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Solving split equality monotone inclusion problem of maximal monotone mappings in Banach spaces

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Abstract

A new iterative scheme for approximating a solution of the split equality monotone inclusion problem (SEMIP) of maximal monotone mappings in the setting of Banach spaces is introduced. Strong convergence of the sequence generated by the proposed scheme to a solution of the SEMIP is then derived without prior knowledge of operator norms of the linear operators involved. In addition, we give some applications of our method and provide numerical examples to illustrate the convergence of the proposed scheme. Our results generalize, improve and extend many results in the literature.

Keywords: Maximal monotone map, zero point, monotone map, split equality inclusion problem
2020 MSC: 47JO5, 47J25, 47H10, 49M27, 47H09, 90C25

1 Introduction

Let E be a real Banach space with its dual E^*. A mapping $S : E \to 2^{E^*}$ is is called monotone if

$$\langle p^* - q^*, p - q\rangle \geq 0,\ \forall (p, p^*), (q, q^*) \in Gph(S),$$

where $Gph(S) = \{(p, p^*) \in E \times E^* : p^* \in Sp\}$ is a graph of S. A monotone mapping is called maximal monotone provided that its graph is not properly contained in a graph of any other monotone mapping. The resolvent of maximal monotone mapping S denoted by Res^g_S, is defined as

$$Res^g_S(p) := (\nabla g + \gamma S)^{-1}\nabla g(p),$$

where ∇g is the gradient of a convex function $g : E \to (\infty, \infty]$ satisfies certain conditions. This resolvent operator enjoys important properties such as single valued and Bregman firmly nonexpansive (see, [5] Prop. 3.8 (iv), pp. 604). Let $S : E \to 2^{E^*}$ be a monotone mapping. The problem of finding a point p^* in E such that

$$0 \in Sp^*, \tag{1.1}$$

is called monotone inclusion problem. This problem has been studied extensively by several authors (see, e.g., [8, 12, 13, 16, 18, 22, 32, 34, 35]). The solution set of the problem (1.1), above is denoted by $S^{-1}(0)$ and $S^{-1}(0) = F(Res^g_S)$, where $F(Res^g_S)$ is the set of fixed points of Res^g_S and S is maximal monotone mapping. One of the generalization of

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the problem (1.1) above is split equality monotone inclusion problem and formulated as the problem of finding p^* and q^* with the property:

$$p^* \in C, q^* \in D, \text{ such that } Ap^* = Bq^*, \tag{1.2}$$

where C and D are closed and convex subsets of H_1 and H_2, respectively, $S : H_1 \to H_1$ and $T : H_2 \to H_2$ are nonlinear mappings and $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear operators. This problem was introduced by Moudafi [19] and has received much attention of researchers due to its applications in many disciplines (see, e.g., [1, 2]). Consequently, it has been studied by several authors in both Hilbert and Banach spaces (see, e.g., [10, 12, 14, 15, 19, 30, 36]).

In 2015, Gua et al. [14] established an iterative algorithm and proved strong convergence of a sequence proposed in their method to a solution of problem (1.2) above in Hilbert spaces. In 2020, Wega and Zegeye [30] introduced an iterative scheme for problem (1.2) and proved a sequence proposed in their algorithm convergence strongly to a solution of the problem for the sum of two maximal monotone mappings in Hilbert spaces. Recently, in 2021 Jolaoso et al. [15] constructed an scheme for problem (1.2) and proved strong convergence of a sequence proposed in their scheme to a solution of the problem for maximal monotone mappings in p-uniformly convex and uniformly smooth Banach spaces.

We note that if in problem (1.2), we take $H_2 = H_3$ and $B = I$, where I is identity map on H_2, the SEMIP reduced to split monotone inclusion problem (SMIP). The SMIP mathematically formulated as as the problem of finding p^* with the property:

$$p^* \in C \text{ such that } q^* = Ap^* \in D. \tag{1.3}$$

Now, we consider the following split equality monotone inclusion problem (SEMIP). Let $S : E_1 \to 2^{E_1^*}$ and $T : E_2 \to 2^{E^*}$ be maximal monotone mappings, where E_1, E_2 and E_3 are Banach spaces with their dual spaces E_1^*, E_2^* and E_3^*, respectively. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear operators with their adjoints $A : E_3^* \to E_1$ and $B : E_3^* \to E_2$, respectively. The SEMIP in the setting of can be formulated as the problem of finding $(p^*, q^*) \in E_1 \times E_2$ such that

$$p^* \in S^{-1}(0) \text{ and } q^* \in T^{-1}(0) \text{ such that } Ap^* = Bq^*. \tag{1.4}$$

Question Can we introduce an iterative algorithm which converges strongly to a solution of SEMIP for maximal monotone mappings in real reflexive Banach spaces?

Inspired and motivated by the research works of Moudafi [19], Gua et al. [14], Wega and Zegeye [30] and Jolaoso et al. [15] it is our purpose in this paper to introduce and study new iterative scheme for solving SEMIP for maximal monotone mappings in the setting of Banach spaces. Strong convergence of the sequence generated by the proposed scheme to a solution of the SEMIP for maximal monotone mappings is proved without prior knowledge of operator norms of the linear operators involved. Some applications of our main result is also provided. Numerical examples are given to illustrate the convergency of the sequence generated by the proposed scheme. Our main result provide an affirmative answer to our concern. Our result generalize, improve and extend many research works in the literature.

2 Preliminaries

In this section, we recall some useful results which will be used in the sequel. Hereafter, in this paper let E be a real reflexive Banach space with its dual space E^*, C be a nonempty, convex and closed subset of E and let $\mathcal{G}$ be a family of proper, lower semi-continuous and convex functions.

Let g be an element of $\mathcal{G}$. The domain of g, *dom* g, is given by $dom\ g = \{p \in E : g(p) < \infty\}$, the Fenchel conjugate of g at p^*, $g^*(p^*)$, is given by $g^*(p^*) = sup\{\langle p^*, p\rangle - g(p) : p \in E \text{ and } p^* \in E^*\}$, the subdifferential of g at p, $\partial g(p)$, is given by $\partial g(p) = \{p^* \in E^* : g(q) \geq g(p) + \langle p^*, q - p\rangle, \forall p \in E\}$, the right hand derivative of g at u in the direction of q, $g'(p, q)$, is given by

$$g'(p, q) = \lim_{s \to 0_+} \frac{g(p + sq) - g(p)}{s}, \tag{2.1}$$

and the gradient of g, at p is a linear function, $\nabla\ g$, is given by $\langle \nabla g(p), q\rangle = g'(p, q)$.

Definition 2.1. The function g is called:

(i) Gâteaux differentiable at p element of E if the limit in (2.1) exists for any q in E as $s \to 0$.

(ii) Gâteaux differentiable if it is Gâteaux differentiable at every element u in *int domg*.
(iii) uniformity Fréchet differentiable on C if the limit as $s \to 0$ in (2.1) attained uniformly for $p \in C$ and $||q|| = 1$.
(iv) Strongly coercive if $\lim_{||p||\to\infty} \frac{g(p)}{||p||} = \infty$.

Definition 2.2. Gâteaux differentiable function g is called Legendre if g^* is Gâteaux differentiable, both *int dom g* and *int dom* g^* are nonempty, *dom* ∇ $g =$ *int dom g* and *dom* ∇ $g^* =$ *int dom* g^*.

