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A trigonometric functional equation with an automorphism

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Abstract

Let S be a semigroup. In the present paper, we determine the complex-valued solutions (f, g) of the functional equation

$$g(x\sigma(y)) = g(x)g(y) - f(x)f(y) + \alpha f(x\sigma(y)),\ \ x, y \in S,$$

where $\sigma : S \to S$ is an automorphism that need not be involutive, and $\alpha \in \mathbb{C}$ is a fixed constant. Our results generalize and extend the ones by Stetkær in The cosine addition law with an additional term. Aequat Math., no. 6, 90, 1147-1168 (2016), and also the ones by Aserrar and Elqorachi in A generalization of the cosine addition law on semigroups. Aequat Math. 97, 787–804 (2023). Some consequences of our results are presented.

Keywords: Functional equation, Semigroup, Addition law, Automorphism
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1 Set up, Notation and terminology

Throughout this paper S denotes a semigroup, i.e. a set equipped with an associative binary operation. $\mathbb{C}^*$ denotes the multiplicative group of non-zero complex numbers.

Definition 1.1. Let $f : S \to \mathbb{C}$.
f is multiplicative, if $f(xy) = f(x)f(y)$ for all $x, y \in S$.
f is additive, if $f(xy) = f(x) + f(y)$ for all $x, y \in S$.
f is central, if $f(xy) = f(yx)$ for all $x, y \in S$.
f is abelian, if f is central and $f(xyz) = f(xzy)$ for all $x, y, z \in S$.

We define the set $S^2 := \{xy \mid x, y \in S\}$. The map $\sigma : S \to S$ denotes an automorphism of S, i.e. $\sigma(xy) = \sigma(x)\sigma(y)$ for all $x, y \in S$, and $\sigma^{-1} : S \to S$ its inverse, i.e. $\sigma \circ \sigma^{-1} = \sigma^{-1} \circ \sigma = id$. For any function $f : S \to \mathbb{C}$ we define the function $f^* := f \circ \sigma$.

If $\chi : S \to \mathbb{C}$ is a multiplicative function, we always denote by ϕ a solution of the special sine addition law

$$\phi(xy) = \phi(x)\chi(y) + \phi(y)\chi(x), \quad x, y \in S. \tag{1.1}$$

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2 Introduction

In 2003, Butler [5] posed the problem: Show that for $d < -1$ there are exactly two real-valued functions g such that

$$g(x+y) - g(x)g(y) = d\sin x \sin y \quad \text{for all } x, y \in \mathbb{R}. \tag{2.1}$$

One year later, Rassias [8] solved this problem by showing that the two functions are

$$g(x) = \cos x \pm \sqrt{-d-1}\sin x.$$

Takahasi, Miura and Takagi [12] treated complex-valued solutions of Eq. (2.1). In [7], Jung, Rassias and Mortici studied the more general functional equation

$$g(x+y) + \lambda g(x)g(y) = \psi(x)\psi(y) + \beta\psi(x+y), \quad x, y \in \mathbb{R}, \tag{2.2}$$

where $g, \psi : \mathbb{R} \to \mathbb{C}$ are the unknown functions, $\lambda, \beta \in \mathbb{C}^*$ are constants such that $\lambda\beta^2 \neq 1$ and $\psi \neq 0$. Their results were extended to the case of semigroups by Stetkær [10] who solved the functional equation

$$g(xy) = g(x)g(y) - f(x)f(y) + \alpha f(xy), \quad x, y \in S, \tag{2.3}$$

where $g, f : S \to \mathbb{C}$, and $\alpha \in \mathbb{C}$ is a fixed constant. He expressed the solutions in terms of multiplicative functions on S, solutions $\phi : S \to \mathbb{C}$ of the special sine addition law (1.1), and sometimes arbitrary functions (See [10, Theorem 3.1]). The functional equation (2.3) contains the cosine addition law

$$g(xy) = g(x)g(y) - f(x)f(y), \quad x, y \in S, \tag{2.4}$$

which have been treated by many authors. Aczél's classic monograph [1] discusses the real valued, continuous solutions of Eq. (2.4) and contains references to earlier works. Stetkær [9] solved Eq. (2.4) on general groups and monoids.

Recently, Aserrar and Elqorachi [4, Theorem 3.4] extended Stetkær's result about solutions of Eq. (2.3) by solving the functional equation

$$g(x\sigma(y)) = g(x)g(y) - f(x)f(y) + \alpha f(x\sigma(y)), \quad x, y \in S, \tag{2.5}$$

on a semigroup S, where $\sigma : S \to S$ is an involutive automorphism, i.e. $\sigma(xy) = \sigma(x)\sigma(y)$ for all $x, y \in S$, and $\sigma \circ \sigma = id$, where id denotes the identity map.

The purpose of the present paper is to solve the functional equation (2.5) on semigroups without using the condition $\sigma \circ \sigma = id$. We impose no further conditions on the automorphism σ. The condition $\sigma \circ \sigma = id$ played an important role in the proofs given in [4]. The present paper shows that this condition is not crucial even in the setting of semigroups, so our results are natural extensions of the previous results. Results from Stetkær [9, Chapter 3], Stetkær [10] and from [3, 4] are indispensable for the present paper. As a consequence of our main result, we solve the functional equation

$$f(x\sigma(y)) = f(x)g(y) + f(y)g(x) - g(x)g(y), \quad x, y \in S, \tag{2.6}$$

which was solved by Ebanks [6, Lemma 3.2] on monoids generated by their squares, and on semigroups by Stetkær [11] with $\sigma = id$. Aserrar and Elqorachi [2] obtained its solutions on semigroups where σ is an involutive automorphism, so the new result here is that $\sigma \circ \sigma \neq id$. We show that Eq. (2.6) is related to Eq. (2.5) with $\alpha = 0$.

All solutions found in this paper are expressed in terms of multiplicative functions, solutions ϕ of Eq. (1.1), and arbitrary functions in some cases.

3 Main result

This section consists of three subsections. Subsection 3.1 contains some preliminaries. In subsection 3.2, we solve the functional equation (2.5), and some consequences are given in subsection 3.3. Theorem 3.7 is the main result of the paper.

