Int. J. Nonlinear Anal. Appl. 15 (2024) 12, 259–269
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.28660.3958

On convergence theorem for the fuzzy sequential Henstock integrals

Victor Odalochi Iluebe, Adesanmi Alao Mogbademu*

Department of Mathematics, University of Lagos, Akoka, Lagos, Nigeria

(Communicated by Reza Saadati)

Abstract

In this paper, we study the idea of fuzzy sequential Henstock integrals for interval-valued functions and also prove some convergence theorems like the fuzzy sequential uniform convergence theorem, convergence theorem for fuzzy sequential uniform Henstock integrable functions and fuzzy sequential monotone convergence theorem for the fuzzy sequential Henstock integrals.

Keywords: Fuzzy Sequential Henstock integral, Convergence Theorem; Guages, Equi-integrability
2020 MSC: Primary 28B05; Secondary 28B10, 28B15, 46G10

1 Introduction

The concept of generalized Riemann integral is known to be Henstock integral, and is more powerful and simpler than the Lebesgue integral. This integral was introduced independently by R. Henstock and J. Kursweil in 1955 and 1957 respectively. It is also known that Henstock integral is equivalent to the Denjoy and Perron integrals and is easier and more reliable than the Wiener, Feynmann and Lebesgue integrals (see [1]-[19]). Wu and Gong [18] established the concept of the Henstock (H) integrals of interval valued functions and fuzzy number-valued functions and obtained some basic properties of the integral. Other recent works on fuzzy Henstock integrals are found in Bongiorno, Piazza and Musial [2], Wu and Gong [16], Musial [12] and Uzzal [18].

Paxton [13] developed an alternative sequential definition of the Henstock integral which he denotes as the Sequential Henstock (SH) integral, and then discussed its properties. The authors in [9] studied the dominated and bounded convergence results for Sequential Henstock integral. Convergence theory is one of the fundamental concepts in measure theory which has various applications in integration theory. In this paper, we establish the convergence results for the fuzzy-Sequential Henstock integral.

2 Preliminaries

Let $\mathbb{R}, \mathbb{R}^+$ and $\mathbb{N}$ denote the real line (with usual topology), the set of all positive real numbers and the set of all positive integers respectively. $\{\delta_n(x)\}_{n=1}^{\infty}$, as sequence of gauge functions. For any two given sets A and B, B^A denotes the set of all mappings with domain A and codomain B. The following definitions will be used in the sequel.

*Corresponding Author: Iluebe Victor Odalochi
Email addresses: victorodalochi1960@gmail.com (Victor Odalochi Iluebe), amogbademu@unilag.edu.ng (Adesanmi Alao Mogbademu)

Received: October 2022 *Accepted:* October 2023

Definition 2.1. [13] A gauge on $[a, b]$ is a positive real-valued function $\delta : [a, b] \to \mathbb{R}^+$. This gauge is δ-fine if $[u_{i-1}, u_i] \subset [t_i - \delta(t_i), t_i + \delta(t_i)]$.

Definition 2.2. [13] A sequence of tagged partition P_n of $[a, b]$ is a finite collection of ordered pairs

$$P_n = \{(u_{(i-1)_n}\, u_{i_n}), t_{i_n}\}_{i=1}^{m_n},$$

where $[u_{i-1}, u_i] \in [a, b]$, $u_{(i-1)_n} \leq t_{i_n} \leq u_{i_n}$ and $a = u_0 < u_{i_1} < ... < u_{m_n} = b$.

Now we recall the following definitions.

Definition 2.3. [13] (Henstock integral). A function $f : [a, b] \to \mathbb{R}$ is Henstock integrable if there exists a number $\alpha \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a positive gauge function $\delta(x) > 0$ such that

$$|\sum_{i=1}^{m_n \in \mathbb{N}} f(t_i)(u_i - u_{i-1}) - \alpha| < \varepsilon,$$

whenever $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$ is a $\delta_n(x) - fine$ tagged partition on $[a, b]$. We say that α is Henstock integral of f on $[a, b]$ i.e $\alpha = (H) \int_a^b f$. We use $SH[a, b]$ to denote the set of all Sequential Henstock integrable functions on $[a, b]$. Any function $f : [a, b] \to E^1$ is called a fuzzy function defined on $[a, b]$.

Wu and Gong [16] introduced the concept of fuzzy Henstock integral of fuzzy function defined on closed interval $[a, b]$. by stating the following.

Definition 2.4. A mapping $\eta \in [0, 1]^{\mathbb{R}}$ is called fuzzy number If

(i) η is normal, i.e. $\eta(r) = 1$ for some $r \in \mathbb{R}$,

(ii) η is convex, i.e. $\eta(\lambda r_1 + (1 - \lambda) r_2) \geq \min\{\eta(r_1), \eta(r_2)\}$ for all $r_1, r_2 \in \mathbb{R}$ and $\lambda \in [0, 1]$

(iii) η is semi-continuous, i.e. for every $\lambda \in [0, 1]$, the set $\{x \in \mathbb{R} : \eta(x) \geq \lambda\}$ is closed.

(iv) $cl([\eta]^0) = cl(\{x \in \mathbb{R} : \eta(x) > 0\})$ is compact, where $cl(A)$ is the closure of $A \in \mathbb{R}$.

The set of all fuzzy numbers is denoted by E^1.

Definition 2.5. (Fuzzy Henstock integral). A fuzzy function $f : [a, b] \to E^1$ is sequential fuzzy Henstock integrable to $\alpha \in E^1$ on $[a, b]$ if for any $\varepsilon > 0$ there exists a sequence of gauge function $\{\delta_n(x)\}_{n=1}^{\infty}$ such that

$$d(\sum_{i=1}^{n \in \mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}), \alpha) < \varepsilon,$$

whenever $P = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}$ $\delta(x) - fine$ tagged partitions of $[a, b]$ and $\alpha = (FH) \int_a^b f$. The set of all fuzzy Henstock integrable fuzzy functions defined on $[a, b]$ is denoted by $FH[a, b]$

Definition 2.6. [13] A function $f : [a, b] \to \mathbb{R}$ is sequential Henstock integrable if there exists a number $\alpha \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a sequence of positive gauge functions $\{\delta_n(x)\}_{n=1}^{\infty}$ such that

$$|\sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - \alpha| < \varepsilon,$$

whenever $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$ is a $\delta_n(x) - fine$ tagged partition on $[a, b]$. We say that α is sequential Henstock integral of f on $[a, b]$ i.e $\alpha = (SH) \int_a^b f$. We use $SH[a, b]$ to denote the set of all sequential Henstock integrable functions on $[a, b]$.

