}$$

which implies that $\{u_n\}$ is bounded in E. Up to a subsequence, we suppose that $u_n \rightharpoonup u$ in E. Since E is compactly embedded in $L^p(\mathbb{R}^3)$ for $2 \leq p < 2_\alpha^*$, then $u_n \to u$ in $L^p(\mathbb{R}^3)$, $2 \leq p < 2_\alpha^*$. Obviously, we can show that the following holds

$$\langle J'(u_n) - J'(u), u_n - u\rangle \to 0 \text{ and } \|u_n - u\|_{L^2}^2 \to 0 \text{ as } n \to \infty. \tag{3.2}$$

Combining the generalization of Hölder inequality, Lemma 2.4 and (2.7), we obtain

$$\begin{aligned} \left|\int_{\mathbb{R}^3} \phi_{u_n}^\beta u_n (u_n - u)\, dx\right| & \leq \left\|\phi_{u_n}^\beta\right\|_{L^{2_\beta^*}} \|u_n\|_{L^{\frac{12}{3+2\beta}}} \|u_n - u\|_{L^{\frac{12}{3+2\beta}}} \\ & \leq K\left\|\phi_{u_n}^\beta\right\|_{D^{\beta,2}} \|u_n\|_{L^{\frac{12}{3+2\beta}}} \|u_n - u\|_{L^{\frac{12}{3+2\beta}}} \\ & \leq K\|u_n\|^3_{L^{\frac{12}{3+2\beta}}} \|u_n - u\|_{L^{\frac{12}{3+2\beta}}} \\ & \leq K\|u_n\|_E^3 \|u_n - u\|_{L^{\frac{12}{3+2\beta}}}. \end{aligned}$$

Similarly, we prove that

$$\left|\int_{\mathbb{R}^3} \phi_u^\beta u (u_n - u)\, dx\right| \leq K\|u\|_E^3 \|u_n - u\|_{L^{\frac{12}{3+2\beta}}}.$$

We have,

$$\left|\int_{\mathbb{R}^3} \left(\phi_{u_n}^\beta u_n - \phi_u^\beta u\right)(u_n - u)\, dx\right| \leq \left|\int_{\mathbb{R}^3} \phi_{u_n}^\beta u_n (u_n - u)\, dx\right| + \left|\int_{\mathbb{R}^3} \phi_u^\beta u (u_n - u)\, dx\right| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.3}$$

From (H_1) and Hölder inequality, we obtain

$$\begin{aligned} \left|\int_{\mathbb{R}^3} (f(x,u_n) - f(x,u))(u_n - u)\, dx\right| & \leq K_1 \int_{\mathbb{R}^3} (|u_n| + |u|)|u_n - u|\, dx + K_1 \int_{\mathbb{R}^3} \left(|u_n|^{p-1} + |u|^{p-1}\right)|u_n - u|\, dx \\ & \leq K_1 (\|u_n\|_{L^2} + \|u\|_{L^2})\|u_n - u\|_{L^2} + K_1\left(\|u_n\|_{L^p}^{p-1} + \|u\|_{L^p}^{p-1}\right)\|u_n - u\|_{L^p} \\ & \leq K(\|u_n\|_E + \|u\|_E)\|u_n - u\|_{L^2} + K\left(\|u_n\|_E^{p-1} + \|u\|_E^{p-1}\right)\|u_n - u\|_{L^p} \end{aligned}$$

$\to 0$ as $n \to \infty$. By (H_4), we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} (g(x)|u_n|^{r-2}u_n - g(x)|u|^{r-2}u)(u_n - u)dx = 0.$$

Thus, we conclude that

$$\|u_n - u\|_E^2 = \langle J'(u_n) - J'(u), u_n - u\rangle - \int_{\mathbb{R}^3} (V(x)u_n(u_n-u) - V(x)u(u_n-u))\,dx$$
$$- \int_{\mathbb{R}^3} \left(\phi_{u_n}^\beta u_n - \phi_u^\beta u\right)(u_n-u)\,dx + \int_{\mathbb{R}^3} (f(x,u_n) - f(x,u))(u_n-u)dx$$
$$+t\int_{\mathbb{R}^3} \Big(g(x)(|u_n|^{r-2}u_n - |u|^{r-2}u)(u_n-u)dx\Big)\,dx \to 0 \quad \text{as } n\to\infty.$$

It follows that $\{u_n\}$ converges strongly in E. □

Corollary 3.2. Under assumptions (H_1) and (H_3), for every finite dimensional subspace $\tilde{E} \subset E$, there is a constant $K = K(\tilde{E}) > 0$ such that

$$J(u) \le 0 \quad \text{for all} \quad u \in \tilde{E} \quad \text{with} \quad \|u\|_E \ge K.$$