3.1 Preliminaries

Lemma 3.1. Let $\alpha \in \mathbb{C}^*\backslash\{-1,1\}$. The solution of the system of equations

$$(E_\alpha):\begin{cases}\dfrac{\alpha b_1}{c_1+\alpha c_2} = c_2 - b_2\\ \dfrac{b_1}{c_1+\alpha c_2} = b_1+1-c_1\\ \dfrac{\alpha b_2}{c_1+\alpha c_2} = \alpha b_2 + \dfrac{\alpha c_2}{c_1+\alpha c_2}\\ \dfrac{b_2}{c_1+\alpha c_2} = \dfrac{c_2}{c_1+\alpha c_2} - \alpha b_1\end{cases},$$

where $c_2 \in \mathbb{C}^*$ and $c_1+\alpha c_2 \neq 0$ is the quadruple

$$(c_1,c_2,b_1,b_2) = \left(\frac{\alpha^2+1}{\alpha^2-1}, \frac{2\alpha}{1-\alpha^2}, \frac{1}{\alpha^2-1}, \frac{\alpha}{1-\alpha^2}\right).$$

Proof . If we compare the first two equations of (E_α), and then the last two we can see that

$$c_2 - b_2 = \alpha b_1 + \alpha - \alpha c_1$$

$$b_2 = -\alpha b_1. \tag{3.1}$$

This implies that

$$c_2 + \alpha c_1 = \alpha. \tag{3.2}$$

Now, by adding the first equation of (E_α) to the last one, and using Eq. (3.1) we obtain

$$c_2 + \frac{c_2}{c_1+\alpha c_2} = 0.$$

Then, since $c_2 \neq 0$, we get $c_1 + \alpha c_2 = -1$. Thus, Eq. (3.2) becomes

$$c_2 + \alpha\,(-1-\alpha c_2) = \alpha.$$

Therefore, $c_2 = \dfrac{2\alpha}{1-\alpha^2}$, since $\alpha \neq 0$. Then $c_1 = -1-\alpha c_2 = \dfrac{\alpha^2+1}{\alpha^2-1}$. So, since $c_1+\alpha c_2 = -1$, we get from the second equation of (E_α) that

$$b_1 = \frac{c_1-1}{2} = \frac{1}{\alpha^2-1},$$

and then

$$b_2 = -\alpha b_1 = \frac{\alpha}{1-\alpha^2}.$$

Conversely, we can check easily that the quadruple given in Lemma 3.1 is a solution of (E_α). This completes the proof of Lemma 3.1. □

Remark 3.2. If (c_1,c_2,b_1,b_2) is a solution of (E_α) such that $\alpha, c_2 \in \mathbb{C}^*$ and $c_1+\alpha c_2 \neq 0$, then $\alpha \neq \pm 1$.

Lemma 3.3. Let $F, g : S \to \mathbb{C}$ be a solution of the functional equation

$$F(xy) = F(x)g(y) - F(y)g(x) \quad x,y \in S. \tag{3.3}$$

Then, F and g are linearly dependent.

Proof . Case 1: $F = 0$ on S^2. In this case, we get from Eq. (3.3) that

$$F(x)g(y) = F(y)g(x), \text{ for all } x,y \in S.$$

Then, g and F are linearly dependent.

Case 2: $F \neq 0$ on S^2. From Eq. (3.3), we can see that

$$F(xy) = -F(yx), \text{ for all } x, y \in S.$$

Then, for all $x, y, z \in S$ we have

$$F(xyz) = -F(zxy) = F(yzx) = -F(xyz).$$

Thus, $F(xyz) = 0$ for all $x, y, z \in S$. So, by applying Eq. (3.3) to (x, yz), we obtain that

$$F(xy)g(z) = F(z)g(xy), \text{ for all } x, y, z \in S.$$

Hence, g and F are linearly dependent since $F \neq 0$ on S^2. This completes the proof of Lemma 3.3. □

The following lemma contains some key properties of the solutions of Eq. (2.5).

Lemma 3.4. Let $g, f : S \to \mathbb{C}$ be a solution of Eq. (2.5) such that f and g are linearly independent. The following statements hold:

(1) There exist two contants $c_1, c_2 \in \mathbb{C}$ such that

$$g = (1 + \alpha c_2)g^* - c_2 f^* \tag{3.4}$$

$$f = c_1 f^* + (\alpha - \alpha c_1)g^* \tag{3.5}$$

(2) If $\alpha = 0$, then $g = g^*$ and $f = f^*$ or $f = -f^*$.

(3) If $c_2 = 0$, then $g = g^*$ and $f = f^*$ or $f = -f^*$.

(4) If $c_2 \neq 0$, then $\alpha \in \mathbb{C}^* \backslash \{-1, 1\}$, and the pair $\left(\frac{1}{2\alpha}f + \frac{1}{2}g, \frac{1}{2\alpha}f - \frac{1}{2}g\right)$ is a solution of the functional equation (2.4) with the roles of g and f in (2.4) played by $\frac{f}{2\alpha} + \frac{g}{2}$ and $\frac{f}{2\alpha} - \frac{g}{2}$ respectively. In addition $f^* - \alpha g^* = \alpha g - f$ and $g^* - \alpha f^* = g - \alpha f$.