Remark 2.7. If $\delta_n = \delta$ for all $n \in \mathbb{N}$ in Definition 2.3, we have our definition for the Henstock integral [10].

The sequential Henstock integral, a theory developed by Paxton [13] is a sequential approach of defining and proving theorems on Henstock-type integral, in which both had been shown to be equivalent. Moreso, it has the potential of expanding the overall theory of Henstock integration into more abstract mathematical settings which in turn may lead to further applications of the Henstock integral. Hence we define newly the following:

Definition 2.8. (Fuzzy sequential Henstock integral) A fuzzy function $f : [a,b] \to E^1$ is fuzzy sequential Henstock integrable to $\alpha \in E^1$ on $[a,b]$ if for any $\varepsilon > 0$ there exists a sequence of gauge function $\{\delta_n(x)\}_{n=1}^{\infty}$ such that

$$d(\sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}), \alpha) < \varepsilon,$$

whenever $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}$ is a $\delta_n(x) - fine$ tagged partitions of $[a,b]$ and $\alpha = (SFH)\int_a^b f$. The set of all fuzzy sequential Henstock integrable functions defined on $[a,b]$ is denoted by $SFH[a,b]$.

We state the following useful lemma proved in Goetschel and Voxman [3].

Lemma 2.9. Let $f_1, f_2 \in \mathbb{R}^{[a,b]}$ be two mapping sending each $\lambda \in [0,1]$ to $f_1(\lambda) = f_1^\lambda$ and $f_2(\lambda) = f_2^\lambda$ respectively with the properties:

(i) f_1 is a bounded increasing function,

(ii) f_2 is a bounded decreasing function,

(iii) $f_1(1) \leq f_2(1)$ and

(iii) f_1 and f_2 are both left continuous on $[0,1]$ and right continuous at 0. Then there exists a unique fuzzy number $\alpha \in E^1$ such that $|\alpha|^\lambda = [f_1^\lambda, f_2^\lambda]$ for each $\lambda \in [0,1]$.

Let $\Omega = \{a = [\underline{a}, \bar{a}] : \underline{a}, \bar{a} \in \mathbb{R}, \underline{a} \leq \bar{a}\}$ be the family of all bounded closed intervals. Let $a, b \in \Omega$ we define

(i) $a = b$ iff $\underline{a} = \underline{b}, \bar{a} = \bar{b}$,

(ii) $a \leq b$ iff $\underline{a} \leq \underline{b}, \bar{a} \leq \bar{b}$,

(iii) $a + b = [\underline{a}, \bar{a}] + [\underline{b}, \bar{b}] = [\underline{a} + \underline{b}, \bar{a} + \bar{b}]$,

(iv) $a.b = \{st : s \in a, t \in b\}$,

(v) $\underline{a.b} = \min\{\underline{a}.\underline{b}, \underline{a}.\bar{b}, \bar{a}.\underline{b}, \bar{a}.\bar{b}\}$,

(vi) $\bar{a.b} = \max\{\underline{a}.\underline{b}, \underline{a}.\bar{b}, \bar{a}.\underline{b}, \bar{a}.\bar{b}\}$.

Here we observe that $'' \leq''$ is a partial order in Ω and the mapping $\rho : \Omega \times \Omega \to \mathbb{R}$ defined by $\rho(a,b) = \max\{|\underline{a} - \underline{b}|, |\bar{a} - \bar{b}|\}$ for all $a, b \in \Omega$ is a metric(called Hausdorff metric) on Ω. Now, it is easy to verify that the mapping $\bar{\rho} : E^1 \times E^1 \to \mathbb{R}$ defined by $\bar{\rho}(\alpha, \beta) = \sup\{\rho([\alpha]^\lambda, [\beta]^\lambda) : \lambda \in [0,1]\}$ for all $\alpha, \beta \in E^1$ is a metric on E^1.

Definition 2.10. [13] A fuzzy number $\alpha_0 \in E^1$ is called the least upper bound (or supremum) of $A \in E^1$ if

(i) $\alpha \leq \alpha_0$ for all $\alpha \in A$ (i.e. α is an upper bound of A),

(ii) for any $\varepsilon > 0$ there exists at least one $\beta \in A$ such that $\alpha_0 < \beta + \varepsilon$. We write $\alpha = supA$.

Similarly, the greatest lower bound(or infinum) of $A \in E^1$ if $\alpha \geq \alpha_0$ for all $\alpha \in A$ (i.e. α is an lower bound of A) and is denoted by inf A . Then A sequence $\{\alpha_n\}, \alpha_n \in E^1$ is said to be monotonically increasing (resp. monotonically decreasing), if of gauge function $\{\delta_n(x)\}_{n=1}^{\infty}$ such that $\alpha_n \leq \alpha_{n+1}$ (resp. $\alpha_{n+1} \leq \alpha_n$) for all $n \in \mathbb{N}$.

The following simple but important theorem was proved by Guang-Quan [6].

Theorem 2.11. Every monotonically increasing (resp. monotonically decreasing) sequence. $\{f_n\}$, $f \in E^1$ on $[a,b]$ with an upper bound (resp. lower bound) converges to $\sup\{\alpha_n : n \in \mathbb{N}\}$, (resp. $\inf\{\alpha_n : n \in \mathbb{N}\}$) in the metric space (E^1, d) on $[a,b]$.

