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Two new regularity criteria for the 3D magneto-micropolar equations in anisotropic Lorentz spaces

Amjad Hussain[a], Muhammad Naqeeb[a,b,*]

[a] *Department of Mathematics, Quaid-I-Azam University 45320, Islamabad, Pakistan*

[b] *IBADAT International University, 6.7 km Japan Rd, Sihala, Islamabad, Pakistan*

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Abstract

In this study, we present two new regularity criteria based on pressure and its gradient to the Cauchy problem of the 3D magneto-micropolar system in anisotropic Lorentz spaces.

Keywords: Weak solutions, Pressure regularity, Inequalities, 3D magneto-micropolar equations, Anisotropic Lorentz spaces
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1 Introduction and preliminaries

We analyze the following 3D magneto-micropolar system in $\mathbb{R}^3 \times [0, T]$:

$$\begin{cases} \frac{\partial \mathcal{U}}{\partial t} + \mathcal{U} \cdot \nabla \mathcal{U} - (\kappa_1 + \kappa_2) \triangle \mathcal{U} + \nabla(\Psi + \mathcal{V}^2) - \kappa_2 \nabla \times \mathcal{W} - \mathcal{V} \cdot \nabla \mathcal{V} = 0, \\ \frac{\partial \mathcal{W}}{\partial t} - \kappa_3 \triangle \mathcal{W} + \mathcal{U} \cdot \nabla \mathcal{W} - \kappa_2 \nabla \times \mathcal{U} + 2\kappa_2 \mathcal{W} - \kappa_4 \nabla div\ \mathcal{W} = 0, \\ \frac{\partial \mathcal{V}}{\partial t} - \kappa_5 \triangle \mathcal{V} + \mathcal{U} \cdot \nabla \mathcal{V} - \mathcal{V} \cdot \nabla \mathcal{U} = 0, \\ \nabla \cdot \mathcal{U} = 0, \quad \nabla \cdot \mathcal{V} = 0, \\ \mathcal{U}(x,0) = \mathcal{U}_0(x), \quad \mathcal{W}(x,0) = \mathcal{W}_0(x), \quad \mathcal{V}(x,0) = \mathcal{V}_0(x), \end{cases} \tag{1.1}$$

where $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ are, respectively, the fluid velocity, magnetic field and the micro-rotational fluid velocity of the flow. The symbol Ψ represents pressure, while $\mathcal{U}_0$, $\mathcal{V}_0$ and $\mathcal{W}_0$ are the given initial velocity, magnetic field and micro-rotation velocity with $\nabla \cdot \mathcal{U}_0 = 0$ and $\nabla \cdot \mathcal{V}_0 = 0$ in the distributional sense. As the varying values of various viscosites and diffusitivity would not effect our system, so through out this article we take $\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = \kappa_5 = 1$.

In [7], Eringen initiated the study of system (1.1) for the case of zero magnetic field that is for micropolar fluids. These microstructured fluids can be used to simulate physiological fluids such as the cerebrospinal fluids that circulate through the brain. Polymers, suspensions, rheological materials, and other microstructural fluids require microscale spin simulation. See [18] and the references therein for further information on these types of fluids. Because of their vast industrial applications, the theory of micropolar fluids, for the last few decades has been a hot topic of

*Corresponding author
Email addresses: a.hussain@qau.edu.pk (Amjad Hussain), mnaqeem@math.qau.edu.pk (Muhammad Naqeeb)

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discussion among the researchers working in this field. In 1977, Galdi and Rionero [11] demonstrated the existence and uniqueness of weak solutions of the micropolar fluid equations. For various other regularity results regarding the micropolar fluid model (see [5, 10, 15, 23, 26]). Ignoring the micro-rotational effects, our system (1.1) models the magnetohydrodynamics flows. The finite-time singularity problem for MHD flows has been extensively tackled by different authors (see [6, 13, 14, 16, 30, 33, 34]) but it is still an important open problem.

Ahmadi and Shahinpoor [1] proposed magneto-micropolar model, which were based on Eringen's theory of micropolar fluids and investigated the stability of solutions in bounded domains. Later on, Torres and Medar [22] and Rojas-Medar [25] proved the global existence of the strong solutions and existence of weak solutions respectively for the magneto micropolar equations. In several function spaces, such as Morrey–Campanato spaces [9, 12] Besov spaces, homogeneous Besov spaces and on different domains the blow-up criteria for smooth solutions and regularity conditions for weak solutions has been obtained (see [19, 20, 27, 28, 29, 32]).

Before presenting our main findings, first, we will go over the problem's history. For the Navier-Stokes equations Berselli and Galdi [3] showed the smoothness of weak solutions on [0,T], if conditions

$$\Psi \in L^m(0,T,L^l) \ \text{ with } \frac{2}{m}+\frac{3}{l}=2, \ \ \frac{3}{2}<l\leq\infty,$$

and

$$\nabla\Psi \in L^m(0,T,L^l) \ \text{ with } \frac{2}{m}+\frac{3}{l}=3, \ \ 1<l\leq\infty,$$

are satisfied. Similarly, for the magneto-hydrodynamics system Zhou [35] obtained the conditions

$$\Psi\in L^{l,m}, \ \ \mathcal{V}\in L^{2l,2m}, \text{ or } \|\Psi\|_{L^{\infty,\frac{3}{l}}}, \ \ \|\mathcal{V}\|_{L^{\infty,3}}, \quad \text{where } \frac{2}{l}+\frac{3}{m}\leq 2, \ \ \frac{3}{2}<m\leq\infty,$$

and

$$\nabla\Psi\in L^{l,m}, \ \ \mathcal{V}\in L^{3l,3m}, \text{ or } \|\nabla\Psi\|_{L^{\infty,3}}, \ \ \|\nabla\mathcal{V}\|_{L^{\infty,3}}, \quad \text{where } \frac{2}{l}+\frac{3}{m}\leq 3, \ \ 1<m\leq\infty.$$

This important result in Lorentz space for micropolar fluid system was presented by Yuan [31] as

$$\nabla\Psi \in L^m(0,T,L^{l,\infty}) \ \text{ with } \frac{2}{m}+\frac{3}{l}\leq 3, \quad 1<l\leq\infty.$$

Feng-Ping and Guang-Xia [8] presented the following regularity criteria for the magneto-micropolar system

$$\nabla\Psi \in L^m(0,T,L^{l,\infty}) \ \text{ with } \frac{2}{m}+\frac{3}{l}\leq 3, \ \ m\geq 2, \quad l>1, \tag{1.2}$$

$$\nabla\Psi\in L^{\frac{2}{3}}(0,T,BMO).$$

Recently, Li and Niu [17] presented the regularity criteria in Lorentz spaces

$$\Psi \in L^{m,\infty}(0,T,L^{l,\infty}) \ \text{ with } \frac{2}{m}+\frac{3}{l}=2, \ \ m\geq 2, \quad \frac{3}{2}<l\leq\infty. \tag{1.3}$$

Motivated by the above results specifically in Lebesgue and Lorentz spaces we will establish two new conditions for the regularity of system (1.1) in generalised Lorentz spaces i.e., in anisotropic Lorentz spaces.