Remark 2.3. $\nabla\ g^* = (\nabla\ g)^{-1}$ (see, [9]) provided that g is Legendre function and the gradient of Legendre function g defined by $g(u) = \frac{||u||^p}{p}$ is coincides with the generalized duality map, that is, $\nabla g\ = J_p$, where $(1 < p, q < \infty)$ and q is a conjugate of p (see, e.g., [4]).

Definition 2.4. The Bregman distance with respect to g (see, e.g., [11]) is a function $D_g : dom\ g \times int\ dom\ g \to [0, \infty)$ defined by

$$D_g(q, p) = g(q) - g(p) - \langle \nabla g(p), q - p\rangle, \tag{2.2}$$

where g is Gâteaux differentiable. The Bregman projection with respect to g at u in *int dom g* onto C is denoted by $P^g_C p$ defined by $D_g(P^g_C p, p) = \inf\{D_g(q, p) : \forall q \in C\}$.

Remark 2.5. We note that the Bregiman distance is not distance in the usual sense. However, it has the following properties (see, e.g., [7, 25, 26]):

(i) The three point identity:

$$D_g(p, q) + D_g(q, w) - D_g(p, w) = \langle\nabla\ g(w) - \nabla\ g(q), p - q\rangle \tag{2.3}$$

for all $q \in dom\ g$ and $p, w \in int\ dom\ g$.

(ii) The four point identity:

$$D_g(q, p) + D_g(q, z) - D_g(w, p) + D_g(w, z) - \langle\nabla\ g(z) - \nabla\ g(p), q - w\rangle, \tag{2.4}$$

for all $q, w \in dom\ g$ and $p, z \in int\ dom\ g$.

Lemma 2.6. [6] Let g be a totally convex and Gáteaux differentiable on *int domg*. Let $p \in int\ domg$. Then, the P^g_c from E onto C is a unique point with the following properties:

(i) $\langle\nabla\ g(p) - \nabla\ g(z), q - z \leq 0$ if and only if $z = P^g_C p$, $\forall q \in C$.
(ii) $D_g(p, q) \geq D_g(q, P^g_C p) + D_g(P^g_C p, p)$, $\forall q \in C$.

Let g be a Legendre and $V_g : E \times E^* \to [0, \infty)$ be a function defined by

$$V_g(p, p^*) = g(p) - \langle p^*, p\rangle + \nabla\ g^*(q^*), \forall p \in E, p^* \in E^*. \tag{2.5}$$

Then, V_g is nonnegative which satisfies (see, e.g., [28])

$$V_g(p, p^*) = D_g(p, \nabla\ g^*(p^*)) \tag{2.6}$$

and

$$V_g(p, p^*) \leq V_g(p, p^* + q^*) - \langle q^*, \nabla\ g^*(p^*) - p\rangle, \tag{2.7}$$

for all $p \in E$ and $p^* \in E^*$.

Lemma 2.7. [23] If g is lower, convex, semi-convex proper function, then g^* is a weak* lower semi-convex and proper function and hence, we have

$$D_g\Big(w, \nabla\ g^*\Big(\sum_{i=1}^{N} s_i \nabla g(p_i)\Big)\Big) \leq \sum_{i=1}^{N} s_i D_g(w, p_i),$$

for all w in E, where $\{p_i\} \subseteq E$ and $\{s_i\} \subseteq (0, 1)$ with $\sum_{i=1}^{N} s_i = 1$.

Definition 2.8. A Gâteaux differentiable function g is called

(i) uniformly convex function (see, [33]), provided that for all p and q $dom\ g$ and $s \in [0,1]$, we have

$$g(sp + (1-s)q) \leq sg(p) + (1-s)g(p) - (1-s)s\phi(||p-q||), \tag{2.8}$$

where ϕ is a function that is increasing and vanishes only at zero.

(ii) strongly convex with constant $\alpha > 0$ for all u and q elements of $domg$ (see, [21])

$$\langle \nabla\ g(p) - \nabla\ g(p), p-q\rangle \geq \alpha||p-q||^2. \tag{2.9}$$

(iii) totaly convex if $\nu_g(p,s) = \inf_{\{p\in E:||p-q||=s\}} D_g(q,p) > 0$, for all $p \in E$ and $s > 0$.

We note that g is uniformly convex if and only if g is totally convex on bounded subsets of E (see, [6], Theorem 2.10 p. 9). Moreover, the class of uniformly convex function functions contains the class of strongly convex functions.

Definition 2.9. A mapping $T: C \to E$ with $\widehat{F}(T) \neq \emptyset$ is said to be

(i) Bregman strongly nonexpansive [25] with respect to $\widehat{F}(T)$, if

$$D_g(p^*,\ Tp) \leq D_g(p^*,\ p),\ \forall\ p \in C,\ p^* \in \widehat{F}(T)$$

and, whenever $\{p_n\} \subseteq C$ is bounded $p^* \in\ \widehat{F}(T)$, and

$$\lim_{n\to\infty}(D_g(p^*,\ p_n) - D_g(p^*,\ Tp_n)) = 0,$$

it follow that $\lim_{n\to\infty} D_g(p_n,\ Tp_n) = 0$.

(ii) Bregman firmly nonexpansive [24] if for each $p, q \in\ C$

$$\langle \nabla\ g(Tp) - \nabla\ g(Tq), Tp - Tq\rangle \leq \langle \nabla\ g(p) - \nabla\ g(q), Tp - Tq\rangle.$$

We remark that if T is a Bregman firmly nonexpansive map and g is a Legender function which is bounded uniformly Frèchet differentiable and totaly convex on bounded subset of E, then it is knowen that $F(T) = \widehat{F}(T)$ and $F(T)$ is closed and convex (see, [24]) and hence every Bregman firmly nonexpansive map is Bregman strongly nonexpansive with respect to $F(T) = \widehat{F}(T)$.

3 Main results

Hereafter, let E_1, E_2 and E_3 be real reflexive Banach spaces with its dual E_1^*, E_2^* and E_3^*, respectively. In the sequel, we shall make the following conditions.

Conditions:

(C1) Let $g_1 : E_1 \to (-\infty, +\infty] \in \mathcal{G}(E_1)$, $g_2 : E_2 \to (-\infty, +\infty] \in \mathcal{G}(E_2)$ and $g_3 : E_3 \to (-\infty, +\infty] \in \mathcal{G}(E_3)$ be bounded, strongly coercive, uniformly Fréchet differentiable Legendre function on bounded subsets of E_1, E_2 and E_3, respectively and strongly convex with constants α_1, α_2 and α_3, respectively, and $\alpha = \min\{\alpha_1, \alpha_2, \alpha_3\}$.

(C2) Let $S : E_1 \to 2^{E_1^*}$ and $T : E_2 \to 2^{E^*}$ be maximal monotone mappings with $Res_S^{g_1} := (\nabla g_1 + \lambda S)^{-1}\nabla g_1$ and $Res_T^{g_2} := (\nabla g_2 + \lambda T)^{-1}\nabla g_2$, respectively, where $\lambda > 0$.

(C3) Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear operators with adjoints $A : E_3^* \to E_1$ and $B : E_3^* \to E_2$, respectively.