3 Convergence Theorems

Let $\{f_n\}$ be a sequence of fuzzy sequential Henstock integral function in E^1 on $[a,b]$ that fuzzy converges to the fuzzy function $f \in E^1$ on $[a,b]$ in the metric space (E^1, d). It is quite natural to expect that $f \in SFH[a,b]$ and

$$(H)\int_a^b f = \lim (H)\int_a^b f_n.$$

But [18, Example 3.1] shows that this is not true in general.

Definition 3.1. A sequence $\{f_n\}$ in E^1 on $[a,b]$ is said to fuzzy uniformly converge to $f \in E^1$ on $[a,b]$ if for each $\varepsilon > 0$ there exists $n_{i_0} \in \mathbb{N}$ such that $d(f_n(x), f(x)) < \varepsilon,$, for all $n \geq n_{i_0}$ and for all $x \in [a,b]$.

Definition 3.2. A sequence $\{f_n\}$ of fuzzy sequential Henstock integrable function in E^1 on $[a,b]$ is called fuzzy uniform Henstock integrable on $[a,b]$ if for each $\varepsilon > 0$ there exists $\{\delta_n(x)\}_{n=1}^{\infty}$ such that

$$d(\sum_{i=1}^{m_n \in \mathbb{N}} f_n(t_{i_n})(u_{i_n} - u_{(i-1)_n}), (SH)\int_a^b f_n) < \varepsilon,$$

whenever $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}$ is a $\delta_n(x) - fine$ tagged partitions of $[a,b]$. for all $n \geq \mathbb{N}$.

Theorem 3.3. (Fuzzy Sequential Uniform Convergence) Let $\{f_n\}$ be a sequence of fuzzy sequential Henstock integrable functions E^1 on $[a,b]$ that fuzzy uniformly converges to the fuzzy function $f \in E^1$ on $[a,b]$. Then

(i) f is Henstock integrable on $[a,b]$,

(ii) $(SH)\int_a^b f = \lim (SH)\int_a^b f_n$.

Proof . Firstly, we prove that $(SH)\int_a^b f_n$. is a Cauchy sequential in (E^1, d). Let $\varepsilon > 0$ Since for each $n_{i_0} \in \mathbb{N}$ there exists a sequence of gauge function $\{\delta_n(x)\}_{n=1}^{\infty}$ such that

$$d(\sum_{i=1}^{m_n \in \mathbb{N}} f_n(t_{i_n})(u_{i_n} - u_{(i-1)_n}), (SH)\int_a^b f) < \frac{\varepsilon}{2},$$

whenever $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}$ is a $\delta_n(x) - fine$ tagged partitions of $[a,b]$. Again, since $\{f_n\}$ in E^1 on $[a,b]$ fuzzy sequential uniformly converges to $\{f\}$ in E^1 on $[a,b]$, there exists a $n_{i_0} \in \mathbb{N}$ such that

$$d(\sum_{i=1}^{m_n \in \mathbb{N}} f_n(t_{i_n})(u_{i_n} - u_{(i-1)_n}), (SH)\int_a^b f) < \frac{\varepsilon}{3(b-a)},$$

for all $n \geq n_{i_0}$ and for all $i = 1, 2, \ldots$. So, for all $n, m \geq n_{i_0}$, taking arbitrary sequence of partition $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}$ simultaneously $\delta_n(x)$ and $\delta_m(x)$-fine, we

$$\begin{aligned}
d((SH)\int_a^b f_n, (SH)\int_a^b f_m) &\leq d(\sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}), \alpha) \\
&\leq d(\sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}), (SH)\int_a^b f_n) \\
&\quad + d(\sum_{i=1}^{m_n \in \mathbb{N}} f_n(t_{i_n})(u_{i_n} - u_{(i-1)_n}), \sum_{i=1}^{n \in \mathbb{N}} f_m(t_{i_n})(u_{i_n} - u_{(i-1)_n})) \\
&\quad + d(\sum_{i=1}^{m_n \in \mathbb{N}} f_m(t_{i_n})(u_{i_n} - u_{(i-1)_n}), (SH)\int_a^b f_m) \\
&\leq \frac{\varepsilon}{3} + \sum_{i=1}^{n \in \mathbb{N}} d((f_n(t_{i_n}), f_m(t_{i_n}))(u_{i_n} - u_{(i-1)_n}) + \frac{\varepsilon}{3} \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3(b-a)}(b-a) + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Thus $\{(SH)\int_a^b f_n\}$ is Cauchy in the complete metric space (E^1, d). Therefore $\{(SH)\int_a^b f_n\}$ converges metric space (E^1, d). Since by [18, Theorem 2.3], (E^1, d) is complete, $(SH)\int_a^b f_n$ converges in the metric space (E^1, d). Suppose

$$\lim (SH)\int_a^b f_n = \alpha.$$

Hence $(SH)\int_a^b f = \alpha$. Let $\varepsilon > 0$ be given. By the condition of the theorem there exists a $n_{i_0} \in N$ such that

$$d(\sum_{i=1}^{m_n\in\mathbb{N}} f_n(t_{i_n})(u_{i_n} - u_{(i-1)_n}), (SH)\int_a^b f) < \frac{\varepsilon}{3(b-a)},$$

for all $n \geq n_{i_0}$ and for all $x \in [a,b]$. Now for any sequence of tagged partition P_n of $[a,b]$ and $n \geq n_{i_0}$, by Theorem 2.3, we get

$$\begin{aligned} d(\sum_{i=1}^{m_n\in\mathbb{N}} f_n(t_{i_n})(u_{i_n} - u_{(i-1)_n}), \sum_{i=1}^{m_n\in\mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n})) &\leq \sum_{i=1}^{m_n\in\mathbb{N}} (u_{i_n} - u_{(i-1)_n}) d(f_n(t_{i_n}), f(t_{i_n})) \\ &\leq (b-a)\sum_{i=1}^{m_n\in\mathbb{N}} d(f_n(t_{i_n}), f(t_{i_n})) \\ &< \frac{\varepsilon}{3}. \end{aligned}$$