Remark 1.1. The results of Theorem 2.1 and Theorem 2.2 are also true for the Navier-Stokes system (putting $\mathcal{W}=\mathcal{V}=0$ in system (1.1), MHD system (putting $\mathcal{W}=0$ in system (1.1) and for Micropolar fluid system (putting $\mathcal{V}=0$ in system (1.1). As pressure plays a significance role in controlling the regularity of weak solutions. Therefore, it is vital to study the regularity of system (1.1) for pressure and its gradient in anisotropic Lorentz spaces.

Remark 1.2. The following embedding relations are helpful in proving our main results.

$$\|f\|_{L^l}\hookrightarrow\|f\|_{L^{l,\infty}}\hookrightarrow\left\|\left\|\left\|f\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}.$$

Definition 1.1. [2] Let $l=(l_1,l_2,l_3)$ and $m=(m_1,m_2,m_3)$ with $0<l_i\le\infty$, $0<m_i\le\infty$. If $l_i=\infty$ then $m_i=\infty$ for every $i=1,2,3$. An anisotropic Lorentz space $L^{l_1,m_1}(\mathbb{R}_{x_1};L^{l_2,m_2}(\mathbb{R}_{x_2};L^{l_3,m_3}((\mathbb{R}_{x_3}))$ is the set of functions defined as

$$\Big\|\Big\|\|f\|_{L^{l_1,m_1}_{x_1}}\Big\|_{L^{l_2,m_2}_{x_2}}\Big\|_{L^{l_3,m_3}_{x_3}}:=\Big(\int_0^\infty\Big(\int_0^\infty\Big(\int_0^\infty[t_1^{\frac{1}{l_1}}t_2^{\frac{1}{l_2}}t_3^{\frac{1}{l_3}}f^{*_1,*_2,*_3}(t_1,t_2,t_3)]^{m_1}\frac{dt_1}{t_1}\Big)^{\frac{m_2}{m_1}}\frac{dt_2}{t_2}\Big)^{\frac{m_3}{m_2}}\frac{dt_3}{t_3}\Big)^{\frac{1}{m_3}}<\infty.$$

For the detailed study on the anisotropic Lorentz spaces and mixed norm spaces (see [2, 4, 21]).

Lemma 1.2. (Holder's inequality for Lorentz spaces)[2, 21] If $1\le l_1,l_2,m_1,m_2\le\infty$, then for any $f\in L^{l_1,m_1}(\mathbb{R}^n)$, $g\in L^{l_2,m_2}(\mathbb{R}^n)$,

$$\|fg\|_{L^{l,m}(\mathbb{R}^n)}\le C\|f\|_{L^{l_1,m_1}(\mathbb{R}^n)}\|g\|_{L^{l_2,m_2}(\mathbb{R}^n)},$$

where $\frac{1}{l}=\frac{1}{l_1}+\frac{1}{l_2}$ and $\frac{1}{m}=\frac{1}{m_1}+\frac{1}{m_2}$.

Lemma 1.3. (Young's inequality for Lorentz spaces)[2, 21] Let $1<l<\infty, 1\le m\le\infty$ and $\frac{1}{l}+\frac{1}{l'}=1$, $\frac{1}{m}+\frac{1}{m'}=1$ with $1<l<l^{'}$ and $m^{'}\le m\le\infty$. If $\frac{1}{l_2}+1=\frac{1}{l}+\frac{1}{l_1}$ and $\frac{1}{m_2}=\frac{1}{m}+\frac{1}{m_1}$ then the convolution operator

$$*:L^{l,m}(\mathbb{R}^n)\times L^{l_1,m_1}(\mathbb{R}^n)\mapsto L^{l_2,m_2}(\mathbb{R}^n)$$

is a bounded bilinear operator.

For any $s\ge 0$, we define homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n)$ as

$$\dot{H}^s(\mathbb{R}^n)=\left\{f\in S^{'}:\hat{f}\in L^1_{loc}(\mathbb{R}^n)\text{ and }\int_{\mathbb{R}^n}|\beta|^{2s}\widehat{|f(\beta)|}^2d\beta<\infty\right\},$$

where S' is the space of tempered distributions.

Lemma 1.4. [24] For $2<l<\infty$, there exists a constant C=C(l) such that $f\in\dot{H}^{\frac{1}{l}}$, then $f\in L^{\frac{2l}{l-2},2}$ and

$$\|f\|_{\frac{2l}{l-2},2}\le C\|f\|_{\dot{H}^{\frac{1}{l}}},$$

where $\dot{H}^{\frac{1}{l}}$ is the homogenous Sobolev space.