(C4) Let $\Omega = \{(p,q) : p \in S^{-1}(0) \text{ and } q \in T^{-1}(0) \text{ such that } Ap = Bq\} \neq \emptyset$.

(C5) Let $\{\alpha_n\} \subset (0,\epsilon) \subset (0,1)$, for some constant $\epsilon > 0$, be such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

(C6) Let $0 < \mu \leq \gamma_n \leq \frac{\alpha^2||Ap_n - Bq_n||^2}{2[||A^*(\nabla\ g_3Ap_n - \nabla\ g_3Bq_n)||^2 + ||B^*(\nabla\ g_3Ap_n - \nabla\ g_3Bq_n)||^2]}$ for $n \in \Upsilon = \{n \in \mathbb{N} : Ap_n - Bq_n \neq 0\}$, otherwise $\gamma_n = \gamma > 0$.

Theorem 3.1. Suppose conditions $(C1)$-$(C6)$ hold. For any $(p_0, q_0), (p,q) \in E_1 \times E_2$, define an iterative algorithm by

$$\begin{cases} x_n = Res_S^{g_1}\big(\nabla\ g_1^*[\nabla\ g_1p_n - \gamma_nA^*(\nabla\ g_3Ap_n - \nabla\ g_3Bq_n)]\big), \\ y_n = Res_T^{g_2}\big(\nabla\ g_2^*[\nabla\ g_2q_n - \gamma_nB^*(\nabla\ g_3Bq_n - \nabla\ g_3Ap_n)]\big), \\ p_{n+1} = \nabla\ g_1^*(\alpha_n\nabla\ g_1p + (1-\alpha_n)\nabla\ g_1x_n), \\ q_{n+1} = \nabla\ g_2^*(\alpha_n\nabla\ g_2q + (1-\alpha_n)\nabla\ g_2y_n). \end{cases} \tag{3.1}$$

Then, the sequence generated by Algorithm 3.1 is bounded in $E_1 \times E_2$.

Proof . Let $(p^*, q^*) \in \Omega$. Then, $p^* \in S^{-1}(0)$, $q^* \in T^{-1}(0)$ and $Ap^* = Bq^*$. Hence, from (3.1) and Lemma 2.7, we get

$$\begin{aligned} D_{g_1}(p^*, p_{n+1}) &= D_{g_1}(p^*, \nabla\ g_1^*(\alpha_n \nabla\ g_1 p + (1-\alpha_n)\nabla\ g_1 x_n)) \\ &\leq \alpha_n D_{g_1}(p^*, p) + (1-\alpha_n) D_{g_1}(p^*, x_n). \end{aligned} \tag{3.2}$$

Similarly, we get

$$\begin{aligned} D_{g_2}(q^*, q_{n+1}) &= D_{g_2}(q^*, \nabla\ g_2^*(\alpha_n \nabla\ g_2 q + (1-\alpha_n)\nabla\ g_2 y_n)) \\ &\leq \alpha_n D_{g_2}(q^*, q) + (1-\alpha_n) D_{g_2}(q^*, y_n). \end{aligned} \tag{3.3}$$

Now, denote $a_n = \nabla\ g_1^*[\nabla\ g_1 p_n - \gamma_n A^*(\nabla\ g_3 A p_n - \nabla\ g_3 B q_n)]$ and
$b_n = \nabla\ g_2^*[\nabla\ g_2 q_n - \gamma_n B^*(\nabla\ g_3 B q_n - \nabla\ g_3 A p_n)]$. Then, from Lemma 2.6 and (2.6), we obtain

$$\begin{aligned} D_{g_1}(p^*, x_n) &= D_{g_1}(p^*, Res_S^{g_1}\big(\nabla\ g_1^*[\nabla\ g_1 p_n - \gamma_n(\nabla\ g_3 A p_n - \nabla\ g_3 B q_n)]\big) \\ &\leq V_{g_1}(p^*, \nabla\ g_1 p_n - \gamma_n A^*(\nabla\ g_3 A p_n - \nabla\ g_3 B q_n)) \\ &\leq D_{g_1}(p^*, p_n) - \gamma_n \langle A a_n - Ap^*, \nabla\ g_3 A p_n - \nabla\ g_3 B q_n\rangle. \end{aligned} \tag{3.4}$$

Similarly, we get that

$$D_{g_2}(q^*, y_n) \leq D_{g_2}(q^*, q_n) - \gamma_n \langle B b_n - Bq^*, \nabla\ g_3 B q_n - \nabla\ g_3 A p_n\rangle. \tag{3.5}$$

Now, from (3.2) and (3.4), we obtain

$$\begin{aligned} D_{g_1}(p^*, p_{n+1}) &\leq \alpha_n D_{g_1}(p^*, p) + (1-\alpha_n)\alpha_n D_{g_1}(p^*, x_n) \\ &\leq \alpha_n D_{g_1}(p^*, p) + (1-\alpha_n) D_{g_1}(p^*, p_n) \\ &\quad -(1-\alpha_n)\gamma_n \langle A a_n - Ap^*, \nabla\ g_3 A p_n - \nabla\ g_3 B q_n\rangle \end{aligned} \tag{3.6}$$

Similarly, from (3.3) and (3.5), we get

$$D_{g_2}(q^*, q_{n+1}) \leq \alpha_n D_{g_2}(q^*, q) + (1-\alpha_n) D_{g_2}(q^*, q_n) - (1-\alpha_n)\gamma_n \langle B b_n - Bq^*, \nabla\ g_3 B q_n - \nabla\ g_3 A p_n\rangle. \tag{3.7}$$

Denote $\Gamma_n = D_{g_1}(p^*, u_n) + D_{g_2}(q^*, v_n)$ and $\Gamma = D_{g_1}(p^*, p) + D_{g_2}(q^*, q)$. Then by adding inequalities (3.6) and (3.7), we get

$$\begin{aligned} \Gamma_{n+1} &\leq \alpha_n \Gamma + (1-\alpha_n)\Gamma_n - (1-\alpha_n)\gamma_n \langle A a_n - Ap^* + Bq^* - B b_n, \nabla\ g_3 A p_n - \nabla\ g_3 B q_n\rangle \\ &= \alpha_n \Gamma + (1-\alpha_n)\Gamma_n - (1-\alpha_n)\gamma_n \langle A a_n - B b_n, \nabla\ g_3 A p_n - \nabla\ g_3 B q_n\rangle \end{aligned} \tag{3.8}$$

Observe that,

$$\begin{aligned} -\gamma_n \langle A a_n - B b_n, \nabla g_3 A p_n - \nabla g_3 B q_n\rangle &= -\gamma_n \langle A p_n - B q_n, \nabla g_3 A p_n - \nabla g_3 B q_n\rangle - \gamma_n \langle a_n - p_n, A^*(\nabla\ g_3 A p_n - \nabla g_3 B q_n)\rangle \\ &\quad - \gamma_n \langle q_n - b_n, B^*(\nabla\ g_3 A p_n - \nabla\ g_3 B q_n)\rangle \\ &\leq -\alpha ||A p_n - B q_n||^2 + ||a_n - p_n|| ||A^*(\nabla\ g_3 A p_n - \nabla\ g_3 B q_n)|| \\ &\quad + ||b_n - q_n|| ||B^*(\nabla\ g_3 A p_n - \nabla\ g_3 B q_n)|| \end{aligned} \tag{3.9}$$