Since $\lim(SH)\int_a^b f_n = \alpha$, there exists a $m(\geq n_{i_0}) \in \mathbb{N}$ such that $d((SH)\int_a^b f_m, \alpha) < \frac{\varepsilon}{3}$. Since $\lim(SH)\int_a^b f_n = \alpha$, there exists a $\{\delta_n(x)\}_{n=1}^{\infty}$ such that

$$d(\sum_{i=1}^{m_n\in\mathbb{N}} f_n(t_{i_n})(u_{i_n} - u_{(i-1)_n}), (SH)\int_a^b f) < \frac{\varepsilon}{3},$$

whenever $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}$ is a $\delta_n(x) - fine$ tagged partitions of $[a,b]$. Thus

$$\begin{aligned} d(\sum_{i=1}^{m_n\in\mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}), \alpha) &\leq d(\sum_{i=1}^{m_n\in\mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}), \sum_{i=1}^{m_n\in\mathbb{N}} f_n(t_{i_n})(u_{i_n} - u_{(i-1)_n})) \\ &\quad + d(\sum_{i=1}^{m_n\in\mathbb{N}} f_n(t_{i_n})(u_{i_n} - u_{(i-1)_n}), (SH)\int_a^b f_n) + d((SH)\int_a^b f_n, \alpha) \\ &< \varepsilon, \end{aligned}$$

whenever $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}$ is a $\delta_n(x) - fine$ tagged partitions of $[a,b]$. Hence $(SH)\int_a^b f$ exists and $(SH)\int_a^b f = \alpha$. □

Theorem 3.4. (Convergence theorem for fuzzy uniform sequential Henstock integrable function) Let $\{f_n\}$ be a sequence of sequential fuzzy uniform Henstock integrable functions E^1 on $[a,b]$ and $f \in E^1$ on $[a,b]$ be such for each $x \in [a,b]$, $\{f_n(x)\}$ converges to $f(x)$ in the metric space (E^1, d) on $[a,b]$. Then

(i) f is sequential Henstock integrable on $[a,b]$,

(ii) $(SH)\int_a^b f = \lim(SH)\int_a^b f_n$.

Proof . Let $\varepsilon > 0$, since $\{f_n\}$ is fuzzy uniform sequential Henstock integrable sequence on $[a,b]$ there exists a sequence of gauge function $\{\delta_n(x)\}_{n=1}^{\infty}$ such that

$$d(\sum_{i=1}^{m_n\in\mathbb{N}} f_n(t_{i_n})(u_{i_n} - u_{(i-1)_n}), (SH)\int_a^b f_n) < \frac{\varepsilon}{2},$$

whenever $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}$ is a $\delta_n(x) - fine$ tagged partitions of $[a,b]$ for all $n \in \mathbb{N}$. Again by above condition, there exists $n \in n_{i_0} \in \mathbb{N}$ such that

$$d(\sum_{i=1}^{m_n\in\mathbb{N}} f_n(t_{i_n})(u_{i_n} - u_{(i-1)_n}), \sum_{i=1}^{m_n\in\mathbb{N}} f_m(t_{i_n})(u_{i_n} - u_{(i-1)_n})) < \frac{\varepsilon}{3(b-a)},$$

for all $n, m \geq n_{i_0}$ and so

$$\begin{aligned} d(\sum_{i=1}^{m_n \in \mathbb{N}} f_n(t_{i_n})(u_{i_n} - u_{(i-1)_n}), \sum_{i=1}^{m_n \in \mathbb{N}} f_m(t_{i_n})(u_{i_n} - u_{(i-1)_n})) &\leq d(\sum_{i=1}^{m_n \in \mathbb{N}} f_n(t_{i_n}), f_m(t_{i_n}))(u_{i_n} - u_{(i-1)_n})) \\ &\leq (b-a)d(\sum_{i=1}^{m_n \in \mathbb{N}} f_n(t_{i_n}), f_m(t_{i_n})) \\ &< \frac{\varepsilon}{3} \end{aligned}$$

for all $n, m \geq n_{i_0}$. Then

$$\begin{aligned} d((SH)\int_a^b f_n, (SH)\int_a^b f_m) &\leq d(\sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}), (SH)\int_a^b f_n) \\ &\quad + d(\sum_{i=1}^{m_n \in \mathbb{N}} f_n(t_{i_n})(u_{i_n} - u_{(i-1)_n}), \sum_{i=1}^{m_n \in \mathbb{N}} f_m(t_{i_n})(u_{i_n} - u_{(i-1)_n})) \\ &\quad + d(\sum_{i=1}^{m_n \in \mathbb{N}} f_m(t_{i_n})(u_{i_n} - u_{(i-1)_n}), (SH)\int_a^b f_m) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3(b-a)}(b-a) + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

for all $n, m \geq n_{i_0}$. So $\{(SH)\int_a^b f_n\}$ is Cauchy in the complete metric space (E^1, d). Therefore $\{(SH)\int_a^b f_n\}$ converges metric space (E^1, d). Suppose $\lim(SH)\int_a^b f_n = \alpha$. Hence $(SH)\int_a^b f = \alpha$. Let $\varepsilon > 0$ be given. Since $\{f_n\}$ is a fuzzy uniformly sequential Henstock integrable sequence on $[a, b]$, there exists a sequence $\{\delta_n(x)\}_{n=1}^{\infty}$ such that

$$d(\sum_{i=1}^{m_n \in \mathbb{N}} f_n(t_{i_n})(u_{i_n} - u_{(i-1)_n}), (SH)\int_a^b f) < \frac{\varepsilon}{3},$$

whenever $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}$ is a $\delta_n(x) - fine$ tagged partitions of $[a, b]$ is satisfied for all $n \in \mathbb{N}$. Since $\lim(SH)\int_a^b f_n = \alpha$., there exists a $n_{i_1} \in \mathbb{N}$ such that $d((SH)\int_a^b f_n, \alpha) < \frac{\varepsilon}{3}$, for all $n_i \in \mathbb{N}$. Again by the given condition of the theorem, there exists $n_i \in \mathbb{N}$. such that

$$\begin{aligned} &d(\sum_{i=1}^{m_n \in \mathbb{N}} f_n(t_{i_n})(u_{i_n} - u_{(i-1)_n}), \sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n})) \\ &\leq (b-a)d(\sum_{i=1}^{m_n \in \mathbb{N}} f_n(t_{i_n})(u_{i_n} - u_{(i-1)_n}), f(t_{i_n})(u_{i_n} - u_{(i-1)_n})) < \frac{\varepsilon}{3}. \end{aligned}$$