Lemma 1.5. Let $2\le l,m,n\le\infty$ and $1-\big(\frac{1}{l}+\frac{1}{m}+\frac{1}{n}\big)\ge 0$, then then there exists $C>0$, such that for all $f\in C_0^\infty(\mathbb{R}^3)$

$$\Big\|\Big\|\|f\|_{L_{x_1}^{\frac{2l}{l-2},2}}\Big\|_{L_{x_2,2}^{\frac{2m}{m-2},2}}\Big\|_{L_{x_3}^{\frac{2n}{n-2},2}}\le C\|\partial_1 f\|_{L^2}^{\frac{1}{l}}\|\partial_2 f\|_{L^2}^{\frac{1}{m}}\|\partial_3 f\|_{L^2}^{\frac{1}{n}}\|f\|_{L^2}^{1-(\frac{1}{m}+\frac{1}{n}+\frac{1}{l})}. \tag{1.4}$$

Let $1\le l,m,n\le\infty$ and $1-\big(\frac{1}{2l}+\frac{1}{2m}+\frac{1}{2n}\big)\ge 0$, then there exists $C>0$, such that for all $f\in C_0^\infty(\mathbb{R}^3)$

$$\Big\|\Big\|\|f\|_{L_{x_1}^{\frac{2l}{l-1},2}}\Big\|_{L_{x_2,2}^{\frac{2m}{m-1},2}}\Big\|_{L_{x_3}^{\frac{2n}{n-1},2}}\le C\|\partial_1 f\|_{L^2}^{\frac{1}{2l}}\|\partial_2 f\|_{L^2}^{\frac{1}{2m}}\|\partial_3 f\|_{L^2}^{\frac{1}{2n}}\|f\|_{L^2}^{1-(\frac{1}{2m}+\frac{1}{2n}+\frac{1}{2l})}. \tag{1.5}$$

Proof . The proof of condition (1.4) is given by Ragusa and Wu [24], for the benefit of reader, we shall give the proof of condition (1.5). Let Λ_1^l be the Fourier multiplier, given as

$$\mathcal{F}_1(\Lambda_1^l f)(\beta_1,x_2,x_3)=|\beta_1|^p\mathcal{F}_1 f(\beta_1,x_2,x_3)$$

with

$$\mathcal{F}_1 f(\beta_1,x_2,x_3)=\int_{\mathbb{R}}e^{-i\beta_1 x_1}f(x_1,x_2,x_3)dx_1.$$

Similarly, we can define Λ_2^m and Λ_3^n. Then by Sobolev embedding, Minkowski's inequality and Holder's inequality and Fourier-Plancherel formula, we can obtain

$$\begin{aligned}\Big\|\Big\|\|f\|_{L^{\frac{2l}{l-1},2}_{x_1}}\Big\|_{L^{\frac{2m}{m-1},2}_{x_2,2}}\Big\|_{L^{\frac{2n}{n-1},2}_{x_3}} \le & C\Big\|\Big\|\|\Lambda_1^{\frac{1}{2l}}f\|_{L^2_{x_1}}\Big\|_{L^{\frac{2m}{m-1},2}_{x_2,2}}\Big\|_{L^{\frac{2n}{n-1},2}_{x_3}} \le C\Big\|\Big\|\|\Lambda_1^{\frac{1}{2l}}f\|_{L^{\frac{2m}{m-1},2}_{x_2}}\Big\|_{L^2_{x_1}}\Big\|_{L^{\frac{2n}{n-1},2}_{x_3}}\\ \le & C\Big\|\Big\|\Lambda_1^{\frac{1}{2l}}\Lambda_2^{\frac{1}{2m}}f\Big\|_{L^2_{x_1,x_2}}\Big\|_{L^{\frac{2n}{n-1},2}_{x_3}} \le C\Big\|\Big\|\Lambda_1^{\frac{1}{2l}}\Lambda_2^{\frac{1}{2m}}f\Big\|_{L^{\frac{2n}{n-1},2}}\Big\|_{L^2_{x_1,x_2}} \le C\Big\|\Lambda_1^{\frac{1}{2l}}\Lambda_2^{\frac{1}{2m}}\Lambda_3^{\frac{1}{2n}}f\Big\|_{L^2_{x_1,x_2,x_3}}\\ \le & C\Big(\int_{\mathbb{R}^3}|\beta_1|^{\frac{1}{l}}|\mathcal{F}f|^{\frac{1}{l}}|\beta_2|^{\frac{1}{m}}|\mathcal{F}f|^{\frac{1}{m}}|\beta_3|^{\frac{1}{n}}|\mathcal{F}f|^{\frac{1}{n}}|\mathcal{F}f|^{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}d\beta_1 d\beta_2 d\beta_3\Big)^{\frac{1}{2}}\\ \le & C\|f\|_{L^2}^{1-(\frac{1}{2l}+\frac{1}{2m}+\frac{1}{2n})}\|\partial_1 f\|_{L^2}^{\frac{1}{2l}}\|\partial_2 f\|_{L^2}^{\frac{1}{2m}}\|\partial_3 f\|_{L^2}^{\frac{1}{2n}}.\end{aligned}$$

Hence the proof is complete. □

The next section is dedicated to prove our main results.

2 Main results and proofs

In this section, we present two new main results and their logarithmic improvements.

Theorem 2.1. Suppose, the initial datum $(\mathcal{U}_0, \mathcal{V}_0, \mathcal{W}_0) \in H^1(\mathbb{R}^3)$ with $\nabla\cdot\mathcal{U}_0 = \nabla\cdot\mathcal{V}_0 = 0$ in the sense of distributions. Suppose that $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ is the weak solution of the system (1.1). If

$$\int_0^T \Big\|\Big\|\|\Psi\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Big\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}} dt < \infty, \tag{2.1}$$

then the solution remains its smoothness upto T. Where $2 \le l, m, n \le \infty$ and 1-$(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})\ge 0$.

Proof . For finding L^4-estimates for $\mathcal{U}$ multiply $(1.1)_1$ with $|\mathcal{U}|^2\mathcal{U}$ integrating by parts and using divergence free condition, we get

$$\begin{aligned}&\frac{1}{4}\frac{d}{dt}\int_{\mathbb{R}^3}|\mathcal{U}|^4dx + \int_{\mathbb{R}^3}|\nabla\mathcal{U}|^2|\mathcal{U}|^2dx + \frac{1}{2}\int_{\mathbb{R}^3}|\nabla|\mathcal{U}|^2|^2dx\\ \le & 2\int_{\mathbb{R}^3}|\Psi||\mathcal{U}|^2|\nabla\mathcal{U}|dx + 3\int_{\mathbb{R}^3}|\mathcal{W}||\mathcal{U}|^2|\nabla\mathcal{U}|dx - \int_{\mathbb{R}^3}|\mathcal{V}||\nabla(|\mathcal{U}|^2\mathcal{U})||\mathcal{V}|dx.\end{aligned} \tag{2.2}$$