Proof . (1) Define $G := g - \alpha f$. The functional equation (2.5) can be written as

$$G(x\sigma(y)) = g(x)g(y) - f(x)f(y), \quad x, y \in S. \tag{3.6}$$

Using the associativity of the semigroup operation we get from Eq. (3.6) that for all $x, y, z \in S$

$$g(x)g(yz) - f(x)f(yz) = g(x\sigma(y))g(z) - f(x\sigma(y))f(z). \tag{3.7}$$

On the other hand, in view of Eq. (2.5) we have

$$g(x\sigma(y))g(z) = g(x)g(y)g(z) - f(x)f(y)g(z) + \alpha g(z)f(x\sigma(y)).$$

So, Eq. (3.7) becomes

$$g(x)\left[g(yz) - g(y)g(z)\right] + f(x)\left[f(y)g(z) - f(yz)\right] = \left(\alpha g(z) - f(z)\right)f(x\sigma(y)). \tag{3.8}$$

Since f and g are linearly independent, there exists $z_0 \in S$ such that $\alpha g(z_0) - f(z_0) \neq 0$. Thus, if we put $z = z_0$ in Eq. (3.8) we obtain

$$f(x\sigma(y)) = g(x)k(y) + f(x)h(y), \tag{3.9}$$

where

$$k(y) = \frac{g(yz_0) - g(y)g(z_0)}{\alpha g(z_0) - f(z_0)},$$

and

$$h(y) = \frac{f(y)g(z_0) - f(yz_0)}{\alpha g(z_0) - f(z_0)}.$$

By using the new functions h, k defined above, equation (3.8) can be written in the form

$$g(x)\left[g(yz)-g(y)g(z)\right]+f(x)\left[f(y)g(z)-f(yz)\right]=g(x)k(y)\left(\alpha g(z)-f(z)\right)+f(x)h(y)\left(\alpha g(z)-f(z)\right).$$

Therefore, since f and g are linearly independent, we deduce that for all $y, z \in S$

$$g(yz)=g(y)g(z)+\left(\alpha g(z)-f(z)\right)k(y), \tag{3.10}$$

and

$$f(yz)=f(y)g(z)+\left(f(z)-\alpha g(z)\right)h(y). \tag{3.11}$$

Replacing z by $\sigma(z)$ in Eq. (3.10) and Eq. (3.11), we find after some rearrangement that

$$g(y\sigma(z))-\alpha f(y\sigma(z))=g^*(z)\left[g(y)+\alpha\left(k(y)+\alpha h(y)-f(y)\right)\right]-f^*(z)\left[k(y)+\alpha h(y)\right]. \tag{3.12}$$

On the other hand

$$k(y)+\alpha h(y)=\frac{g(yz_0)-\alpha f(yz_0)+\alpha f(y)g(z_0)-g(y)g(z_0)}{\alpha g(z_0)-f(z_0)}.$$

Since σ is a bijection, we get in view of Eq. (2.5) that

$$\begin{aligned} g(yz_0)-\alpha f(yz_0)&=g(y\sigma(\sigma^{-1}(z_0))-\alpha f(y\sigma(\sigma^{-1}(z_0))\\ &=g(y)g(\sigma^{-1}(z_0))-f(y)f(\sigma^{-1}(z_0)). \end{aligned}$$

Thus

$$k(y)+\alpha h(y)=\frac{(g(\sigma^{-1}(z_0))-g(z_0))g(y)+(\alpha g(z_0)-f(\sigma^{-1}(z_0)))f(y)}{\alpha g(z_0)-f(z_0)}.$$

So

$$k+\alpha h=c_1f(y)+c_2g(y), \tag{3.13}$$

where $c_1=\dfrac{\alpha g(z_0)-f(\sigma^{-1}(z_0))}{\alpha g(z_0)-f(z_0)}$ and $c_2=\dfrac{g(\sigma^{-1}(z_0))-g(z_0)}{\alpha g(z_0)-f(z_0)}$ are constants. Then, Eq. (3.12) becomes

$$g(y\sigma(z))-\alpha f(y\sigma(z))=g^*(z)\left[g(y)+\alpha\left((c_1-1)f(y)+c_2g(y)\right)\right]-f^*(z)\left[c_1f(y)+c_2g(y)\right].$$

That is

$$g(y\sigma(z))-\alpha f(y\sigma(z))=g(y)\left[(1+\alpha c_2)g^*(z)-c_2f^*(z)\right]-f(y)\left[(\alpha-\alpha c_1)g^*(z)+c_1f^*(z)\right]. \tag{3.14}$$

Now, by comparing Eq. (3.14) with Eq. (2.5) and using the linear independence of f and g, we get Eq. (3.4) and Eq. (3.5).

(2) is [3, Lemma 3.1], where f is replaced by if.

(3) Suppose $c_2=0$. It follows from Eq. (3.4) that $g=g^*$. If $c_1=0$ then Eq. (3.5) becomes $f=\alpha g^*=\alpha g$, which contradicts the linear independence of f and g. So, $c_1\neq 0$. Thus, we get from Eq. (3.5) that

$$f^*-\alpha g^*=\frac{f-\alpha g}{c_1}. \tag{3.15}$$

Now, replacing (y,z) by $(x,\sigma(y))$ in Eq. (3.11) and using Eq. (3.15), we obtain

$$f(x\sigma(y))=f(x)g(y)+\frac{1}{c_1}f(y)h(x)-\frac{\alpha}{c_1}g(y)h(x). \tag{3.16}$$

On the other hand, since $c_2=0$ we get $k+\alpha h=c_1f$. That is $k=c_1f-\alpha h$. So, Eq. (3.9) becomes

$$f(x\sigma(y))=c_1g(x)f(y)-\alpha g(x)h(y)+f(x)h(y). \tag{3.17}$$

Therefore, if we compare Eq. (3.16) with Eq. (3.17) we can see that

$$f(x\sigma(y)) = c_1 f(y\sigma(x)) \quad \text{for all } x, y \in S.$$

Then

$$f(x\sigma(y)) = c_1 f(y\sigma(x)) = c_1 \left(c_1 f(x\sigma(y))\right) = c_1^2 f(x\sigma(y)).$$

If $f = 0$ on S^2, we get from Eq. (3.17), since $c_1 \neq 0$, f and g are linearly independent that $h = 0$ and $f = 0$, but this is not possible. So $f \neq 0$ on S^2. Thus, $c_1 = \pm 1$.

First case: $c_1 = 1$. In this case, Eq. (3.5) becomes $f = f^*$.