Thus,

$$\begin{aligned} d(\sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}), \alpha) &\leq d(\sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}), \sum_{i=1}^{m_n \in \mathbb{N}} f_n(t_{i_n})(u_{i_n} - u_{(i-1)_n})) \\ &\quad + d(\sum_{i=1}^{m_n \in \mathbb{N}} f_n(t_{i_n})(u_{i_n} - u_{(i-1)_n}), (SH)\int_a^b f_n) + d((SH)\int_a^b f_n, \alpha) \\ &< \varepsilon, \end{aligned}$$

whenever $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}$ is a $\delta_n(x) - fine$ tagged partitions of $[a, b]$. Hence $(SH)\int_a^b f$ exists and $(SH)\int_a^b f = \alpha$.$\square$

The following fuzzy sequential Sak-Henstock lemma is simple but useful in proving the convergence theorem.

Lemma 3.5. Let $f:[a,b]\to E^1$ be a sequential fuzzy Henstock integrable function in E^1 on $[a,b]$, let

$$\phi(x)=(SH)\int_a^b f.$$

for all $x\in[a,b]$ and let $\varepsilon>0$. Suppose that $\{\delta_n(x)\}_{n=1}^{\infty}$ is a sequence of positive real-valued function such that

$$d(\sum_{i=1}^{m_n\in\mathbb{N}} f_n(t_{i_n})(u_{i_n}-u_{(i-1)_n}),\ \sum_{i=1}^{m_n\in\mathbb{N}} \phi(t_{i_n})(u_{i_n}-u_{(i-1)_n}))<\varepsilon,$$

for every $\delta_n(x)-fine$ division of $[a,b]$. If $\{(u'_{(i-1)_n}\, u'_{i_n}),t'_{i_n}\}_{i=1}^{m_n}=\{(t'_{i_n},[a'_{i_n},b'_{i_n}]):i=1,2,...,k\}$ is a $\delta_n(x)-fine$ subdivision of $[a,b]$, then

$$d(\sum_{i=1}^{m_n\in\mathbb{N}} f_n(t'_{i_n})(u'_{i_n}-u'_{(i-1)_n}),\ \sum_{i=1}^{m_n\in\mathbb{N}} \phi(t'_{i_n})(u'_{i_n}-u'_{(i-1)_n}))\le\varepsilon.$$

Definition 3.6. A sequence $\{f_n\}$,$f_n\in E^1$ on $[a,b]$ is called fuzzy sequential increasing(resp. fuzzy sequential decreasing) in $[a,b]$ if $f_n(x)\le f_{n+1}(x)$ (resp. $f_{n+1}(x)\le f_n(x)$) for all $x\in[a,b]$ and $k\in\mathbb{N}$. A sequence $\{f_n\}$ is called fuzzy monotone on $[a,b]$ if it is either fuzzy increasing or fuzzy decreasing in $[a,b]$.

Theorem 3.7. (Fuzzy sequential monotone convergence theorem) Let $\{f_n\}$ be a fuzzy sequential monotone sequence of fuzzy sequential Henstock integrable functions E^1 on $[a,b]$ and $f\in E^1$ on $[a,b]$ be such for each $x\in[a,b]$, $\{f_n(x)\}$ converges to $f(x)$ in the metric space (E^1,d) on $[a,b]$. Then

(i) f is sequential Henstock integrable on $[a,b]$,

(ii) $(SH)\int_a^b f=\lim(SH)\int_a^b f_n$.

Proof . Let $\{f_n\}$ be a fuzzy sequential increasing sequence of fuzzy sequential Henstock integrable functions E^1 on $[a,b]$, then $(SH)\int_a^b f_n$ is bounded and fuzzy increasing. Then by Theorem 2.11, $(SH)\int_a^b f_n$ must fuzzy converges to $\alpha=\sup(SH)\int_a^b f_n$. Let $\varepsilon>0$, then we choose $r\in\mathbb{N}$ such that $\frac{1}{2^{r-2}}<\frac{\varepsilon}{3}$ and $d((SH)\int_a^b f_n,\alpha)<\frac{\varepsilon}{3}$. Again since $\{f_n\}$ is a sequence of fuzzy sequential Henstock integrable function on $[a,b]$, for each $n\in\mathbb{N}$ there exists a sequence of $\{\delta_n(x)\}_{n=1}^{\infty}$ such that every $\delta_n(x)-fine$ tagged partitions $P_n=\{(u_{(i-1)_n},u_{i_n}),t_{i_n}\}$ of $[a,b]$ satisfies

$$d(\sum_{i=1}^{m_n\in\mathbb{N}} f_n(t_{i_n})(u_{i_n}-u_{(i-1)_n}),(SH)\int_a^b f_n)<\frac{1}{2^n}.$$

Again, for each $t\in[a,b]$, we can select a $n_t(\ge r)\in\mathbb{N}$ such that $d(f_{n_t},f_n)<\frac{\varepsilon}{3(b-a)}$. Consider the function $\delta_n=\delta_{n_x}$ and let $P_n=\{(u_{(i-1)_n},u_{i_n}),t_{i_n}\}_{i=1}^{m_n}$ be any δ_n-fine partitions of $[a,b]$. Then, we have

$$\begin{aligned}d(\sum_{i=1}^{m_n\in\mathbb{N}} f(t_{i_n})(u_{i_n}-u_{(i-1)_n}),\alpha)\le& d(\sum_{i=1}^{m_n\in\mathbb{N}} f(t_{i_n})(u_{i_n}-u_{(i-1)_n}),\ \sum_{i=1}^{m_n\in\mathbb{N}} f_n(t_{i_n})(u_{i_n}-u_{(i-1)_n}))\\ &+d(\sum_{i=1}^{m_n\in\mathbb{N}} f_n(t_{i_n})(u_{i_n}-u_{(i-1)_n}),\ \sum_{i=1}^{m_n\in\mathbb{N}}(SH)\int_a^b f_{n_{t_{i_n}}})d(\sum_{i=1}^{m_n\in\mathbb{N}}(SH)\int_a^b f_n,\alpha).\end{aligned}$$

Now we estimate the three values in the right-handed sum of the last inequality.