Similarly, evaluating L^4-estimates for $\mathcal{W}$, taking the inner product to $(1.1)_2$ with $|\mathcal{W}|^2\mathcal{W}$, we obtain

$$\frac{1}{4}\frac{d}{dt}\int_{\mathbb{R}^3}|\mathcal{W}|^4dx+\int_{\mathbb{R}^3}|\nabla\mathcal{W}|^2|\mathcal{W}|^2dx+\frac{1}{2}\int_{\mathbb{R}^3}|\nabla|\mathcal{W}|^2|^2dx+\int_{\mathbb{R}^3}|div\ \mathcal{W}|^2dx+2\int_{\mathbb{R}^3}|\mathcal{W}|^4dx \le 3\int_{\mathbb{R}^3}|\mathcal{U}||\mathcal{W}|^2|\nabla\mathcal{W}|dx. \tag{2.3}$$

In case of $(1.1)_3$, we take inner product with $|\mathcal{V}|^2\mathcal{V}$, and obtain L^4-estimates for $\mathcal{V}$

$$\frac{1}{4}\frac{d}{dt}\int_{\mathbb{R}^3}|\mathcal{V}|^4dx + \int_{\mathbb{R}^3}|\nabla\mathcal{V}|^2|\mathcal{V}|^2dx + 2\int_{\mathbb{R}^3}|\nabla|\mathcal{V}||\mathcal{V}||^2dx \le \int_{\mathbb{R}^3}|\mathcal{V}||\nabla(|\mathcal{V}|^2\mathcal{V})|\mathcal{U}||dx. \tag{2.4}$$

Adding (2.2), (2.3) and (2.4) implies

$$\begin{aligned}&\frac{1}{4}\frac{d}{dt}(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{W}\|_{L^4}^4 + \|\mathcal{V}\|_{L^4}^4) + \||\nabla\mathcal{U}||\mathcal{U}|\|_{L^2}^2 + \frac{1}{2}\|\nabla|\mathcal{U}|^2\|_{L^2}^2\\ &+\||\nabla\mathcal{W}||\mathcal{W}|\|_{L^2}^2 + \frac{1}{2}\|\nabla|\mathcal{W}|^2\|_{L^2}^2 + \|div\ \mathcal{W}\|_{L^2}^2 + 2\|\mathcal{W}\|_{L^4}^4 + \||\nabla\mathcal{V}||\mathcal{V}|\|_{L^2}^2 + 2\|\nabla|\mathcal{V}||\mathcal{V}|\|_{L^2}^2\\ \le & 2\int_{\mathbb{R}^3}|\Psi||\mathcal{U}|^2|\nabla\mathcal{U}|dx + 3\int_{\mathbb{R}^3}|\mathcal{W}||\mathcal{U}|^2|\nabla\mathcal{U}|dx + 3\int_{\mathbb{R}^3}|\mathcal{U}||\mathcal{W}|^2|\nabla\mathcal{W}|dx\\ &-\int_{\mathbb{R}^3}|\mathcal{V}||\nabla(|\mathcal{U}|^2\mathcal{U})||\mathcal{V}|dx + \int_{\mathbb{R}^3}|\mathcal{V}||\nabla(|\mathcal{V}|^2\mathcal{V})|\mathcal{U}||dx\\ = & L_1 + L_2 + L_3 + L_4 + L_5.\end{aligned} \tag{2.5}$$

Now, estimating L_1 by employing Holder's and Young's inequality

$$2\int_{\mathbb{R}^3}|\Psi||\mathcal{U}|^2|\nabla\mathcal{U}|dx \leq \frac{1}{4}\|\nabla|\mathcal{U}|^2\|_{L^2}^2 + C\int_{\mathbb{R}^3}|\Psi||\Psi||\mathcal{U}|^2dx = P_1 + P_2.$$

Now, for P_2 using Holder's and Young's inequality as given by Lemma 1.4., Lemma 1.5. and applying Lemma 1.7. we obtain

$$\begin{aligned}
P_2 =&C\int_{\mathbb{R}^3}|\Psi||\Psi||\mathcal{U}|^2dx \leq C\left\|\left\|\left\|\Psi\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}\left\|\left\|\left\|\Psi\right\|_{L^{\frac{2l}{l-2},2}_{x_1}}\right\|_{L^{\frac{2m}{m-2},2}_{x_2}}\right\|_{L^{\frac{2n}{n-2},2}_{x_3}}\||\mathcal{U}|^2\|_{L^2}\\
\leq&C\left\|\left\|\left\|\Psi\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}\|\Psi\|_{L^2}^{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}\|\partial_1\Psi\|_{L^2}^{\frac{1}{l}}\|\partial_2\Psi\|_{L^2}^{\frac{1}{m}}\|\partial_3\Psi\|_{L^2}^{\frac{1}{n}}\|\mathcal{U}\|_{L^4}^2\\
\leq&C\left\|\left\|\left\|\Psi\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}\|\Psi\|_{L^2}^{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}\|\nabla\Psi\|_{L^2}^{\frac{1}{l}+\frac{1}{m}+\frac{1}{n}}\|\mathcal{U}\|_{L^4}^2\\
\leq&C\left\|\left\|\left\|\Psi\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}\||\mathcal{U}|\nabla\mathcal{U}\|_{L^2}^{\frac{1}{l}+\frac{1}{m}+\frac{1}{n}}\|\mathcal{U}\|_{L^4}^{4-(\frac{2}{l}+\frac{2}{m}+\frac{2}{n})}\\
\leq&\frac{1}{4}\||\mathcal{U}|\nabla\mathcal{U}\|_{L^2}^2 + C\left\|\left\|\left\|\Psi\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\|\mathcal{U}\|_{L^4}^4.
\end{aligned}$$

The final estimates for L_1 are

$$L_1 \leq \frac{1}{4}\|\nabla|\mathcal{U}|^2\|_{L^2}^2 + \frac{1}{4}\||\mathcal{U}|\nabla\mathcal{U}\|_{L^2}^2 + C\left\|\left\|\left\|\Psi\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\|\mathcal{U}\|_{L^4}^4. \tag{2.6}$$