Second case: $c_1 = -1$. Equation (3.5) implies that $f^* = 2\alpha g - f$. Replacing (x, y) by $(\sigma(x), y)$ in Eq. (2.5), we find that

$$g(xy) = g(x)g(y) - \left(2\alpha g(x) - f(x)\right) f(y) + 2\alpha^2 g(xy) - \alpha f(xy), \tag{3.18}$$

for all $x, y, z \in S$. Now, by applying Eq.(3.18) to $(\sigma(x), \sigma(y))$, we obtain

$$g(xy) = g(x)g(y) - f(x)\left(2\alpha g(y) - f(y)\right) + \alpha f(xy). \tag{3.19}$$

Therefore, comparing Eq. (3.18) with Eq. (3.19), we deduce that

$$2\alpha \left(f(xy) - \alpha g(xy)\right) = 2\alpha \left(f(x)g(y) - f(y)g(x)\right). \tag{3.20}$$

If $\alpha \neq 0$, Eq. (3.20) implies that

$$f(xy) - \alpha g(xy) = f(x)g(y) - f(y)g(x). \tag{3.21}$$

Defining $F := \dfrac{1}{\alpha} f - g$, Eq. (3.21) can be written as

$$F(xy) = F(x)g(y) - F(y)g(x).$$

Then, according to Lemma 3.3, F and g are linearly dependent, which implies that f and g are linearly dependent. This is a contradiction, since f an g are linearly independent. Thus $\alpha = 0$, and so $f^* = -f$. This is case (3).

(4) Suppose $c_2 \neq 0$. If $\alpha = 0$, then by Lemma 3.4 (1), $g = g^*$. So, from Eq. (3.4) we get that $c_2 f^* = 0$. That is, $c_2 = 0$, since $f^* \neq 0$ (because $f \neq 0$, since f and g are linearly independent), but this is a contradiction. Therefore, $\alpha \neq 0$.

Multiplying Eq. (3.4) by c_1 and Eq. (3.5) by c_2, we get respectively

$$c_1 g = c_1(1 + \alpha c_2)g^* - c_1 c_2 f^*, \tag{3.22}$$

$$c_2 f = c_1 c_2 f^* + c_2(\alpha - \alpha c_1)g^*. \tag{3.23}$$

Then, by adding Eq. (3.22) to (3.23), we obtain

$$c_2 f + c_1 g = (c_1 + \alpha c_2)g^*. \tag{3.24}$$

So, $c_1 + \alpha c_2 \neq 0$, since f and g are linearly independent and $c_2 \neq 0$. It follows from Eq.(3.24) that

$$g^* = \frac{c_2}{c_1 + \alpha c_2} f + \frac{c_1}{c_1 + \alpha c_2} g, \tag{3.25}$$

and from Eq. (3.4) we get, since $c_2 \neq 0$ that

$$f^* = \frac{1 + \alpha c_2}{c_2} g^* - \frac{1}{c_2} g.$$

That is, in view of Eq. (3.25)

$$f^* = \frac{1 + \alpha c_2}{c_1 + \alpha c_2} f + \frac{\alpha c_1 - \alpha}{c_1 + \alpha c_2} g. \tag{3.26}$$

Now, replacing (y, z) by $(x, \sigma(y))$ in Eq. (3.10), and using Eq. (3.25) and Eq. (3.26), we find that

$$g(x\sigma(y)) = \frac{g(x)}{c_1 + \alpha c_2} \left[c_2 f(y) + c_1 g(y)\right] + \frac{k(x)}{c_1 + \alpha c_2} \left[\alpha g(y) - f(y)\right]. \tag{3.27}$$

On the other hand, we see from Eq. (3.13) that $\alpha h = c_1 f + c_2 g - k$, multiplying Eq. (3.9) by α, we get that

$$\alpha f(x\sigma(y)) = \alpha g(x)k(y) + f(x)\left[c_1 f(y) + c_2 g(y) - k(y)\right].$$

So, Eq. (2.5) becomes

$$g(x\sigma(y)) = g(x)\left(g(y) + \alpha k(y)\right) + f(x)\left((c_1 - 1)f(y) + c_2 g(y) - k(y)\right). \tag{3.28}$$

Thus, comparing Eq. (3.27) with Eq. (3.28), we can see that

$$\frac{k(x)}{c_1 + \alpha c_2}\left[\alpha g(y) - f(y)\right] = f(x)\left[(c_1 - 1)f(y) + c_2 g(y) - k(y)\right] + g(x)\left[\frac{\alpha c_2}{c_1 + \alpha c_2} g(y) - \frac{c_2}{c_1 + \alpha c_2} f(y) + \alpha k(y)\right]. \tag{3.29}$$

Therefore, if we put $y = z_0$ such that $\alpha g(z_0) - f(z_0) \neq 0$, we get from Eq. (3.29) that

$$k = b_1 f + b_2 g,$$

for some constants $b_1, b_2 \in \mathbb{C}$. Inserting this into Eq. (3.29) and using the linear independence of f and g we find that (c_1, c_2, b_1, b_2) is a solution of the system (E_α) from Lemma 3.1. Thus, $\alpha \neq \pm 1$ by Remark 3.2. So, according to Lemma 3.1

$$(c_1, c_2, b_1, b_2) = \left(\frac{\alpha^2 + 1}{\alpha^2 - 1}, \frac{2\alpha}{1 - \alpha^2}, \frac{1}{\alpha^2 - 1}, \frac{\alpha}{1 - \alpha^2}\right).$$

Then

$$k = \frac{1}{\alpha^2 - 1} f + \frac{\alpha}{1 - \alpha^2} g,$$

and $h = \dfrac{c_1 f + c_2 g - k}{\alpha}$, i.e.

$$h = \frac{\alpha}{\alpha^2 - 1} f + \frac{1}{1 - \alpha^2} g.$$

Thus, inserting these forms into Eq. (3.10) and Eq. (3.11) we find respectively that

$$g(yz) = \frac{g(y)}{1 - \alpha^2}\left(g(z) - \alpha f(z)\right) + \frac{f(y)}{1 - \alpha^2}\left(f(z) - \alpha g(z)\right), \tag{3.30}$$

$$f(yz) = \frac{g(y)}{1 - \alpha^2}\left(f(z) - \alpha g(z)\right) + \frac{f(y)}{1 - \alpha^2}\left(g(z) - \alpha f(z)\right). \tag{3.31}$$

Multiplying Eq. (3.30) by α and adding the identity obtained to Eq. (3.31) we get that

$$f(yz) + \alpha g(yz) = f(y)g(z) + f(z)g(y).$$

Dividing this equation by 2α, we get

$$\frac{1}{2\alpha} f(yz) + \frac{1}{2} g(yz) = \frac{1}{2\alpha} f(y)g(z) + \frac{1}{2\alpha} f(z)g(y). \tag{3.32}$$