(a) Estimation of $d(\sum_{i=1}^{m_n\in\mathbb{N}} f(t_{i_n})(u_{i_n}-u_{(i-1)_n}),\sum_{i=1}^{m_n\in\mathbb{N}} f_{n_{t_{i_n}}}(t_{i_n})(u_{i_n}-u_{(i-1)_n}))$. By [18, Theorem 2.3]:

$$\begin{aligned}d(\sum_{i=1}^{m_n\in\mathbb{N}} f(t_{i_n})(u_{i_n}-u_{(i-1)_n}),\ \sum_{i=1}^{m_n\in\mathbb{N}} f_{n_{t_{i_n}}}(t_{i_n})(u_{i_n}-u_{(i-1)_n}))\ &\le\ \sum_{i=1}^{m_n\in\mathbb{N}} d((f(t_{i_n}),f_{n_{t_{i_n}}}))(u_{i_n}-u_{(i-1)_n})\\ &<\ \frac{\varepsilon}{3(b-a)}(b-a)\\ &=\ \frac{\varepsilon}{3}.\end{aligned}$$

(b) Estimation of $d(\sum_{i=1}^{m_n\in\mathbb{N}} f_{n_{t_{i_n}}}(t_{i_n})(u_{i_n}-u_{(i-1)_n}), \sum_{i=1}^{m_n\in\mathbb{N}}(SH)\int_a^b f_{n_{t_{i_n}}})$. Suppose that $p=\max\{n_{t_{i_n}} : i = 1,2,...,m\}$, then

$$d(\sum_{i=1}^{m_n\in\mathbb{N}} f_{n_{t_{i_n}}}(t_{i_n})(u_{i_n}-u_{(i-1)_n}), \sum_{i=1}^{m_n\in\mathbb{N}}(SH)\int_a^b f_{n_{t_{i_n}}})$$
$$\leq \sum_{i=r}^{p}\left(\sum_{i\in\{1,2,...,m:n_{t_{i_n}}\}} d(f_{n_{t_{i_n}}}(t_{i_n})(u_{i_n}-u_{(i-1)_n}), (SH)\int_a^b f_{n_{t_{i_n}}}\right).$$

By applying Lemma 3.5,

$$\sum_{\{i\in 1,2,...,m:n_{t_{i_n}}\}} d\left(f_{n_{t_{i_n}}}(t_{i_n})(u_{i_n}-u_{(i-1)_n}), (SH)\int_a^b f_{n_{t_{i_n}}}\right) < \sum_{i=r}^{p}\frac{1}{2^{i-1}} < \frac{1}{2^{r-2}} < \frac{\varepsilon}{3}.$$

(c) Estimation of $d(\sum_{i=1}^{m_n\in\mathbb{N}}(SH)\int_a^b f_{n_{t_{i_n}}}, \alpha)$. Here $r\leq n_{t_{i_n}}\leq p$ implies $f_r(x)\leq f_{n_{t_{i_n}}}(x)\leq f_p(x)$, for all $x\in[a,b]$ and so

$$\int_a^b f_r(x)\leq \int_a^b f_{n_{t_{i_n}}}(x)\leq \int_a^b f_p(x).$$

Hence

$$\int_a^b f_r(x)\leq \sum_{i=1}^{m_n\in\mathbb{N}}\int_a^b f_{n_{t_{i_n}}}(x)\leq \int_a^b f_p(x)\leq\alpha.$$

Thus by [18, Theorem 2.3],

$$d(\sum_{i=1}^{m_n\in\mathbb{N}}(SH)\int_a^b f_{n_{t_{i_n}}}, \alpha)\leq d(\int_a^b f_r, \alpha) < \frac{\varepsilon}{3}.$$

Hence

$$d(\sum_{i=1}^{m_n\in\mathbb{N}} f(t_{i_n})(u_{i_n}-u_{(i-1)_n}), \alpha) < \varepsilon.$$

So f is sequential Henstock on $[a,b]$ and $(SH)\int_a^b f = \lim(SH)\int_a^b f_n$. □

The next theorem provides the necessary and sufficient condition such that the pointwise limit $f\in E^1$ on $[a,b]$ of a sequence $\{f_n\}$ of sequential fuzzy Henstock integrable functions is fuzzy sequential Henstock integrable on $[a,b]$ and the equality

$$(SH)\int_a^b f = \lim(SH)\int_a^b f_n.$$

holds.

Theorem 3.8. Let $\{f_n\}$ be a sequence of sequential fuzzy Henstock integrable functions in E^1 on $[a,b]$ and $f\in E^1$ on $[a,b]$ be such for each $x\in[a,b]$, $\{f_n(x)\}$ converges to $f(x)$ in the metric space (E^1,d) on $[a,b]$. Then the following conditions are equivalent:

(i) f is fuzzy Sequential Henstock integrable on $[a,b]$ and

$$(SH)\int_a^b f = \lim(SH)\int_a^b f_n.$$

(ii) for each $\varepsilon>0$, there exists $m\in\mathbb{N}$ such that for each $n\geq m$, there exists a sequence of $\{\delta_n(x)\}_{n=1}^{\infty}$ such that every $\delta_n(x)-fine$ tagged partitions $P_n=\{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}$ of $[a,b]$ satisfies

$$d(\sum_{i=1}^{m_n\in\mathbb{N}} f_n(t_{i_n})(u_{i_n}-u_{(i-1)_n}), \sum_{i=1}^{m_n\in\mathbb{N}} f(t_{i_n})(u_{i_n}-u_{(i-1)_n})) < \varepsilon.$$