Assessing L^4-estimates for L_2 and L_3

$$L_2 \leq \frac{1}{2}\||\mathcal{U}||\nabla\mathcal{U}|\|_{L^2}^2 + C\Big(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{W}\|_{L^4}^4\Big) \tag{2.7}$$

and

$$L_3 \leq \frac{1}{2}\||\mathcal{W}||\nabla\mathcal{W}|\|_{L^2}^2 + C\Big(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{W}\|_{L^4}^4\Big). \tag{2.8}$$

Now, assessing L^2-estimates for L_4 and L_5

$$\begin{aligned}
L_4 \leq& C\||\mathcal{V}|^2|\mathcal{U}|\|_{L^2}\|\nabla|\mathcal{U}|^2\|_{L^2} \leq C\||\mathcal{V}|^2|\mathcal{U}|\|_{L^2}^2 + \frac{1}{4}\|\nabla|\mathcal{U}|^2\|_{L^2}^2\\
\leq& C\||\mathcal{V}|^2\|_{L^6}^2\||\mathcal{U}|\|_{L^3}^2 + \frac{1}{4}\|\nabla|\mathcal{U}|^2\|_{L^2}^2 \leq C\|\nabla|\mathcal{V}|^2\|_{L^2}^2\|\nabla\mathcal{U}|\|_{L^2}\||\mathcal{U}|\|_{L^2} + \frac{1}{4}\|\nabla|\mathcal{U}|^2\|_{L^2}^2\\
\leq& C\||\mathcal{V}|\nabla|\mathcal{V}|\|_{L^2}^2 + \frac{1}{4}\|\nabla|\mathcal{U}|^2\|_{L^2}^2
\end{aligned} \tag{2.9}$$

and

$$L_5 \leq C\||\mathcal{V}^2|\mathcal{U}|\|_{L^2}^2 + \frac{1}{8}\|\nabla|\mathcal{V}|^2\|_{L^2} \leq C\||\mathcal{V}|\nabla|\mathcal{V}|\|_{L^2}^2. \tag{2.10}$$

Putting all the estimates (2.6), (2.7), (2.8), (2.10), (2.9) in (2.5)results as

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{W}\|_{L^4}^4 + \|\mathcal{V}\|_{L^4}^4) + \||\nabla\mathcal{U}||\mathcal{U}|\|_{L^2}^2 + \frac{1}{2}\|\nabla|\mathcal{U}|^2\|_{L^2}^2\\
&+ \||\nabla\mathcal{W}||\mathcal{W}|\|_{L^2}^2 + \frac{1}{2}\|\nabla|\mathcal{W}|^2\|_{L^2}^2 + \|div\ \mathcal{W}\|_{L^2}^2 + 2\|\mathcal{W}\|_{L^4}^4 + \||\nabla\mathcal{V}||\mathcal{V}|\|_{L^2}^2 + 2\|\nabla|\mathcal{V}||\mathcal{V}|\|_{L^2}^2\\
\leq&\frac{1}{4}\|\nabla|\mathcal{U}|^2\|_{L^2}^2 + \frac{1}{4}\||\mathcal{U}|\nabla\mathcal{U}\|_{L^2}^2 + C\left\|\left\|\left\|\Psi\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\|\mathcal{U}\|_{L^4}^4 + \frac{1}{2}\||\mathcal{U}||\nabla\mathcal{U}|\|_{L^2}^2\\
&+ \frac{1}{2}\||\mathcal{W}||\nabla\mathcal{W}|\|_{L^2}^2 + C\Big(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{W}\|_{L^4}^4\Big) + C\||\mathcal{V}|\nabla|\mathcal{V}|\|_{L^2}^2 + \frac{1}{4}\|\nabla|\mathcal{U}|^2\|_{L^2}^2 + C\||\mathcal{V}|\nabla|\mathcal{V}|\|_{L^2}^2.
\end{aligned}$$

Further simplifications yield

$$
\begin{aligned}
&\frac{1}{4}\frac{d}{dt}(\|\mathcal{U}\|_{L^4}^4+\|\mathcal{W}\|_{L^4}^4+\|\mathcal{V}\|_{L^4}^4)+\||\nabla\mathcal{U}||\mathcal{U}|\|_{L^2}^2+\frac{1}{2}\|\nabla|\mathcal{U}|^2\|_{L^2}^2\\
&+\||\nabla\mathcal{W}||\mathcal{W}|\|_{L^2}^2+\frac{1}{2}\|\nabla|\mathcal{W}|^2\|_{L^2}^2+\|div\ \mathcal{W}\|_{L^2}^2+2\|\mathcal{W}\|_{L^4}^4+\||\nabla\mathcal{V}||\mathcal{V}|\|_{L^2}^2+2\|\nabla|\mathcal{V}||\mathcal{V}|\|_{L^2}^2\\
\le& C\Big(1+\Big\|\Big\|\|\Psi\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Big\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\Big)\Big(\|\mathcal{U}\|_{L^4}^4+\|\mathcal{W}\|_{L^4}^4+\|\mathcal{V}\|_{L^4}^4\Big). \qquad (2.11)
\end{aligned}
$$

Gronwall's Lemma to (2.11) results in

$$
\sup_{0\le t\le T}(\|\mathcal{U}\|_{L^4}^4+\|\mathcal{W}\|_{L^4}^4+\|\mathcal{V}\|_{L^4}^4)\le C\exp\int_0^t\Bigg\{\Big(1+\Big\|\Big\|\|\Psi\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Big\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\Big)\Bigg\}\Big(\|\mathcal{U}_0\|_{L^4}^4+\|\mathcal{W}_0\|_{L^4}^4+\|\mathcal{V}_0\|_{L^4}^4\Big)<\infty.
$$

Which proves Theorem 2.1. as desired. □

Theorem 2.2. Let the initial datum $(\mathcal{U}_0,\mathcal{V}_0,\mathcal{W}_0)\in H^1(\mathbb{R}^3)$ with $\nabla\cdot\mathcal{U}_0=\nabla\cdot\mathcal{V}_0=0$ in the distributional sense. Suppose $(\mathcal{U},\mathcal{V},\mathcal{W})$ be the weak solution to system (1.1). If

$$
\int_0^T\Big\|\Big\|\|\nabla\Psi\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Big\|_{L^{n,\infty}_{x_3}}^{\frac{2}{3-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}dt<\infty, \qquad (2.12)
$$

then the solution remains its smoothness upto T. Where $1\le l,m,n\le\infty$ and 1-$(\frac{1}{2l}+\frac{1}{2m}+\frac{1}{2n})\ge 0$.