Moreover, from definitions of g_1 and a_n, we obtain

$$\begin{aligned} ||p_n - a_n|| &= ||\nabla\ g_1^*(\nabla\ g_1 p_n) - \nabla\ g_1^*[\nabla\ g_1 p_n - \gamma_n A^*(\nabla\ g_3 A p_n - \nabla\ g_3 B q_n)]|| \\ &\leq \frac{\gamma_n}{\alpha} ||A^*(\nabla\ g_3 A p_n - \nabla\ g_3 B q_n)||. \end{aligned} \tag{3.10}$$

Similarly, from definitions of g_2 and b_n, we get

$$||q_n - b_n|| \leq \frac{\gamma_n}{\alpha} ||B^*(\nabla\ g_3 A p_n - \nabla\ g_3 B q_n)||. \tag{3.11}$$

Hence, from (3.9), (3.10) and (3.11), we get

$$\begin{aligned}
-\gamma_n\langle Aa_n - Bb_n, \nabla\, g_3Ap_n - \nabla\, g_3Bq_n\rangle \leq & -\gamma_n\alpha||Ap_n - Bq_n||^2 + \frac{\gamma_n^2}{\alpha}||A^*(\nabla\, g_3Ap_n - \nabla\, g_3Bq_n)||^2 \\
& + \frac{\gamma_n^2}{\alpha}||B^*(\nabla\, g_3Ap_n - \nabla\, g_3Bq_n)||^2 \\
\leq & -\frac{\mu\alpha}{2}||Ap_n - Bq_n||^2 - \gamma_n\Big[\frac{\alpha}{2}||Ap_n - Bq_n||^2 \\
& - \frac{\gamma_n}{\alpha}\Big(||A^*(\nabla\, g_3Ap_n - \nabla\, g_3Bq_n)||^2 + ||B^*(\nabla\, g_3Ap_n - \nabla\, g_3Bq_n)||^2\Big)\Big] \\
\leq & -\frac{\mu\alpha}{2}||Ap_n - Bq_n||^2. \qquad (3.12)
\end{aligned}$$

Finally, from (3.8) and (3.12), we get

$$\Gamma_{n+1} \leq \alpha_n\Gamma + (1-\alpha_n)\Gamma_n - (1-\alpha_n)\frac{\mu\alpha}{2}||Ap_n - Bq_n||^2 \qquad (3.13)$$

$$\leq \alpha_n\Gamma + (1-\alpha_n)\Gamma_n \qquad (3.14)$$

and hence by induction we get

$$\Gamma_n \leq \max\{\Gamma_0, \Gamma\}. \qquad (3.15)$$

Hence, the sequence $\{\Gamma_n\}$ is bounded. Thus, by Lemma 7 in [29], $\{p_n\}$ and $\{q_n\}$ are bounded and hence the sequences $\{x_n\}$, $\{y_n\}$, $\{Sx_n\}$, $\{Ty_n\}$ are bounded. □

Theorem 3.2. Suppose conditions $(C1)$-$(C6)$ hold. For any $(p_0, q_0), (p, q) \in E_1 \times E_2$, define an iterative algorithm by

$$\begin{cases}
x_n = Res_S^{g_1}\big(\nabla\, g_1^*[\nabla\, g_1p_n - \gamma_nA^*(\nabla\, g_3Ap_n - \nabla\, g_3Bq_n)]\big), \\
y_n = Res_T^{g_2}\big(\nabla\, g_2^*[\nabla\, g_2q_n - \gamma_nB^*(\nabla\, g_3Bq_n - \nabla\, g_3Ap_n)]\big), \\
p_{n+1} = \nabla\, g_1^*(\alpha_n\nabla\, g_1p + (1-\alpha_n)\nabla\, g_1x_n), \\
q_{n+1} = \nabla\, g_2^*(\alpha_n\nabla\, g_2q + (1-\alpha_n)\nabla\, g_2y_n).
\end{cases} \qquad (3.16)$$

Then, the sequence generated by Algorithm 3.16 converges strongly to an element $(p^*, q^*) = P_\Omega^g(p, q)$.

Proof . Let $(p^*, q^*) \in \Omega$ such that $(p^*, q^*) = P_\Omega^g(p, q)$. From Theorem 3.1, we have that the sequence $\{(p_n, q_n)\}$ is bounded. Then, using the same techniques of Theorem 2 of [34], we obtain

$$D_{g_1}(p^*, p_{n+1}) \leq (1-\alpha_n)D_{g_1}(p^*, p_n) + \alpha_n||p_n - p_{n+1}||||\nabla\, g_1p - \nabla\, g_1p^*|| + \alpha_n\langle\nabla\, g_1p - \nabla\, g_1p^*, p_n - p^*\rangle \qquad (3.17)$$

and

$$D_{g_2}(q^*, q_{n+1}) \leq (1-\alpha_n)D_{g_2}(q^*, q_n) + \alpha_n||q_n - q_{n+1}||||\nabla\, g_2q - \nabla\, g_2q^*|| + \alpha_n\langle\nabla\, g_2q - \nabla\, g_2q^*, q_n - q^*\rangle \qquad (3.18)$$

Thus, by adding inequalities (3.17) and (3.18), we obtain

$$\begin{aligned}
\Gamma_{n+1} \leq & (1-\alpha_n)\Gamma_n + \alpha_n||q_n - p_{n+1}||||\nabla\, g_1p - \nabla\, g_1p^*|| + \alpha_n||q_n - q_{n+1}||||\nabla\, g_2q - \nabla\, g_2q^*|| \\
& + \alpha_n\langle\nabla\, g_1p - \nabla\, g_1p^*, p_n - p^*\rangle + \alpha_n\langle\nabla\, g_2q - \nabla\, g_2q^*, v_n - q^*\rangle, \qquad (3.19)
\end{aligned}$$

where $\Gamma_n = D_{g_1}(p^*, p_n) + D_{g_2}(q^*, q_n)$. Now, to complete our proof we consider the following two cases.

Case 1. Suppose there exists $n_0 \in \mathbb{N}$ such that the sequence of real numbers Γ_n is decreasing for all $n \geq n_0$. Thus, the sequence Γ_n is convergence and hence $\Gamma_n - \Gamma_{n+1} \to 0$ as $n \to \infty$. Hence, from (3.13) and the conditions on the sequence $\{\alpha_n\}$, we get

$$\lim_{n\to\infty}||Ap_n - Bq_n|| = 0, \qquad (3.20)$$

and the fact that $\nabla\ g_1^*$ is uniformly continuous on bounded subset of E_1^* (see, [33]) yields

$$\lim_{n\to\infty} ||p_{n+1}-x_n|| = 0. \tag{3.21}$$

Similarly, we obtain

$$\lim_{n\to\infty} ||q_{n+1}-y_n|| = 0. \tag{3.22}$$

In addition, from Lemma 2.6, we obtain

$$\begin{aligned} D_{g_1}(p_n,x_n) =& D_{g_1}(p_n, Res_S^{g_1}\left(\nabla\ g_1^*[\nabla\ g_1p_n-\gamma_nA^*(\nabla\ g_3Ap_n-\nabla\ g_3Bq_n)]\right) \\ \leq& V_{g_1}(p_n,\nabla\ g_1p_n-\gamma_nA^*(\nabla\ g_3Ap_n-\nabla\ g_3Bq_n)) \\ \leq& V_{g_1}(p_n,\nabla\ g_1p_n)-\langle\gamma_nA^*(\nabla\ g_3Ap_n-\nabla\ g_3Bq_n),\nabla\ g_1^*[\nabla\ g_1p_n-\gamma_nA^*(\nabla\ g_3Ap_n-\nabla\ g_3Bq_n)]-p_n\rangle \\ \leq& D_{g_1}(p_n,p_n)+\frac{\gamma_n^2}{\alpha}||A||^2||Ap_n-Bq_n||^2. \end{aligned} \tag{3.23}$$