On the other hand, using the identity $(a + b)(c + d) - (a - b)(c - d) = 2ad + 2bc$ for $a, b, c, d \in \mathbb{C}$, we can see that

$$\left(\frac{1}{2\alpha} f(y) + \frac{1}{2} g(y)\right)\left(\frac{1}{2\alpha} f(z) + \frac{1}{2} g(z)\right) - \left(\frac{1}{2\alpha} f(y) - \frac{1}{2} g(y)\right)\left(\frac{1}{2\alpha} f(z) - \frac{1}{2} g(z)\right) = \frac{1}{2\alpha} f(y)g(z) + \frac{1}{2\alpha} f(z)g(y). \tag{3.33}$$

Thus, using Eq. (3.33), Eq. (3.32) becomes

$$\frac{1}{2\alpha}f(yz)+\frac{1}{2}g(yz)=\left(\frac{1}{2\alpha}f(y)+\frac{1}{2}g(y)\right)\left(\frac{1}{2\alpha}f(z)+\frac{1}{2}g(z)\right)-\left(\frac{1}{2\alpha}f(y)-\frac{1}{2}g(y)\right)\left(\frac{1}{2\alpha}f(z)-\frac{1}{2}g(z)\right).$$

This verifies the statement about (2.4) in Lemma 3.4. In addition, Eq.(3.25) and Eq. (3.26) yields

$$g^*=\frac{2\alpha}{\alpha^2-1}f+\frac{\alpha^2+1}{1-\alpha^2}g,$$

and

$$f^*=\frac{\alpha^2+1}{\alpha^2-1}f+\frac{2\alpha}{1-\alpha^2}g.$$

Therefore, $f^*-\alpha g^*=\alpha g-f$ and $g^*-\alpha f^*=g-\alpha f$. This occurs in case (4). This completes the proof of Lemma 3.4. □

Remark 3.5. In Lemma 3.4 (4), we can see from the equations $f^*-\alpha g^*=\alpha g-f$ and $g^*-\alpha f^*=g-\alpha f$ that $f\neq f^*$ and $g\neq g^*$, since f and g are linearly independent.

In the final preliminary, we state some needed properties of solutions of Eq. (1.1), and of multiplicative functions.

Lemma 3.6. (1) Let $\chi,\chi':S\to\mathbb{C}$ be two different non-zero multiplicative functions, and $\phi:S\to\mathbb{C}$ a non-zero solution of Eq. (1.1). Then $\{\chi,\chi',\phi\}$ is linearly independent.

(2) Let $\chi_1,\chi_2:S\to\mathbb{C}$ be two different non-zero multiplicative functions, and $f,g:S\to\mathbb{C}$ two functions such that

$$f=a_1\chi_1+a_2\chi_2, \tag{3.34}$$

$$g=b_1\chi_1+b_2\chi_2, \tag{3.35}$$

for some constants $a_1,a_2,b_1,b_2\in\mathbb{C}$, where $a_1-\alpha b_1=\alpha b_2-a_2\neq 0$ and $f^*-\alpha g^*=\alpha g-f$. Then $\chi_1^*=\chi_2$ and $\chi_2^*=\chi_1$.

Proof . (1) Suppose $a\phi+b\chi+c\chi'=0$ for some $a,b,c\in\mathbb{C}$. Then

$$\begin{aligned}0&=(a\phi+b\chi+c\chi')(xy)=a\phi(x)\chi(y)+a\phi(y)\chi(x)+b\chi(x)\chi(y)+c\chi'(x)\chi'(y)\\&=(a\phi+b\chi)(x)\chi(y)+a\phi(y)\chi(x)+c\chi'(x)\chi'(y)\\&=c\chi'(x)(\chi'(y)-\chi(y))+a\phi(y)\chi(x),\quad x,y\in S.\end{aligned}$$

Thus, by [9, Theorem 3.18 (b)] we see that $a=c=0$, since $\phi\neq 0$, and it follows that $b=0$.

(2) Since $f^*-\alpha g^*=\alpha g-f$, we get in view of the forms (3.34) and (3.35) that

$$(a_1-\alpha b_1)\chi_1^*+(a_2-\alpha b_2)\chi_2^*=(\alpha b_1-a_1)\chi_1+(\alpha b_2-a_2)\chi_2.$$

This implies that, since $a_1-\alpha b_1=\alpha b_2-a_2\neq 0$

$$\chi_1^*-\chi_2^*=\chi_2-\chi_1.$$

That is $\chi_1+\chi_1^*=\chi_2+\chi_2^*$. Now, the result follows easily from [9, Corollary 3.19]. This completes the proof of Lemma 3.6. □

3.2 Solutions of Eq. (2.5)

The next theorem gives the general solution of Eq. (2.5) on semigroups. It generalizes [10, Theorem 3.1] and [4, Theorem 3.4], and the news in the formulation of Theorem 3.7 compared with [10, Theorem 3.1] and [4, Theorem 3.4] is the condidtion $(\chi^*)^*=\chi$ in (8). We recall that $\sigma:S\to S$ denotes an automorphism of the semigroup S, that $F^*=F\circ\sigma$ for any function F on S and that $\alpha\in\mathbb{C}$ is a constant. Furthermore that $\phi:S\to\mathbb{C}$ denotes a solution of the functional equation (1.1).

Theorem 3.7. Any solution $g, f : S \to \mathbb{C}$ of the functional equation (2.5) falls into at least one of the eight categories listed below.