Proof . We will prove Theorem 2.2 by finding a priori estimates, in that regards, we continue our calculations from equation (2.5), and rewrite it as

$$
\begin{aligned}
&\frac{1}{4}\frac{d}{dt}(\|\mathcal{U}\|_{L^4}^4+\|\mathcal{W}\|_{L^4}^4+\|\mathcal{V}\|_{L^4}^4)+\||\nabla\mathcal{U}||\mathcal{U}|\|_{L^2}^2+\frac{1}{2}\|\nabla|\mathcal{U}|^2\|_{L^2}^2+\||\nabla\mathcal{W}||\mathcal{W}|\|_{L^2}^2\\
&+\frac{1}{2}\|\nabla|\mathcal{W}|^2\|_{L^2}^2+\|div\ \mathcal{W}\|_{L^2}^2+2\|\mathcal{W}\|_{L^4}^4+\||\nabla\mathcal{V}||\mathcal{V}|\|_{L^2}^2+2\|\nabla|\mathcal{V}||\mathcal{V}|\|_{L^2}^2\\
\le&2\int_{\mathbb{R}^3}|\Psi||\mathcal{U}|^2|\nabla\mathcal{U}|dx+3\int_{\mathbb{R}^3}|\mathcal{W}||\mathcal{U}|^2|\nabla\mathcal{U}|dx+3\int_{\mathbb{R}^3}|\mathcal{U}||\mathcal{W}|^2|\nabla\mathcal{W}|dx\\
&-\int_{\mathbb{R}^3}|\mathcal{V}||\nabla(|\mathcal{U}|^2\mathcal{U})||\mathcal{V}|dx+\int_{\mathbb{R}^3}|\mathcal{V}||\nabla(|\mathcal{V}|^2\mathcal{V})|\mathcal{U}||dx\\
=&\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4+\Gamma_5. \qquad (2.13)
\end{aligned}
$$

Estimating Γ_1 by employing Holder's and Young's inequality and Lemma 1.7.

$$
\begin{aligned}
\Gamma_1=&2\int_{\mathbb{R}^3}|\Psi||\mathcal{U}|^2|\nabla\mathcal{U}|dx\le C\int_{\mathbb{R}^3}|\nabla\Psi||\mathcal{U}|^3dx\le C\int_{\mathbb{R}^3}|\nabla\Psi|^{\frac{1}{2}}|\nabla\Psi|^{\frac{1}{2}}|\mathcal{U}|^2|\mathcal{U}|dx\\
\le& C\Big\|\Big\|\||\nabla\Psi|^{\frac{1}{2}}\|_{L^{2l,\infty}_{x_1}}\Big\|_{L^{2m,\infty}_{x_2}}\Big\|_{L^{2n,\infty}_{x_3}}\||\nabla\Psi|^{\frac{1}{2}}\|_{L^4}\Big\|\Big\|\||\mathcal{U}|^2\|_{L^{\frac{2l}{l-1},2}_{x_1}}\Big\|_{L^{\frac{2m}{m-1},2}_{x_2}}\Big\|_{L^{\frac{2n}{n-1},2}_{x_3}}\|\mathcal{U}\|_{L^4}\\
\le& C\Big\|\Big\|\|\nabla\Psi\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Big\|_{L^{n,\infty}_{x_3}}^{\frac{1}{2}}\|\nabla|\mathcal{U}|^2\|_{L^2}^{\frac{1}{2l}+\frac{1}{2m}+\frac{1}{2n}}\||\mathcal{U}||\nabla\mathcal{U}|\|_{L^2}^{\frac{1}{2}}\|\mathcal{U}\|_{L^4}^{3-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}\\
\le& C\Big\|\Big\|\|\nabla\Psi\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Big\|_{L^{n,\infty}_{x_3}}^{\frac{2}{3-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\|\mathcal{U}\|_{L^4}^4+\frac{1}{4}(\|\nabla|\mathcal{U}|^2\|_{L^2}^2+\||\mathcal{U}||\nabla\mathcal{U}|\|_{L^2}^2). \qquad (2.14)
\end{aligned}
$$

For Γ_2

$$
\Gamma_2\le\frac{1}{2}\||\mathcal{U}||\nabla\mathcal{U}|\|_{L^2}^2+C\Big(\|\mathcal{U}\|_{L^4}^4+\|\mathcal{W}\|_{L^4}^4\Big).
$$

For Γ_3

$$\Gamma_3 \leq \frac{1}{2}\||\mathcal{W}||\nabla \mathcal{W}|^2\|_{L^2}^2 + C\Big(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{W}\|_{L^4}^4\Big).$$

Γ_4 and Γ_5 are estimated same as L_4 and L_5. Putting all the estimates in (2.13) results in

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{W}\|_{L^4}^4 + \|\mathcal{V}\|_{L^4}^4) + \||\nabla \mathcal{U}||\mathcal{U}|\|_{L^2}^2 + \frac{1}{2}\|\nabla|\mathcal{U}|^2\|_{L^2}^2 \\
&+ \||\nabla \mathcal{W}||\mathcal{W}|\|_{L^2}^2 + \frac{1}{2}\|\nabla|\mathcal{W}|^2\|_{L^2}^2 + \|div\ \mathcal{W}\|_{L^2}^2 + 2\|\mathcal{W}\|_{L^4}^4 + \||\nabla \mathcal{V}||\mathcal{V}|\|_{L^2}^2 + 2\|\nabla|\mathcal{V}||\mathcal{V}|\|_{L^2}^2 \\
\leq& \frac{1}{4}\|\nabla|\mathcal{U}|^2\|_{L^2}^2 + \frac{1}{4}\||\mathcal{U}|\nabla \mathcal{U}|\|_{L^2}^2 + C\left\|\left\|\left\|\nabla\Psi\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{3-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\|\mathcal{U}\|_{L^4}^4 + \frac{1}{2}\||\mathcal{U}||\nabla\mathcal{U}|\|_{L^2}^2 \\
&+ \frac{1}{2}\||\mathcal{W}||\nabla \mathcal{W}|\|_{L^2}^2 + C\Big(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{W}\|_{L^4}^4\Big) + C\||\mathcal{V}|\nabla|\mathcal{V}|\|_{L^2}^2 + \frac{1}{4}\|\nabla|\mathcal{U}|^2\|_{L^2}^2 + C\||\mathcal{V}|\nabla|\mathcal{V}|\|_{L^2}^2. \qquad (2.15)
\end{aligned}$$