Proof . $(i) \Rightarrow (ii)$. Let $\varepsilon > 0$. Since $(SH)\int_a^b f = \lim(SH)\int_a^b f_n$, there exists $m \in \mathbb{N}$ such that $d((SH)\int_a^b f_n = (SH)\int_a^b f) < \frac{\varepsilon}{3}$, for all $n \geq m$. Again since $\{f_n\}$ is a sequence of fuzzy sequential Henstock integrable functions on $[a,b]$, for each $n(\geq m)$, we can find a sequence of $\{\delta_n(x)\}_{n=1}^{\infty}$ such that every $\delta_n(x) - fine$ tagged partitions $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}$ of $[a,b]$ satisfies

$$d(\sum_{i=1}^{m_n \in \mathbb{N}} f_n(t_{i_n})(u_{i_n} - u_{(i-1)_n}), (SH)\int_a^b f_n) < \frac{\varepsilon}{3}.$$

Again, since f is a fuzzy sequential Henstock integrable function on $[a,b]$, we can find a sequence of $\{\delta_0(x)\}_{n=1}^{\infty}$ such that every $\delta_0(x) - fine$ tagged partitions $P_n = \{(u_{(i-1)_0}, u_{i_0}), t_{i_0}\}$ of $[a,b]$ satisfies

$$d(\sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_0})(u_{i_0} - u_{(i-1)_0}), (SH)\int_a^b f) < \frac{\varepsilon}{3}.$$

We take $\delta_i(x)$ defined by the $\min\{\delta_0(x), \delta_n(x)\}$. Then

$$\begin{aligned} d(\sum_{i=1}^{m_n \in \mathbb{N}} f_n(t_{i_n})(u_{i_n} - u_{(i-1)_n}), \sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n})) \leq & d(\sum_{i=1}^{m_n \in \mathbb{N}} f_n(t_{i_n})(u_{i_n} - u_{(i-1)_n}), (SH)\int_a^b f_n) \\ & + d((SH)\int_a^b f_n, (SH)\int_a^b f) \\ & + d((SH)\int_a^b f, \sum_{i=1}^{m_n \in \mathbb{N}} f_n(t_{i_n})(u_{i_n} - u_{(i-1)_n})) \\ < & \varepsilon \end{aligned}$$

for every $\delta_n(x) - fine$ division of $[a,b]$. and $n \geq m$. $(ii) \Rightarrow (i)$ Let $\varepsilon > 0$ and (ii) holds. Then we assert that $(SH)\int_a^b f_n$ is Cauchy. By (ii), there exists $m \in \mathbb{N}$ such that for each $n, l \geq m$, there exist $\delta_n(x), \delta_l(x)$ such that $(SH)\int_a^b f = \lim(SH)\int_a^b f_n$, there exists $m \in \mathbb{N}$ such that $d((SH)\int_a^b f_n, (SH)\int_a^b f) < \frac{\varepsilon}{3}$,

$$d(\sum_{i=1}^{m_n \in \mathbb{N}} f_n(t_{i_n})(u_{i_n} - u_{(i-1)_n}), \sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n})) < \frac{\varepsilon}{4}$$

for every $\delta_n(x) - fine$ partition of $[a,b]$ and

$$d(\sum_{i=1}^{m_k \in \mathbb{N}} f_k(t_{i_k})(u_{i_k} - u_{(i-1)_k}), \sum_{i=1}^{m_k \in \mathbb{N}} f(t_{i_k})(u_{i_l} - u_{(i-1)_k})) < \frac{\varepsilon}{4}$$

for every $\delta_k(x) - fine$ partition $P_k = \{(u_{(i-1)_k}, u_{i_k}), t_{i_k}\}$ of $[a,b]$. Since f_n and f_k are sequential fuzzy Henstock integrable function on $[a,b]$, we can find $\varsigma_n : [a,b] \to \mathbb{R}^+, \varsigma_k : [a,b] \to \mathbb{R}^+ \in$ such that

$$d(\sum_{i=1}^{m_k \in \mathbb{N}} f_n(\zeta_{i_n})(u_{i_n} - u_{(i-1)_n}), (SH)\int_a^b f_n) < \frac{\varepsilon}{4}$$

for every $\varsigma_n(x) - fine$ partition $P_n = \{(u_{(i-1)_n}, u_{i_n}), \zeta_{i_n}\}$of $[a,b]$ and

$$d(\sum_{i=1}^{m_k \in \mathbb{N}} f_k(\zeta_{i_k})(u_{i_k} - u_{(i-1)_k}), (SH)\int_a^b f_k) < \frac{\varepsilon}{4}$$

for every $\varsigma_k(x) - fine$ partition $P_k = \{(u_{(i-1)_k}, u_{i_k}), \zeta_{i_k}\}$ of $[a,b]$. Now define δ_i by $\delta_i(x) = \min\{\delta_l(x), \delta_k(x), \varsigma_n(x), \varsigma_k(x)\}$. Then for all $k, l \geq m$, $d((SH)\int_a^b f_n, (SH)\int_a^b f_k) < \varepsilon$ and so $\{(SH)\int_a^b f_k\}$ is a Cauchy sequence. Completeness of metric space (E^1, d) guarantees that $\lim(SH)\int_a^b f_n$ exists in E^1. Suppose that $\lim(SH)\int_a^b f_n = \alpha$. Then we can choose a

$p(\geq m) \in \mathbb{N}$ such that $d((SH)\int_a^b f_n, \alpha) < \dfrac{\varepsilon}{3}$. Also by (ii), there exists $\delta_{n_1} - fine$ partition $P_{n_1} = \{(u_{(i-1)_{n_1}}, u_{i_{n_1}}), t_{i_{n_1}}\}$ of $[a,b]$ satisfies

$$d(\sum_{i=1}^{m_{n_1}\in\mathbb{N}} f_p(t_{i_{n_1}})(u_{i_{n_1}} - u_{(i-1)_{n_1}}), \sum_{i=1}^{m_{n_1}\in\mathbb{N}} f(t_{i_{n_1}})(u_{i_{n_1}} - u_{(i-1)_{n_1}})) < \frac{\varepsilon}{3}.$$