Thus, from (3.20) and (3.23), we get

$$\lim_{n\to\infty} D_{g_1}(p_n,x_n) = 0, \tag{3.24}$$

which implies

$$\lim_{n\to\infty} ||p_n-x_n|| = 0. \tag{3.25}$$

Similarly, we get

$$\lim_{n\to\infty} ||q_n-y_n|| = 0. \tag{3.26}$$

Consequently, from (3.21) and (3.25), we get

$$||p_{n+1}-p_n|| \leq ||p_{n+1}-x_n||+||x_n-p_n|| \to 0 \quad as \quad n\to\infty. \tag{3.27}$$

Similarly, from (3.22) and (3.26), we get

$$||q_{n+1}-q_n|| \leq ||q_{n+1}-y_n||+||y_n-q_n|| \to 0 \quad as \quad n\to\infty. \tag{3.28}$$

Denote $a_n=\nabla\ g_1^*[\nabla\ g_1p_n-\gamma_nA^*(\nabla\ g_3Ap_n-\nabla\ g_3Bq_n)]$ and $b_n=\nabla\ g_2^*[\nabla\ g_2q_n-\gamma_nB^*(\nabla\ g_3Bq_n-\nabla\ g_3Ap_n)]$. Now, since the sequence $\{(p_n,q_n)\}$ is bounded in $E_1\times E_2$, there exists $(\hat{p},\hat{q})\in E_1\times E_2$ and a subsequence $\{(p_{n_k},q_{n_k})\}$ of (p_n,q_n) such that $\{(p_{n_k},q_{n_k})\}\rightharpoonup(\hat{p},\hat{q})$ and

$$\begin{aligned} &\limsup_{n\to\infty}[\langle p_n-p^*,\nabla\ g_1p-\nabla\ g_1p^*\rangle+\langle q_n-p^*,\nabla\ g_2q-\nabla\ g_2q^*\rangle] \\ &=\lim_{k\to\infty}[\langle p_{n_k}-p^*,\nabla\ g_1p-\nabla\ g_1p^*\rangle+\langle q_{n_k}-q^*,\nabla\ g_2q-\nabla\ g_2q^*\rangle]. \end{aligned} \tag{3.29}$$

It then follows that $p_{n_k}\rightharpoonup\hat{p}$ in E_1 and $q_{n_k}\rightharpoonup\hat{q}$ in E_1. From (3.25) and (3.26) $x_{n_k}\rightharpoonup\hat{p}$ in E_1 and $y_{n_k}\rightharpoonup\hat{q}$ in E_2, respectively. Next, we prove that $(\hat{p},\hat{q})\in S^{-1}(0)\times T^{-1}(0)$ and $A\hat{p}=B\hat{q}$. Suppose $z\in Sw$. For the fact that $x_{n_k}=Res_S^{g_1}a_{n_k}$, for each $\gamma>0$, that is,

$$\nabla\ g_1a_{n_k}\in(\nabla\ g_1+\lambda S)x_{n_k}, \tag{3.30}$$

and hence

$$\nabla\ g_1a_{n_k}-\nabla\ g_1x_{n_k}\in\lambda Sx_{n_k}. \tag{3.31}$$

From (3.31) and the fact S is maximal monotone map, we get

$$\langle z-\frac{\nabla\ g_1a_{n_k}-\nabla\ g_1x_{n_k}}{\lambda},w-x_{n_k}\rangle\geq 0, \tag{3.32}$$

this implies

$$\langle z,w-x_{n_k}\rangle\geq\langle\frac{\nabla\ g_1a_{n_k}-\nabla\ g_1x_{n_k}}{\lambda},w-x_{n_k}\rangle. \tag{3.33}$$

From equation (3.25) and the fact that $\nabla\ g_1$ is uniformly continuous on E_1, we get $\lim_{k\to\infty}||\nabla\ g_1x_{n_k} - \nabla\ g_1a_{n_k}|| = 0$. Hence , inequality (3.33) implies that $\langle w-\hat{p}, z\rangle \geq 0$ as $k\to\infty$. Thus, by maximality of S, we get $0\in S\hat{p}$. Similarly, we get that $0\in T\hat{q}$. Moreover, from (2.9), we obtain

$$\begin{aligned}||A\hat{p}-B\hat{q}||^2 \leq & \frac{1}{\alpha}\langle A\hat{p}-B\hat{q}, \nabla\ g_3A\hat{p}-\nabla\ g_3B\hat{q}\rangle \\ = & \frac{1}{\alpha}\langle Ap_{n_k}-Bq_{n_k}+A\hat{p}-Ap_{n_k}+Bq_{n_k}-B\hat{q}, \nabla\ g_3A\hat{p}-\nabla\ g_3B\hat{q}\rangle \\ \leq & \frac{1}{\alpha}||Ap_{n_k}-Bq_{n_k}||||\nabla\ g_3A\hat{p}-\nabla\ g_3B\hat{q}\rangle|| + \frac{1}{\alpha}[\langle A\hat{p}-Ap_{n_k}+Bq_{n_k}-B\hat{q}, \nabla\ g_3A\hat{p}-\nabla\ g_3B\hat{q}\rangle] \end{aligned} \quad (3.34)$$

From (3.34), (3.20), and the fact that $Ap_{n_k} \rightharpoonup \hat{p}$ and $Bq_{n_k} \rightharpoonup \hat{q}$, we conclude that $A\hat{p} = B\hat{q}$. Consequently, $(\hat{p},\hat{q})\in\Omega$. It follows from Lemma 2.6 (i), that

$$\begin{aligned}&\limsup_{n\to\infty}[\langle p_n-p^*, \nabla\ g_1p-\nabla\ g_1p^*\rangle + \langle q_n-p^*, \nabla\ g_2q-\nabla\ g_2q^*\rangle] \\ &= \lim_{k\to\infty}[\langle p_{n_k}-p^*, \nabla\ g_1p-\nabla\ g_1p^*\rangle + \langle q_{n_k}-q^*, \nabla\ g_2q-\nabla\ g_2q^*\rangle] \\ &= \langle \hat{p}-p^*, \nabla\ g_1p-\nabla\ g_1p^*\rangle + \langle \hat{q}-q^*, \nabla\ g_2q-\nabla\ g_2q^*\rangle \leq 0. \end{aligned} \quad (3.35)$$