Simplify (2.15) further yields

$$\frac{1}{4}\frac{d}{dt}(\|\mathcal{U}\|_{L^4}^4 + \|\mathcal{W}\|_{L^4}^4 + \|\mathcal{V}\|_{L^4}^4) + \||\nabla \mathcal{U}||\mathcal{U}|\|_{L^2}^2 + \frac{1}{2}\|\nabla|\mathcal{U}|^2\|_{L^2}^2 \qquad (2.16)$$

$$+ \||\nabla \mathcal{W}||\mathcal{W}|\|_{L^2}^2 + \frac{1}{2}\|\nabla|\mathcal{W}|^2\|_{L^2}^2 + \|div\ \mathcal{W}\|_{L^2}^2 + 2\|\mathcal{W}\|_{L^4}^4 + \||\nabla \mathcal{V}||\mathcal{V}|\|_{L^2}^2 + 2\|\nabla|\mathcal{V}||\mathcal{V}|\|_{L^2}^2 \qquad (2.17)$$

$$\leq C\left(1 + \left\|\left\|\left\|\nabla\Psi\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{3-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\right)\Big(\|\mathcal{U}\|_{L^4}^4+\|\mathcal{W}\|_{L^4}^4+\|\mathcal{V}\|_{L^4}^4\Big). \qquad (2.18)$$

Applying Gronwall's lemma for (2.16)

$$\sup_{0\leq t\leq T}(\|\mathcal{U}\|_{L^4}^4+\|\mathcal{W}\|_{L^4}^4+\|\mathcal{V}\|_{L^4}^4) \leq \exp\left\{\int_0^T C\Big(1 + \left\|\left\|\left\|\nabla\Psi\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{3-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\Big)d\tau\right\} < \infty. \qquad (2.19)$$

The bounds on (2.19) proved our result. □

Theorem 2.3. Considering same assumptions as for Theorem 2.1 and Theorem 2.2. The sufficient conditions

$$\int_0^T \frac{\left\|\left\|\left\|\Psi\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}}{1+\ln(1+\|\Psi\|_{L^2}^2)}dt < \infty, \qquad (2.20)$$

and

$$\int_0^T \frac{\left\|\left\|\left\|\nabla\Psi\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{3-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}}{1+\ln(1+\|\Psi\|_{L^2}^2)}dt < \infty, \qquad (2.21)$$

are the logarithmic improvements of the conditions (2.1) and (2.12).

Proof . Continuing from inequality (2.11), as $1+\ln(1+\|\Psi\|_{L^2}^2) \leq 1+\ln(\Pi(t))$, where $\Pi(t) = e+\|\mathcal{U}\|_{L^4}^4+\|\mathcal{W}\|_{L^4}^4+\|\mathcal{V}\|_{L^4}^4$,

$$\begin{aligned}\frac{d}{dt}\Pi(t) \leq & C\left(\frac{1+\left\|\left\|\left\|\Psi\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}}{1+\ln(e+\|\Psi\|_{L^2}^2)}\right)(\|\mathcal{U}\|_{L^4}^4+\|\mathcal{W}\|_{L^4}^4+\|\mathcal{V}\|_{L^4}^4)(1+\ln(e+\|\Psi\|_{L^2}^2)\\ \leq & C\left(\frac{1+\left\|\left\|\left\|\Psi\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}}{1+\ln(e+\|\Psi\|_{L^2}^2)}\right)(\Pi(t))(1+\ln(\Pi(t))\end{aligned}$$

Hence,

$$\frac{d}{dt}(1+\ln\Pi(t)) \leq C\left(\frac{1+\left\|\left\|\left\|\Psi\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}}{1+\ln(1+\|\Psi\|_{L^2}^2)}\right)(1+\ln\Pi(t)).$$

Gronwall's lemma results in

$$\sup_{0\leq t\leq T}\ln\Pi(t) \leq (1+\ln\Pi(0))\exp\left\{C\left(\frac{1+\|\|\|\Psi\|_{L^{l,\infty}_{x_1}}\|_{L^{m,\infty}_{x_2}}\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}}{1+\ln(1+\|\Psi\|_{L^2}^2)}\right)d\tau\right\}.$$

This implies that

$$\sup_{0\leq t\leq T}\left(\|\mathcal{U}\|_{L^4}^4+\|\mathcal{W}\|_{L^4}^4+\|\mathcal{V}\|_{L^4}^4\right) \leq \infty, \tag{2.22}$$

which proves our result in the interval $[0,T]$. These bounds together with (2.22) ensure the regularity of weak solutions on the interval $[0,T]$. Following the same steps as for (2.20), condition (2.21) can be proved. Therefore, its proof is omitted. □

3 Conclusion

Regularity conditions (2.1) and (2.12) are new on the framework of anisotorpic Lebesgue spaces and improve or generalize the conditions (1.2) and (1.3) for the system (1.1) . Here we pose a future problem of finding regularity criteria based on one-directional derivative of pressure in anisotropic Lorentz spaces.

References

[1] G. Ahmadi and M. Shahinpoor, *Universal stability of magneto-micropolar fluid motions*, Internat J. Engrg. Sci. **12** (1974), no. 7, 657–663.

[2] K.A. Bekmaganbetov and Y. Toleugazy, *On the Order of the trigonometric diameter of the anisotropic Nikol'skiiBesov class in the metric of anisotropic Lorentz spaces*, Anal. Math. **45** (2019), 237–247.

[3] C. Berselli and P. Galdi, *Regularity criteria involving the pressure for the weak solutions to the Navier-Stokes equations*, Proc. Amer. Math. Soc. **130** (2002), no. 12, 3585–3595.

[4] T. Chen and W. Sun, *Iterated weak and weak mixed-norm spaces with applications to geometric inequalities*, J. Geom. Anal. **30** (2020), no. 4, 4268–4323.