Proof . By (H_1) and (H_3), there exist $K_2, K_3 > 0$ such that

$$F(x,u) \ge K_2|u|^\mu - K_3|u|^2 \qquad (x,u) \in \mathbb{R}^3 \times \mathbb{R}.$$

For all $u \in \tilde{E}$, we have

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^3} \Big(|\nabla^\alpha u|^2 + V(x)u^2\Big)dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^\beta u^2 dx - \int_{\mathbb{R}^3} F(x,u)\,dx - \frac{t}{r}\int_{\mathbb{R}^3} g(x)|u|^r dx$$
$$\le \frac{1}{2}\|u\|_E^2 + K\|u\|_E^4 - K_2\|u\|_{L^\mu}^\mu + K_3\|u\|_{L^2}^2 - \frac{t}{r}\|u\|_{L^r}^r.$$

Then, we assert that $J(u) \to -\infty$ as $\|u\|_E \to \infty$ for $r < 2 < 4 < \mu$. The conclusion follows. □

Lemma 3.3. For $2 \le p < 2_\alpha^*$, we have that

$$\Gamma_k := \sup_{u\in Z_k, \|u\|=1} \|u\|_{L^p} \to 0 \quad as \quad k \to \infty.$$

By Lemma 3.3, we can choose an integer $m \ge 1$ such that

$$\|u\|_{L^2}^2 \le \tfrac{1}{2K_1}\|u\|_E^2 \quad , \quad \|u\|_{L^p}^p \le \tfrac{p}{4K_1}\|u\|_E^p \quad \forall u \in Z_m.$$

Lemma 3.4. Suppose (H_1) and (V) are satisfied, there exist constants $\rho, \delta > 0$ satisfying $J\big|_{\partial B_\rho \bigcap Z_m} \ge \delta > 0$.

Proof . By (H_1), we have

$$|F(x,u)| \le \frac{K_1}{2}|u|^2 + \frac{K_1}{p}|u|^p \quad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}.$$

From (H_1) and Lemma 3.3, we derive

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^3} \Big(|\nabla^\alpha u|^2 + V(x)u^2\Big)dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^\beta u^2 dx - \int_{\mathbb{R}^3} F(x,u)\,dx - \frac{t}{r}\int_{\mathbb{R}^3} g(x)|u|^r dx$$
$$\ge \frac{1}{2}\|u\|_E^2 - \frac{K_1}{2}\|u\|_{L^2}^2 - \frac{K_1}{p}\|u\|_{L^p}^p - t\|g\|_{L^{\frac{2}{2-r}}} C^r \|u\|_E^r$$
$$\ge \frac{1}{4}(\|u\|_E^2 - \|u\|_E^p) - t\|g\|_{L^{\frac{2}{2-r}}} C^r \|u\|_E^r$$
$$\ge \|u\|_E^2 \Big(\frac{1}{4} - \frac{1}{4}\|u\|_E^{p-2} - t\|g\|_{L^{\frac{2}{2-r}}} C^r \|u\|_E^{r-2}\Big).$$

Set $\psi(s) = \frac{1}{4} - \frac{1}{4}s^{p-2} - t\|g\|_{L^{\frac{2}{2-r}}} C^r s^{r-2}, \quad s > 0$. Since $1 < r < 2 < p$, there exists $\rho_t > 0$ such that

$$\rho_t = \left(\frac{4t(2-r)C^r \|g\|_{L^{\frac{2}{2-r}}}}{(p-2)} \right)^{\frac{1}{p-r}},$$

where $\max\limits_{s\in\mathbb{R}^+} \psi(s) = \psi(\rho_t)$. Therefore for any $t < t_0 := \left(\frac{2-r}{p-r}\right)^{\frac{p-r}{p-2}} \cdot \left(\frac{p-2}{4C^r(2-r)\|g\|_{L^{\frac{2}{2-r}}}} \right)$,

$$J(u) \geq \|\rho_t\|_E^2 \, \psi(\rho_t) > 0.$$

Hence, for any $t < t_0$ we can choose $\rho = \rho_t > 0$ and $\delta = \psi(\rho_t) > 0$, then $J\big|_{\partial B_\rho \bigcap Z_m} \geq \delta > 0$. $\square$

Proof of Theorem 1.1 Let $Y = Y_m$ and $Z = Z_m$. Obviously, $J(u)$ is even due to (H_2). Based on Lemma 3.1, Lemma 3.3 and Corollary 3.2, the functional $J(u)$ satisfies all conditions of Theorem 2.7. Thus, the result follows.

4 Conclusion

In this paper, we study a new class of fractional Schrödinger-Poisson system. System (1.5) comes from the interaction of a charged particle with an electromagnetic field in $\mathbb{R}^3$. By applying the symmetric mountain pass theorem, multiple non-trivial solutions were obtained. From our perspective, this paper seems to enrich the related results of this new class of system involving this kind of fractional operator.

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