Since f_p is a fuzzy sequential Henstock integrable function on $[a,b]$, there exists $\delta_{n_2} - fine$ partition $P_{n_2} = \{(u_{(i-1)_{n_2}}, u_{i_{n_2}}), t_{i_{n_2}}\}$ of $[a,b]$ satisfies

$$d(\sum_{i=1}^{m_{n_2}\in\mathbb{N}} f_p(t_{i_{n_1}})(u_{i_{n_1}} - u_{(i-1)_{n_1}}), (SH)\int_a^b f_p) < \frac{\varepsilon}{3}.$$

$\delta_n(x) = \min\{\delta_{n_1}(x), \delta_{n_2}(x)\}$. Then

$$\begin{aligned} d(\alpha, \sum_{i=1}^{m_n\in\mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n})) &\leq d(\alpha, (SH)\int_a^b f_p) + d((SH)\int_a^b f_p, \sum_{i=1}^{m_n\in\mathbb{N}} f_p(t_{i_n})(u_{i_n} - u_{(i-1)_n})) \\ &\quad + d(\sum_{i=1}^{m_n\in\mathbb{N}} f_p(t_{i_n})(u_{i_n} - u_{(i-1)_n}), \sum_{i=1}^{m_n\in\mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n})) \\ &< \varepsilon. \end{aligned}$$

So f is sequential Henstock integrable on $[a,b]$ and $\lim(SH)\int_a^b f_n = \alpha = (SH)\int_a^b f$. $\square$

4 Example

Let $F_n(x) = [\dfrac{1}{n}, \dfrac{1}{n+1}]$, for $x \in [a,b]$ then $\{F_n\}$ tends pointwise on $[a,b]$ to zero function. Furthermore, as $\sum_{i=1}^{m_n\in\mathbb{N}} F_n(t_{i_n})(u_{i_n} - u_{(i-1)_n} = [\dfrac{1}{n}, \dfrac{1}{n+1}]$ for each $n \in N$ and sequence partitions P_n of $[a,b]$, we see that $\int_a^b F_n = [\dfrac{1}{n}, \dfrac{1}{n+1}]$, for each $n \in \mathbb{N}$ and the sequence $\{f_n\}$ is interval fuzzy sequential Henstock uniformly integrable on [a,b]. That is F_n converge uniformly to the function at both lower and upper ends.

5 Conclusion

In this paper, we establish the convergence results for fuzzy Sequential Henstock integrable functions: fuzzy sequential uniform convergence theorem, convergence theorem for fuzzy sequential uniform Henstock integrable functions and fuzzy sequential monotone convergence theorem. We also find a necessary and sufficient condition under which the pointwise limit of a sequence of fuzzy Henstock integrable functions is fuzzy sequential Henstock integrable.

References

[1] R.G. Bartle, *A convergence theorem for generalized Riemann integrals*, Real Anal. Exch. **20** (1994-95), no. 2, 119–124.

[2] R. Bongiorno, I. Di Piazza, and K. Musial, *A decomposition theorem for the fuzzy Henstock integrals*, Fuzzy Sets Syst. **200** (2012), 36–47.

[3] R. Goetschel and W. Voxman, *Elementary fuzzy calculus*, Fuzzy Sets Syst. **18** (1986), 31–43.

[4] Z. Gong and Y, Shao *The controlled convergence theorems for the strong Henstock integrals of fuzzy-number-valued functions*, Fuzzy Sets Syst. **160** (2009), 1528–1546.

[5] R. Gordon, *The Integral of Lebesgues, Denjoy, Perron and Henstock*, Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 1994.

[6] Z. Guang-Quan, *Fuzzy continuous functions and its properties*, Fuzzy Sets Syst. **43** (1991), 159–171.

[7] M.E. Hamid, A.H. Elmuiz, and M.E. Sheima, *On AP-Henstock integral interval of valued functions and fuzzy-valued functions*, Appl. Math. 7 (2016), no. 18, 2285–2295.

[8] R. Henstock, *The General Theory of Integration*, Oxford University Press, Oxford, UK, 1991.

[9] V.O. Iluebe and A.A. Mogbademu, *Dominated and bounded convergence results of sequential Henstock Stieltjes integral in real valued space*, J. Nepal Math. Soc. **3** (2020), no. 1, 17–20.

[10] V.O. Iluebe and A.A. Mogbademu, *Equivalence of Henstock and certain sequential Henstock integral*, Bangmond Int. J. Math. Comput. Sci. **1** (2020). no. 1-2, 9–16.

[11] V.O. Iluebe and A.A. Mogbademu, *Equivalence of P-Henstock type*, Ann. Math. Comput. Sci. **2** (2021), 15–22.

[12] K.A. Musial, *A decomposition theorem for Banach space valued fuzzy Henstock integral*, Fuzzy Sets Syst. **250** (2015), 21–28.

[13] L.A. Paxton, *Sequential approach to the Henstock integral*, arXiv preprint arXiv:1609.05454, 2016 - arxiv.org.

[14] C. Swartz, *Introduction to Gauge Integrals*, World Scientific, Singapore, 2001.

[15] A. Van der Schaft, A.E. Sterk, and R. Van Dijk, *The Henstock-Kurzweil integral*, Bachelor Thesis in Mathematics, Rijkuniversiteit Groningen, 2014.

[16] C.X. Wu, and Z.T. Gong, *On Henstock integrals of interval-valued functions and fuzzy-valued functions*, Fuzzy Set Syst. **115**, (2016), 377-391.

[17] C.X. Wu and Z.T. Ming, *On embedded problem of fuzzy number space: Part I*, Fuzzy Set Syst. **44** (1991), 33–38.

[18] B.M.A. Uzzal, *On convergence theorem for fuzzy Henstock integrals*, Iran. J. Fuzzy Syst. **14** (2017), 87–102.

[19] X.Y. You and D. Zhao, *On convergence theorems for the McShane integrals of interval-valued functions on time scale*, J. Chungcheong Math. Soc. **25** (2012), 109–115.