Therefore, from (3.19), (3.27), (3.28), (3.35) and Lemma 2.5 of [31] p. 243, we conclude that $\Gamma_n\to 0$ as $n\to\infty$. Hence, by Lemma 2.4 of [20] p. 15, $p_n\to p^*$ and $q_n\to q^*$ as $n\to\infty$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\Gamma_{n_i} < \Gamma_{n_i+1}, \forall i\in\mathbb{N}. \quad (3.36)$$

Then, by Lemma 3.1 of [17] p. 904, there exists a nondecreasing sequence $\{l_m\}$ in the set of natural numbers such that $l_m\to\infty$ as $m\to\infty$, $\Gamma_{l_m}\leq\Gamma_{l_m+1}$ and $\Gamma_m\leq\Gamma_{l_m+1}$ for all m element of the set of natural numbers. Thus, from (3.13), we obtain

$$\lim_{m\to\infty}||Ap_{l_m}-Bq_{l_m}|| = 0. \quad (3.37)$$

Moreover, following the methods in Case 1, we get

$$\lim_{m\to\infty}||p_{l_m}-u_{m_l+1}|| = \lim_{m\to\infty}||q_{l_m}-v_{m_l+1}|| = 0 \quad (3.38)$$

and

$$\lim_{m\to\infty}[\langle p_{l_m}-p^*, \nabla\ g_1p-\nabla\ g_1p^*\rangle + \langle q_{l_m}-p^*, \nabla\ g_2q-\nabla\ g_2q^*\rangle] \leq 0. \quad (3.39)$$

In addition, from (3.17) and (3.36), we obtain

$$\begin{aligned}\Gamma_{l_m} \leq\ & ||p_{l_m}-p_{l_m+1}||||\nabla\ g_1p-\nabla\ g_1p^*|| + ||q_{l_m}-q_{l_m+1}||||\nabla\ g_2q-\nabla\ g_2q^*|| \\ & + \langle\nabla\ g_1p-\nabla\ g_1p^*, p_{l_m}-p^*\rangle + \langle\nabla\ g_2q-\nabla\ g_2q^*, q_{l_m}-q^*\rangle. \end{aligned} \quad (3.40)$$

Therefore, from (3.38), (3.39) and (3.40), we get $\lim_{m\to\infty}\Gamma_{l_m} = 0$. But from inequality (3.13), we obtain that $\lim_{m\to\infty}\Gamma_{l_m+1} = 0$ and hence the fact that $\Gamma_m\leq\Gamma_{l_m+1}$ implies $\lim_{m\to\infty}\Gamma_m = 0$. Thus, by Lemma 2.4 of [20] p. 15 $p_m\to p^*$ and $q_m\to q^*$ as $m\to\infty$. □

If in Theorem 3.2, we assume $E_2 = E_3$ and $B = I$, then SEMIP reduces to SMIP for maximal monotone mappings and the method of proof of Theorem 3.2 provides the following corollary for approximating a solution of SMIP for maximal monotone mappings in real reflexive Banach spaces.

Corollary 3.3. Suppose conditions $(C1)$, $(C2)$, and $(C5)-(C6)$ hold with $B = I$ and $E_2 = E_3$. Let $\Omega = \{(p^*,q^*)\in E_1\times E_2 : p^*\in S^{-1}(0)$ and $q^*\in T^{-1}(0)$ such that $Ap^* = q^*\}\neq\emptyset$. For any $(p_0,q_0),(p,q)\in E_1\times E_2$, define an iterative algorithm by

$$\begin{cases} x_n = Res_S^{g_1}\left(\nabla\ g_1^*[\nabla\ g_1p_n-\gamma_nA^*(\nabla\ g_3Ap_n-\nabla\ g_3q_n)]\right), \\ y_n = Res_T^{g_2}\left(\nabla\ g_2^*[\nabla\ g_2q_n-\gamma_n(\nabla\ g_3q_n-\nabla\ g_3Ap_n)]\right), \\ p_{n+1} = \nabla\ g_1^*(\alpha_n\nabla\ g_1p+(1-\alpha_n)\nabla\ g_1x_n), \\ q_{n+1} = \nabla\ g_2^*(\alpha_n\nabla\ g_2q+(1-\alpha_n)\nabla\ g_2y_n). \end{cases} \quad (3.41)$$

Then, the sequence generated by Algorithm 3.41 converges strongly to an element $(p^*, q^*) = P^g_\Omega(p, q)$.

If in Theorem 3.2,, we assume $E_1 = E_2 = E_3$ and $A = I = B$, then SEMIP reduces to common zero point of maximal monotone mappings in real reflexive Banach spaces and the method of proof of Theorem 3.2 provides the following corollary for approximating a solution of common zero point of maximal monotone mappings in real reflexive Banach spaces.

Corollary 3.4. Suppose conditions $(C1)$, $(C2)$ and $(C5) - (C6)$ hold with $E_1 = E_2 = E_3$, $A = I = B$ and $\Omega = \{(p^*, q^*) \in E_1 \times E_2 : p^* \in S^{-1}(0)$ and $q^* \in T^{-1}(0)$ such that $p^* = q^*\} \neq \emptyset$. For any $(p_0, q_0), (p, q) \in E_1 \times E_2$, define an iterative algorithm by

$$\begin{cases} x_n = Res^{g_1}_S\big(\nabla\, g_1^*[\nabla\, g_1 p_n - \gamma_n(\nabla\, g_3 p_n - \nabla\, g_3 q_n)]\big), \\ y_n = Res^{g_2}_T\big(\nabla\, g_2^*[\nabla\, g_2 q_n - \gamma_n(\nabla\, g_3 q_n - \nabla\, g_3 p_n)]\big), \\ p_{n+1} = \nabla\, g_1^*(\alpha_n \nabla\, g_1 p + (1-\alpha_n)\nabla\, g_1 x_n), \\ q_{n+1} = \nabla\, g_2^*(\alpha_n \nabla\, g_2 q + (1-\alpha_n)\nabla\, g_2 y_n). \end{cases} \tag{3.42}$$

Then, the sequence generated by Algorithm 3.42 converges strongly to an element $(p^*, q^*) = P^g_\Omega(p, q)$.

4 Application to Minimization Problem

In this section, we apply our main result Theorem 3.2 to approximate a solution of SEMPP for convex functions in real reflexive Banach spaces. Let $f : E \to \mathbb{R} \in \mathcal{G}(E)$ and $k : E \to \mathbb{R}$ be a convex smooth function. We consider the problem of approximating $p^* \in E_1$ and $q^* \in E_2$ such that

$$f(p^*) = \min_{p \in E_1}\{f(p)\}, h(q^*) = \min_{q \in E_2}\{h(q)\} \text{ and } Ap^* = Bq^*. \tag{4.1}$$

This problem is equivalent, by Fermat's rule, to the problem of finding $p^* \in E_1$ and $q^* \in E_2$ such that

$$0 \in \partial f(p^*) \;\; 0 \in \nabla\, k(q^*) \text{ and } Ap^* = Bq^*, \tag{4.2}$$

where ∂f is a subdifferential of f and $\nabla\, k$ is a gradient of k. We remark that both $\nabla\, k$ and ∂g are maximal monotone mappings (see, e.g., [3, 27]). One way of solving problem (4.2) is finding a solution of SEMIP for maximal monotone mappings of $S = \nabla\, \partial f$ and $T = \nabla\, k$ as zero points of S and T are minimum point of f and k, respectively. Thus, Algorithm 3.16 in Theorem 3.2 reduces to Algorithm 4.3 in Theorem 4.1 given below.