[5] B. Dong, Y. Jia, and Z. Chen, *Pressure regularity criteria of the three-dimensional micropolar fluid flows*, Math. Meth. Appl. Sci. **34** (2011), no. 5, 595–606.

[6] H. Duan, *On regularity criteria in terms of pressure for the 3D viscous MHD equations*, Appl. Anal. **91** (2012), no. 5, 947–952.

[7] A.C. Eringen, *Theory of micropolar fluids*, J. Math. Mech. **16** (1996), no. 1, 1–18.

[8] L. Feng-Ping and C. Guang-Xia, *Regularity criteria for weak solutions to the 3D magneto-micropolar fluid equations*, J. SD. Univ. (Nat Sci.) **50** (2015), 60–67.

[9] G. Gala, *Regularity criteria for the 3D magneto-micropolar fluid equations in the Morrey–Campanato space*, Nonlinear Differ. Equ. Appl. **17** (2010), no. 2, 181—194.

[10] S. Gala, *A remark on the logarithmically improved regularity criterion for the micropolar fluid equations in terms of the pressure*, Math. Meth. Appl. Sci. **34** (2011), no. 34, 1945–1953.

[11] G.P. Galdi and S. Rionero, *A note on the existence and uniqueness of solutions of the micropolar fluid equations*, Int. J. Eng. Sci. **15** (1977), no. 2, 105–108.

[12] J. Geng, X. Chen, and S. Gala, *On regularity criteria for the 3D magneto-micropolar fluid equations in the critical Morrey-Campanato space*, Commun. Pure Appl. Anal. **10** (2011), no. 2, 583–592.

[13] Z. Guo, T. Tong, and W. Wang, *On regularity of the 3D MHD equations based on one velocity component in anisotropic Lebesgue spaces*, Appl. Math. Lett. **120** (2021), 1–7.

[14] Z. Guo and S. Zhang, *An optimal regularity criterion for the 3D MHD equations in homogeneous Besov spaces*, Math. Meth. Appl. Sci. **44** (2021), no. 2, 2130–2139.

[15] Y. Jia, W. Zhang, and B. Dong, *Remarks on the regularity criterion of the 3D micropolar fluid flows in terms of the pressure*, Appl. Math. Lett. **24** (2011), no. 2, 199–203.

[16] X. Jia and Y. Zhou, *On Regularity Criteria for the 3D Incompressible MHD Equations Involving One Velocity Component*, J. Math. Fluid Mech. **18** (2016), no. 1, 187–206.

[17] Z. Li and P. Niu, *New regularity criteria for the 3D magneto-micropolar fluid equations in Lorentz spaces*, Math. Meth. Appl. Sci. **44** (2021), no. 7, 6056–6066.

[18] G. Lukaszewicz, *Micropolar Fluids: Theory and Applications, Modeling and Simulation in Science, Engineering and Technology*, 1st ed., Birkh¨auser Boston, New York, 1999.

[19] E. Mallea-Zepeda and E.E. Ortega-Torres, *Control problem for a magneto-micropolar flow with mixed boundary conditions for the velocity field*, J. Dyn. Control Syst. **25** (2019), no. 4, 599–618.

[20] W.G. Melo, *The magneto-micropolar equations with periodic boundary conditions: solution properties at potential blow-up times*, J. Math. Anal. Appl. **435** (2016), no. 2, 1194–1209.

[21] R. O'Neil, *Convolution operators and L(p,q) spaces*, Duke Math. J, **30** (1963), no. 1, 129–142.

[22] E.E. Ortega-Torres and M.A. Rojas-Medar, *Rojas-Medar, Magneto-micropolar fluid motion: global existence of strong solutions*, Abstr. Appl. Anal. **4** (1999), no. 2, 109–125.

[23] E.E. Ortega-Torres and M.A. Rojas-Medar, *On the regularity for solutions of the micropolar fluid equations*, Rend. Sem. Mat. Padova. **122** (2009), 27–37.

[24] M.A. Ragusa and F. Wu, *Regularity criteria for the 3D magneto-hydrodynamics equations in anisotropic Lorentz spaces*, Symmetry **13** (2021), no. 4, 1–10.

[25] M.A. Rojas-Medar and J.L. Boldrini, *Magneto-micropolar fluid motion: Existence of weak solutions*, Rev. Mat. Complut., **11** (1988), no. 2, 443–460.

[26] M.A. Rojas-Medar and M. Loazya, *A weak-L^p Prodi-Serrin type regularity criterion for the micropolar fluid equations*, J. Math. Phys. **57** (2016), no. 2, 1–6.

[27] P.B. Silva, L. Friz, and M.A. Rojas-Medar, *Exponential stability for magneto-micropolar fluids*, Nonlinear Anal. **143** (2016), 211–223.

[28] Z. Tan, W. Wu, and J. Zhou, *Global existence and decay estimate of solutions to magneto-micropolar fluid equations*, J. Differ. Equ. **266** (2019), no. 7, 4137–4169.

[29] F. Wu, *A refined regularity criteria of weak solutions to the magneto-micropolar fluid equations*, J. Evol. Equ. **21** (2021), no. 1, 725—734.

[30] X. Xu, Z. Ye and Z. Zhang, *Remark on an improved regularity criterion for the 3D MHD equations*, Appl. Math. Lett. **42** (2015), 41–46.

[31] B.Q. Yuan, *On regularity criteria of weak solutions to the micropolar fluid equations in Lorentz space*, Proc. Amer. Math. Soc. **138** (2010), no. 6, 2010–2025.

[32] B. Yuan and X. Li, *Regularity of weak solutions to the 3D magneto-micropolar equations in Besov spaces*, Acta Appl. Math. **163** (2019), no. 1, 207–223.

[33] Z. Zhang, *Regularity criteria for the 3D MHD equations involving one current density and the gradient of one velocity component*, Nonlinear Anal. **115** (2015), 41–49.

[34] Z. Zhang, *Remarks on the global regularity criteria for the 3D MHD equations via two components*, Z. Angew Math. **66** (2015), no. 3, 977–987.

[35] Y. Zhou, *Regularity criteria for the 3D MHD equations in terms of the pressure*, Int. J. Nonlinear Mech. **41** (2006), no. 10, 1174–1180.