(1) $\alpha = \pm 1$, f is any non-zero function and $g = \alpha f$.
(2) $\alpha \neq 1$, $f = g \neq 0$ and $g = 0$ on S^2.
(3) $\alpha \neq -1$, $f = -g \neq 0$ and $g = 0$ on S^2.
(4) $f = (q+\alpha)\dfrac{\chi}{2}$ and $g = \left(1 \pm \sqrt{1+q^2-\alpha^2}\right)\dfrac{\chi}{2}$, where $q \in \mathbb{C}$ is a constant and $\chi : S \to \mathbb{C}$ a non-zero multiplicative function such that $\chi^* = \chi$.
(5) $f = \alpha\dfrac{\chi_1+\chi_2}{2} + q\dfrac{\chi_1-\chi_2}{2}$ and $g = \dfrac{\chi_1+\chi_2}{2} \pm \sqrt{1+q^2-\alpha^2}\dfrac{\chi_1-\chi_2}{2}$, where $\chi_1, \chi_2 : S \to \mathbb{C}$ are two different non-zero multiplicative functions such that $\chi_1^* = \chi_1$, $\chi_2^* = \chi_2$, and $q \in \mathbb{C}\backslash\{\pm\alpha\}$ is a constant.
(6) $\alpha \neq 0$, $f = \alpha\chi_1$ and $g = \chi_2$, where $\chi_1, \chi_2 : S \to \mathbb{C}$ are two different non-zero multiplicative functions such that $\chi_1^* = \chi_1$ and $\chi_2^* = \chi_2$.
(7) $f = \alpha\chi + \phi$ and $g = \chi \pm \phi$, where $\chi : S \to \mathbb{C}$ is a non-zero multiplicative function such that $\phi \neq 0$, $\chi^* = \chi$ and $\phi^* = \phi$.
(8) $\alpha \neq \pm 1$, and
$$f = \frac{1+\alpha}{2}\chi - \frac{1-\alpha}{2}\chi^* \quad \text{and} \quad g = \frac{1+\alpha}{2}\chi + \frac{1-\alpha}{2}\chi^*,$$
where $\chi : S \to \mathbb{C}$ is a multiplicative function such that $\chi \neq \chi^*$ and $(\chi^*)^* = \chi$.
Conversely, each pair (g, f) on the list is a solution of (2.5). Note that, off the exceptional case (1), g and f are abelian.

Proof . Let $g, f : S \to \mathbb{C}$ be a solution of the functional equation (2.5). If $g = 0$, then the functional equation (2.5) yields $\alpha f(x\sigma(y)) - f(x)f(y) = 0$ for all $x, y \in S$. If $\alpha = 0$ then $f = 0$. This is case (4) with $q = -\alpha$ and $\chi = 1$. If $\alpha \neq 0$, then $f(x\sigma(y)) = \dfrac{1}{\alpha}f(x)f(y)$ for all $x, y \in S$. Therefore, according to [3, Lemma 2.4] there exists a multiplicative function $\chi : S \to \mathbb{C}$ such that $f = \alpha\chi$, $\chi^* = \chi$, so we are in family (4) with $q = \alpha$ if $f \neq 0$ and in family (4) with $q = -\alpha$ and $\chi = 1$ if $f = 0$. Henceforth we suppose $g \neq 0$ and we split the discussion into two cases according to whether f and g are linearly dependent or not.

Case 1: f and g are linearly dependent. That is $f = \beta g$ for some constant $\beta \in \mathbb{C}$. The functional equation (2.5) can be written as $(1-\alpha\beta)g(x\sigma(y)) = (1-\beta^2)g(x)g(y)$ for all $x, y \in S$. Thus, by the help of [10, Lemma 4.3] and [3, Lemma 2.4] we get the solution families (1), (2), (3) and (4).

Case 2: f and g are linearly independent. According to Lemma 3.4 (1) there exists two constants $c_1, c_2 \in \mathbb{C}$ such that
$$g = (1+\alpha c_2)g^* - c_2 f^* \quad \text{and} \quad f = c_1 f^* + (\alpha - \alpha c_1)g^*.$$

We discuss the two subcases $c_2 = 0$ and $c_2 \neq 0$ separately as follows.

Case 2.1: $c_2 = 0$. According to Lemma 3.4 (3) we have $g = g^*$ and $f = f^*$ or $f = -f^*$.

Case 2.1.1: $f = f^*$. Replacing (x, y) by $(\sigma(x), y)$ in the functional equation (2.5), we get that
$$g(xy) = g(x)g(y) - f(x)f(y) + \alpha f(xy), \quad x, y \in S.$$

We proceed as in the proof of [4, Theorem 3.4 subcase A] and get the solution families (5), (6) and (7).

Case 2.1.2: $f = -f^*$. In this case, we get if we replace (x, y) by $(\sigma(x), y)$ in the functional equation (2.5) that
$$g(xy) = g(x)g(y) + f(x)f(y) - \alpha f(xy), \quad x, y \in S,$$
which means that the pair (g, if) is a solution of the functional equation (2.3) such that α is replaced by $i\alpha$. Therefore, by proceeding similarly to the proof of [4, Theorem 3.4 subcase B.2], we show that this case leads to the solution family (8) with $\alpha = 0$.

Case 2.2: $c_2 \neq 0$. According to Lemma 3.4 (4), $\alpha \in \mathbb{C}^*\backslash\{-1, 1\}$ and the pair
$$\left(\frac{1}{2\alpha}f + \frac{1}{2}g, \frac{1}{2\alpha}f - \frac{1}{2}g\right),$$

is a solution of the functional equation (2.4), $f^* - \alpha g^* = \alpha g - f$ and $g^* - \alpha f^* = g - \alpha f$. Taking into account that f and g are linearly independent we get from [10, Theorem 3.1] that there are only two possibilities (i) and (ii). They correspond to (d) and (g) of [10, Theorem 3.1]:

(i) $\frac{1}{2\alpha}f - \frac{1}{2}g = q\frac{\chi_1 - \chi_2}{2}$ and $\frac{1}{2\alpha}f + \frac{1}{2}g = \frac{\chi_1 + \chi_2}{2} \pm \sqrt{1+q^2}\frac{\chi_1 - \chi_2}{2}$, where $\chi_1, \chi_2 : S \to \mathbb{C}$ are two different non-zero multiplicative functions, and $q \in \mathbb{C}^*$ is a constant. This implies that

$$g = \frac{\chi_1 + \chi_2}{2} + \left(\pm\sqrt{1+q^2} - q\right)\frac{\chi_1 - \chi_2}{2},$$

and

$$f = \alpha\frac{\chi_1 + \chi_2}{2} + \left(\alpha q \pm \alpha\sqrt{1+q^2}\right)\frac{\chi_1 - \chi_2}{2}.$$