The method of proof of Theorem 3.2 provides the following theorem for approximating a solution of split equality minimum pint problem (SEMPP) for convex functions in real reflexive Banach spaces.

Theorem 4.1. Suppose conditions $(C1)$ and $(C3)-(C6$ hold. Let $f : E_1 \to \mathbb{R} \in \mathcal{G}(E_1)$ and $h : E_2 \to \mathbb{R}$ be a Gâteaux differentiable function such that $\Omega = \{(p^*, q^*) \in E_1 \times E_2 : f(p^*) = \min_{p \in E_1}\{f(p)\},\ k(q^*) = \min_{q \in E}\{k(q)\}$ and $Ap^* = Bq^*\} \neq \emptyset$. For any $(p_0, q_0), (p, q) \in E_1 \times E_2$, define an iterative algorithm by

$$\begin{cases} x_n = Res^{g_1}_{\partial f}\big(\nabla\, g_1^*[\nabla\, g_1 p_n - \gamma_n A^*(\nabla\, g_3 A p_n - \nabla\, g_3 B q_n)]\big), \\ y_n = Res^{g_2}_{\nabla k}\big(\nabla\, g_2^*[\nabla\, g_2 q_n - \gamma_n B^*(\nabla\, g_3 B q_n - \nabla\, g_3 A p_n)]\big), \\ p_{n+1} = \nabla\, g_1^*(\alpha_n \nabla\, g_1 p + (1-\alpha_n)\nabla\, g_1 x_n), \\ q_{n+1} = \nabla\, g_2^*(\alpha_n \nabla\, g_2 q + (1-\alpha_n)\nabla\, g_2 y_n). \end{cases} \tag{4.3}$$

Then, the sequence generated by Algorithm 4.3 converges strongly to an element $(p^*, q^*) = P^g_\Omega(p, q)$.

Proof . Consider $S = \partial f$ and $T = \nabla\, k$. Then, we get that S and T are maximal monotone mappings and zero points of S and T are minimum points of g and k, respectively. Thus, Theorem 3.2 provides the conclusion of Theorem 4.1. □

If in Theorem 4.1, we assume $E_2 = E_3$ and $B = I$, then SEMIP reduced to split minimum point problem (SMPP) for convex functions and the method of proof of Theorem 4.1 provides the following Corollary for approximating a solution of SMPP for convex functions in real reflexive Banach spaces.

Corollary 4.2. Suppose conditions $(C1)$ and $(C3)-(C6)$ hold with $B = I$ and $E_2 = E_3$. Let $f : E_1 \to \mathbb{R} \in \mathcal{G}(E_1)$ and $k : E_2 \to \mathbb{R}$ be a Gâteaux differentiable function such that $\Omega = \{(p^*, q^*) \in E_1 \times E_2 : f(p^*) = \min_{p\in E_1}\{f(p)\},\ k(q^*) = \min_{q\in E}\{k(q)\}$ and $Ap^* = q^*\} \neq \emptyset$. For any $(p_0, q_0), (p, q) \in E_1 \times E_2$, define an iterative algorithm by

$$\begin{cases} x_n = Res^{g_1}_{\partial f}\big(\nabla\ g_1^*[\nabla\ g_1 p_n - \gamma_n A^*(\nabla\ g_3 A p_n - \nabla\ g_3 q_n)]\big),\\ y_n = Res^{g_2}_{\nabla k}\big(\nabla\ g_2^*[\nabla\ g_2 q_n - \gamma_n(\nabla\ g_3 q_n - \nabla\ g_3 A p_n)]\big),\\ p_{n+1} = \nabla\ g_1^*(\alpha_n \nabla\ g_1 p + (1-\alpha_n)\nabla\ g_1 x_n),\\ q_{n+1} = \nabla\ g_2^*(\alpha_n \nabla\ g_2 q + (1-\alpha_n)\nabla\ g_2 y_n). \end{cases} \tag{4.4}$$

Then, the sequence generated by Algorithm 4.4 converges strongly to an element $(p^*, q^*) = P^g_\Omega(p, q)$.

If in Theorem 4.1, we assume $E_1 = E_2 = E_3$ and $A = I = B$, then SEMIP reduced tho a common minimum point problem (MPP) for convex functions and the method of proof of Theorem 4.1 provides the following corollary for approximating a solution of common minimum point problem for convex functions in Banach spaces.

Corollary 4.3. Suppose conditions $(C1)$ and $(C3)-(C6)$ hold with $E_1 = E_2 = E_3$ and $A = I = B$. Let $f : E_1 \to \mathbb{R} \in \mathcal{F}(E_1)$ and $k : E_2 \to \mathbb{R}$ be a Gâteaux differentiable function such that $\Omega = \{(p^*, q^*) \in E_1 \times E_2 : g(p^*) = \min_{p\in E_1}\{g(p)\},\ k(q^*) = \min_{q\in E}\{k(q)\}$ and $p^* = q^*\} \neq \emptyset$. For any $(p_0, q_0), (p, q) \in E_1 \times E_2$, define an iterative algorithm by

$$\begin{cases} x_n = Res^{g_1}_{\partial f}\big(\nabla\ g_1^*[\nabla\ g_1 p_n - \gamma_n A^*(\nabla\ g_3 A p_n - \nabla\ g_3 q_n)]\big),\\ y_n = Res^{g_2}_{\nabla k}\big(\nabla\ g_2^*[\nabla\ g_2 q_n - \gamma_n(\nabla\ g_3 q_n - \nabla\ g_3 A p_n)]\big),\\ p_{n+1} = \nabla\ g_1^*(\alpha_n \nabla\ g_1 p + (1-\alpha_n)\nabla\ g_1 x_n),\\ q_{n+1} = \nabla\ g_2^*(\alpha_n \nabla\ g_2 q + (1-\alpha_n)\nabla\ g_2 y_n). \end{cases} \tag{4.5}$$

Then, the sequence generated by Algorithm 4.5 converges strongly to an element $(p^*, q^*) = P^g_\Omega(p, q)$.

5 Numerical Examples

In this section, we provide numerical examples to illustrate the convergence of the sequence generated by the proposed scheme. The following numerical examples verify the conclusion of Theorem 3.2.

Example 5.1. Let $E = \mathbb{R}^2$ with the standard topology. Define $g_1, g_2, g_3 : \mathbb{R}^2 \to \mathbb{R}$, by $g_i(x) = \frac{x^2}{2}$, then $g_i(x^*) = \frac{1}{2}x^{*2}$ and $\nabla\ g_i(x) = x = \nabla\ g_i(x^*) = x^*$ for all $i = 1, 2, 3$, where $x \in \mathbb{R}^2$. Let $S, T, A, B : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $S(x) = (2x_1, 3x_2) - (1, 1)$, $T(x) = (3x_1, 2x_2) + (1, 1)$, $Ax = (2x_1, 3x_2)$ and $Bx = -(3x_1, 2x_2)$ were $x = (x_1, x_2) \in \mathbb{R}^2$, then S and T are maximal monotone mappings with $S^{-1}(0) = \{(\frac{1}{2}, \frac{1}{3})\}$, $T^{-1}(0) = \{-(\frac{1}{3}, \frac{1}{2})\}$ and $A(\frac{1}{2}, \frac{1}{3}) = (1, 1) = B(\frac{1}{3}, \frac{1}{2})$. Thus, $\Omega \neq \emptyset$. Now, if we assume $\lambda = 1$, $\alpha = 1$, $(p, q) = ((0.5, 1), (0.2, -1))$, $\alpha_n = \frac{1}{n+10^5}$ for all $n \geq 0$, and take different initial points $(p_0, q_0) = ((0, 1), (1, -1))$, $(p_0^{'}, q_0^{'}) = ((-1, 0), (0.5, 1))$ and $(p_0^{''}, q_0^{''}) = ((0.5, 1), (0.4, 0.5)$, then in all cases, the numerical experiment results using MATLAB provide that the sequence $\{(p_n, q_n)\}$ generated by Algorithm 3.16 converges strongly to $(p^*, q^*) = ((\frac{1}{2}, \frac{1}{3}), -(\frac{1}{3}, \frac{1}{2}))$ (see, Figure 5.1).

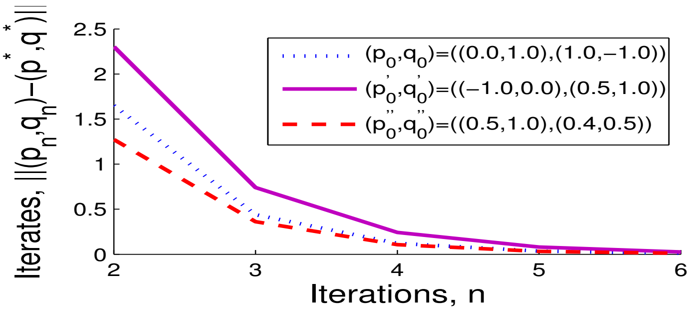

Figure 1: The graph of $||(p_n, q_n) - (p^*, q^*)||$ versus number of iterations with different choices of (p_0, q_0)

In addition, we have sketched the difference term $||Ap_n - Bq_n||$ for each initial point. From the sketch we observe that $||Ap_n - Bq_n|| \to 0$ as $n \to \infty$ (see, Figure 5.1).

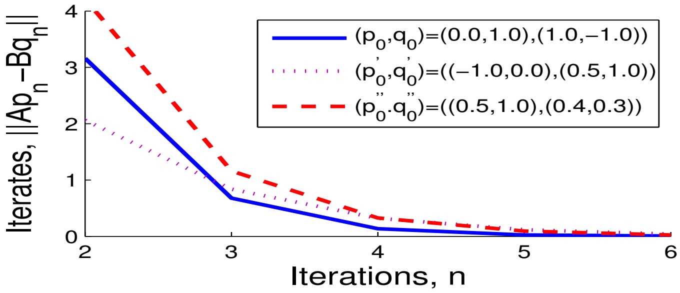

Figure 2: The graph of $||Ap_n - Bq_n||$ versus number of iterations with different choices of (p_0, q_0)

Example 5.2. Let $E = l_2$ with the standard topology. Define $g_1, g_2, g_3 : l_2 \to l_2$, by $g_i(x) = \frac{x^2}{2}$, then $g_i(x^*) = \frac{1}{2}x^{*2}$ and $\nabla\ g_i(x) = x = \nabla\ g_i(x^*) = x^*$ for all $i = 1, 2, 3$, where $x \in l_2$. Let $S, T, A, B : l_2 \to l_2$ be defined by

$$S(x) = (\frac{2x_1 - 1}{2}, \frac{3x_2 - 1}{4}, \frac{4x_3 - 1}{8}, \frac{x_4}{16}, \frac{x_5}{32}, ...),$$

$$T(x) = (\frac{x_1 - 2}{3}, \frac{2x_2 - 3}{9}, \frac{3x_3 - 4}{27}, \frac{x_4}{81}, \frac{x_5}{243}, ...),$$

$$Ax = (2x_1, 3x_2, 4x_3, 0, 0, ...) \ and \ Bx = (\frac{1}{2}x_1, \frac{2}{3}x_2, \frac{3}{4}x_3, 0, 0...),$$

were $x = (x_1, x_2, x_3, ...) \in l_2$, then the mappings S and T are maximal monotone with $S^{-1}(0) = \{(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 0, 0...)\}$, $T^{-1}(0) = \{(2, \frac{3}{2}, \frac{4}{3}, 0, 0...)\}$ and

$$A(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 0, 0...) = (1, 1, 1, 0, 0, ...) = B(2, \frac{3}{2}, \frac{4}{3}, 0, 0...).$$

Thus, $\Omega \neq \emptyset$. Now, if we assume $\lambda = 1$, $\alpha = 1$, $(p, q) = ((0, 0, 0, ...), (0, 0, 0, ...))$, $\alpha_n = \frac{1}{n+10^6}$ for all $n \geq 0$, and take different initial points

$$(p_0, q_0) = ((-1, -1, -2, 0, 0, ...), (-2, -1, 3, 0, 0...)),$$

$$(p_0^{'}, q_0^{'}) = ((2, 1, -1, 0, 0, ...), (1, 1, 1, 0, 0, ...))$$

and

$$(p_0^{''}, q_0^{''}) = ((1, 0, -1, 0, 0, ...), (-1, 2, 0, 0, 0, ...),$$

then in all cases, the numerical experiment result provide that the sequence $\{(p_n, q_n)\}$ generated by Algorithm 3.16 converges strongly to

$$(p^*, q^*) = ((\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 0, 0...), (2, \frac{3}{2}, \frac{4}{3}, 0, 0...)) \ \text{(see, Figure 5.2).}$$

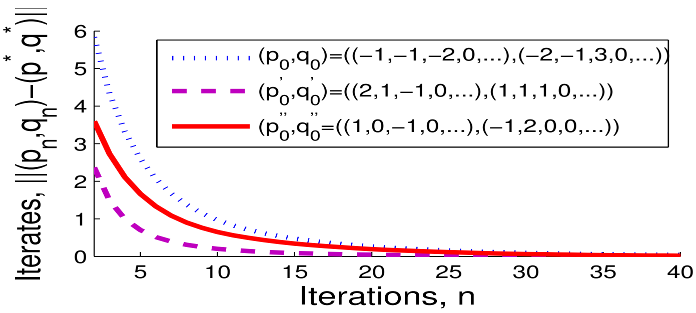

Figure 3: The graph of $||(p_n, q_n) - (p^*, q^*)||$ versus number of iterations with different choices of (p_0, q_0)

6 Conclusion

In this paper, we have established an iterative algorithm for solving SEMIP of maximal monotone mappings in real reflexive Banach spaces. We also proved a strong convergence theorem without prior knowledge of norm operators of the bounded linear operators A and B. We gave some applications of our main results. Numerical examples which validate the conclusion of our main result were provided. Our result generalize and extend many results in the literature. In particular, Theorem 3.2 extends the results of Moudafi [19], Gua et al. [14] and Wega and Zegeye [30] from Hilbert spaces to real reflexive Banach spaces and Theorem 3.1 of Jolaoso et al. [15] from uniformly convex and uniformly smooth Banach spaces to real reflexive Banach spaces.

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