We let $Q := \pm\sqrt{1+q^2}$, $a_1 := \frac{\alpha + \alpha q + \alpha Q}{2}$, $a_2 := \frac{\alpha - \alpha q - \alpha Q}{2}$, $b_1 := \frac{1 - q + Q}{2}$ and $b_2 := \frac{1 + q - Q}{2}$. So, $f = a_1\chi_1 + a_2\chi_2$ and $g = b_1\chi_1 + b_2\chi_2$. In addition $a_1 - \alpha b_1 = \alpha b_2 - a_2 = \alpha q \neq 0$, since $\alpha, q \neq 0$. Thus, by Lemma 3.6 (2), we get that $\chi_1^* = \chi_2$ and $\chi_2^* = \chi_1$. So, if we let $\chi := \chi_1$, i.e. $\chi^* = \chi_1^* = \chi_2$, we can see that $\chi \neq \chi^*$ and $(\chi^*)^* = \chi$. Then

$$f = a_1\chi + a_2\chi^* \quad \text{and} \quad g = b_1\chi + b_2\chi^*.$$

On the other hand, since $g^* - \alpha f^* = g - \alpha f$, we obtain that

$$[(b_1 - \alpha a_1) - (b_2 - \alpha a_2)]\,\chi^* + [(b_2 - \alpha a_2) - (b_1 - \alpha a_1)]\,\chi = 0.$$

Therefore, by the help of [9, Theorem 3.18 (b)], we get that $b_1 - \alpha a_1 = b_2 - \alpha a_2$, which implies that $Q - q = \alpha^2(Q+q)$. Multiplying this by $(Q-q)$, we get that $(Q-q)^2 = \alpha^2$. That is $Q - q = \pm\alpha$ and so $Q + q = \alpha^{-2}(Q-q) = \pm\alpha^{-1}$. Thus

$$\begin{cases} f = \frac{\alpha+1}{2}\chi + \frac{\alpha-1}{2}\chi^* \\ g = \frac{\alpha+1}{2}\chi + \frac{1-\alpha}{2}\chi^* \end{cases} \quad \text{or} \quad \begin{cases} f = \frac{\alpha-1}{2}\chi + \frac{\alpha+1}{2}\chi^* \\ g = \frac{1-\alpha}{2}\chi + \frac{1+\alpha}{2}\chi^* \end{cases}.$$

The pair $(f;g)$ on the left falls into case (8). So does the pair on the right, except that there χ is replaced by χ^*.

(ii) $\frac{1}{2\alpha}f - \frac{1}{2}g = \phi$ and $\frac{1}{2\alpha}f + \frac{1}{2}g = \chi \pm \phi$, where $\chi : S \to \mathbb{C}$ is a non-zero multiplicative function and $\phi \neq 0$. Then

$$\begin{cases} f = \alpha\chi + 2\alpha\phi \\ g = \chi \end{cases} \quad \text{or} \quad \begin{cases} f = \alpha\chi \\ g = \chi - 2\phi \end{cases}.$$

By Remark 3.5, we see that $\chi \neq \chi^*$. In addition, $f^* - \alpha g^* = \alpha g - f$ yields $\phi^* = -\phi$ since $\alpha \neq 0$. On the other hand, since $g^* - \alpha f^* = g - \alpha f$, we get

$$(1-\alpha^2)\chi^* - 2\alpha^2\phi^* = (1-\alpha^2)\chi - 2\alpha^2\phi$$

or

$$(1-\alpha^2)\chi^* - 2\phi^* = (1-\alpha^2)\chi - 2\phi.$$

That is, since $\phi^* = -\phi$

$$(1-\alpha^2)\chi^* + 4\alpha^2\phi - (1-\alpha^2)\chi = 0$$

or

$$(1-\alpha^2)\chi^* + 4\phi - (1-\alpha^2)\chi = 0.$$

Hence, by Lemma 3.6 (1), we obtain $\alpha = \pm 1$ and $\alpha = 0$ or $\alpha = \pm 1$ and $4 = 0$, which is impossible. So, this case does not occur.

Conversely, we check by elementary computations that the pairs (g, f) described in Theorem 3.7 are solutions of Eq. (2.5). This completes the proof. □

3.3 Some Consequences

An immediate consequence of our main result is the solution of the cosine addition law

$$g(x\sigma(y)) = g(x)g(y) - f(x)f(y), \quad x, y \in S, \tag{3.36}$$

which is the case $\alpha = 0$.

Corollary 3.8. The solutions $g, f : S \to \mathbb{C}$ of the functional equation (3.36) can be listed as follows.

(1) $f = g \neq 0$ and $g = 0$ on S^2.
(2) $f = -g \neq 0$ and $g = 0$ on S^2.
(3) $f = q\dfrac{\chi}{2}$ and $g = \left(1 \pm \sqrt{1+q^2}\right)\dfrac{\chi}{2}$, where $q \in \mathbb{C}$ is a constant and $\chi : S \to \mathbb{C}$ a non-zero multiplicative function such that $\chi^* = \chi$.
(4) $f = q\dfrac{\chi_1 - \chi_2}{2}$ and $g = \dfrac{\chi_1 + \chi_2}{2} \pm \sqrt{1+q^2}\dfrac{\chi_1 - \chi_2}{2}$, where $\chi_1, \chi_2 : S \to \mathbb{C}$ are two different non-zero multiplicative functions such that $\chi_1^* = \chi_1$, $\chi_2^* = \chi_2$, and $q \in \mathbb{C}^*$ is a constant.
(5) $f = \phi$ and $g = \chi \pm \phi$, where $\chi : S \to \mathbb{C}$ is a non-zero multiplicative function such that $\phi \neq 0$, $\chi^* = \chi$ and $\phi^* = \phi$.
(6) $f = \dfrac{\chi - \chi^*}{2}$ and $g = \dfrac{\chi + \chi^*}{2}$, where $\chi : S \to \mathbb{C}$ is a multiplicative function such that $\chi \neq \chi^*$ and $(\chi^*)^* = \chi$.
Note that, g and f are abelian in each case.

An other interesting consequence is the following corollary about solutions of Eq. (2.6), namely

$$f(x\sigma(y)) = f(x)g(y) + f(y)g(x) - g(x)g(y), \quad x, y \in S.$$

Corollary 3.9. The solutions $f, g : S \to \mathbb{C}$ of the functional equation (2.6) are the following pairs.

(1) $g = 0$, and f is any non-zero function such that $f = 0$ on S^2.
(2) $g = 2f$, and f is any non-zero function such that $f = 0$ on S^2.
(3) $g = \left(1 - q + \beta\sqrt{1+q^2}\right)\dfrac{\chi}{2}$ and $f = \left(1 + \beta\sqrt{1+q^2}\right)\dfrac{\chi}{2}$, where $q \in \mathbb{C}$ is a constant, $\beta \in \{-1, 1\}$ and $\chi : S \to \mathbb{C}$ a non-zero multiplicative function such that $\chi^* = \chi$.
(4) There exist a constant $q \in \mathbb{C}^*$, $\beta \in \{-1, 1\}$ and two different non-zero multiplicative functions $\chi_1, \chi_2 : S \to \mathbb{C}$ such that $\chi_1^* = \chi_1$, $\chi_2^* = \chi_2$ and

$$g = \frac{\chi_1 + \chi_2}{2} + \left(\beta\sqrt{1+q^2} - q\right)\frac{\chi_1 - \chi_2}{2}, \text{ and}$$

$$f = \frac{\chi_1 + \chi_2}{2} + \beta\sqrt{1+q^2}\frac{\chi_1 - \chi_2}{2}.$$

(5) $g = \chi$ and $f = \chi + \phi$, where $\chi : S \to \mathbb{C}$ is a non-zero multiplicative function such that $\phi \neq 0$, $\chi^* = \chi$ and $\phi^* = \phi$.
(6) $g = \chi - 2\phi$ and $f = \chi - \phi$, where $\chi : S \to \mathbb{C}$ is a non-zero multiplicative function such that $\phi \neq 0$, $\chi^* = \chi$ and $\phi^* = \phi$.
(7) $g = \chi^*$ and $f = \dfrac{\chi + \chi^*}{2}$, where $\chi : S \to \mathbb{C}$ is a multiplicative function such that $\chi \neq \chi^*$ and $(\chi^*)^* = \chi$.

Note that, g and f are abelian in each case.

Proof . We check by elementary computations that the pairs (f, g) described in Corollary 3.9 are solutions of Eq. (2.6). Now, let $f, g : S \to \mathbb{C}$ be a solution of Eq. (2.6), define $\delta := f - g$. We remark that for all $x, y \in S$

$$\begin{aligned} f(x)f(y) - \delta(x)\delta(y) &= f(x)f(y) - (f(x) - g(x))(f(y) - g(y)) \\ &= f(x)f(y) - f(x)f(y) + f(x)g(y) + g(x)f(y) - g(x)g(y) \\ &= f(x)g(y) + f(y)g(x) - g(x)g(y). \end{aligned}$$

So, the functional equation (2.6) can be written as

$$f(x\sigma(y)) = f(x)f(y) - \delta(x)\delta(y), \quad x, y \in S, \tag{3.37}$$

which means that the pair (f, δ) is a solution of Eq. (3.36). The rest of the proof follows easily from Corollary 3.8. □

4 An application

As it is mentioned in the introduction, Jung, Rassias and Mortici [7] studied the functional equation (2.2), namely

$$g(x+y)+\lambda g(x)g(y)=\psi(x)\psi(y)+\beta\psi(x+y), \quad x,y\in\mathbb{R},$$

where $g,\psi:\mathbb{R}\to\mathbb{C}$ are the unknown functions, $\lambda,\beta\in\mathbb{C}^*$ are constants such that $\lambda\beta^2\neq 1$ and $\psi\neq 0$. In this section, we apply our theory to solve a more general functional equation.

Let $S=(\mathbb{R},+)$, let $\gamma\in\mathbb{R}\backslash\{0\}$ be a fixed element and let $\sigma(x)=\gamma x$ for all $x\in\mathbb{R}$. The functional equation (2.5) can be written as follows

$$g(x+\gamma y)=g(x)g(y)-f(x)f(y)+\alpha f(x+\gamma y), \quad x,y\in\mathbb{R}. \tag{4.1}$$

We can see that if $\gamma=1$, Eq. (4.1) is equivalent to Eq. (2.2), so we are interested to determine the solutions of (4.1) when $\gamma\in\mathbb{R}\backslash\{0,-1,1\}$. For this we apply Theorem 3.7 to Eq. (4.1). Let $\chi:S\to\mathbb{C}$ be a non-zero multiplicative function such that $\chi^*=\chi$, i.e.

$$\chi(\gamma x)=\chi(x), \text{ for all } x\in\mathbb{R}.$$

Since S is a group, then χ is a character. So we get $\chi\left((\gamma-1)x\right)=1$ for all $x\in\mathbb{R}$. Since $\gamma\neq 1$, we obtain $\chi=1$. By the same way we show that the only non-zero multiplicative function χ satisfying $\chi(\gamma^2 x)=\chi(x)$ for all $x\in\mathbb{R}$ is $\chi=1$ because $\gamma\neq\pm 1$. So the special sine addition law (1.1) becomes

$$\phi(x+y)=\phi(x)+\phi(y), \quad x,y\in\mathbb{R}.$$

That is ϕ additive. In addition if $\phi(\gamma x)=\phi(x)$ for all $x\in\mathbb{R}$, then $\phi=0$ since $\gamma\neq 1$. The solutions $f,g:S\to\mathbb{C}$ of Eq. (4.1) are the following when $\gamma\in\mathbb{R}\backslash\{0,-1,1\}$.

(1) $\alpha=\pm 1$, f is any non-zero function and $g=\alpha f$.

(2) $f=\dfrac{q+\alpha}{2}$ and $g=\dfrac{1\pm\sqrt{1+q^2-\alpha^2}}{2}$, where $q\in\mathbb{C}$ is a constant.

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