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One step hybrid block method for solving nonlinear second order Dirichlet value problems of ordinary differential equations directly

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Abstract

The aim of this article is to approximate the solution of nonlinear second-order Dirichlet boundary value problems of ordinary differential equations directly using the hybrid block method. To derive this method, we first transform the boundary value problem to its corresponding second-order initial value problem via the nonlinear shooting method. Then, a direct one-step hybrid block with three off-step points is derived using a collocation and interpolation approach. The numerical results clearly show that the developed method is able to generate good results when it is compared with the existing method in terms of error.

Keywords: Block method, Hybrid method, Second order differential equation, Collocation and interpolation
2020 MSC: 35J25, 35Q60, 65L06, 65L05

1 Introduction

In this article, we are interesting in finding the numerical solution of second order nonlinear Dirichlet of BVP in the following form

$$y^{''}=f(x,y,y^{'}),\quad y(a)=\alpha,\ y(b)=\beta,\ x\in[a,b]. \tag{1.1}$$

Boundary value problems (BVP) have been of great concentration due to their vast applications in numerous sciences fields such as mechanics, medicine, physics, engineering and chemistry. Many of researchers (see,[1, 3, 2]) have proposed several method for solving (1.1). However, as for as we known, none has ever attempted to solve (1.1)by using one step hybrid block method. The purpose of this paper is to develop one step hybrid block method for solving second order Dirichlet boundary value directly. In this work, the three-step iterative method proposed by [10] is employed in shooting method in order to obtain initial guess condition.

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2 Development of the Method

2.1 Transformation from BVPs to IVPs

According to [1], the difference between linear and nonlinear shooting methods for solving second-order boundary-value problem is that in the later case the solution of nonlinear problem cannot be expressed as a linear combination of the solutions to two initial-value problems. As an alternative, a sequence of the following initial-value problems containing a parameter t are used to approximate the solution to the boundary-value problem

$$y^{''} = f(x, y, y^{'}),\ y(a) = \alpha,\ \ y'(a) = t, \quad x \in [a, b]. \tag{2.1}$$

We do this by choosing the parameters $t = t_k$ in a manner to ensure that

$$\lim_{x\to\infty} y(b, t_k) = y(b) = \beta, x \in [a, b]. \quad y(a) = \alpha,\ \ y(b) = t, \tag{2.2}$$

where $y(x, t_k)$ presents the solution to the initial-value problem (11.7) with $t = t_k$, and $y(x)$ denotes the solution of (1). In this research, the solution of the first initial value problem in the demanded sequence will be found by imposing initial guess $t_0 = \frac{\beta-\alpha}{b-a}$. In order to find the solutions of remaining sequence, the following guess points t_k , $k = 1, 2, \cdots$ with nonlinear function $y(b, t) - \beta = 0$ are determined using three-step iterative method which was proposed by [1] as given below

$$\begin{aligned} T_k &= t_k - \frac{y(b, t_k) - \beta}{y^{'}(b, t_k)} \\ S_k &= \frac{y(b, T_k) - \beta}{y^{'}(b, t_k)} \\ t_{k+1} &= T_k + S_k - \frac{y(b, T_k + S_k) - \beta}{y^{'}(b, t_k)} \end{aligned} \tag{2.3}$$

In finding the value of $y^{'}(b, t_k)$, equation (2.1) is reformed depending on two variables x and t as below

$$y^{''}(x, t) = f(x, y(x, t), y^{'}(x, t)) =,\ y(a, t) = \alpha,\ \ y'(a, t) = t, \quad x \in [a, b]. \tag{2.4}$$

Differentiating equation above partially with respect to t gives

$$\begin{aligned} \frac{\partial y^{''}}{\partial t}(x, t) &= \frac{\partial f}{\partial t}(x, y(x, t), y^{'}(x, t)) \\ &= \frac{\partial f}{\partial t}(x, y(x, t), y^{'}(x, t))\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}(x, y(x, t), y^{'}(x, t))\frac{\partial y}{\partial t}(x, t) + \frac{\partial f}{\partial y^{'}}(x, y(x, t), y^{'}(x, t))\frac{\partial y^{'}}{\partial t}(x, t) \end{aligned}$$

which leads to following second order initial value problem since $\frac{\partial x}{\partial t} = 0$.

$$\begin{aligned} \frac{\partial y^{''}}{\partial t}(x, t) &= \frac{\partial f}{\partial y}(x, y(x, t), y^{'}(x, t))\frac{\partial y}{\partial t}(x, t) + \frac{\partial f}{\partial y^{'}}(x, y(x, t), y^{'}(x, t))\frac{\partial y^{'}}{\partial t}(x, t) \\ \frac{\partial y}{\partial t}(a, t) &= 0,\ \frac{\partial y^{'}}{\partial t}(a, t) = 0, \quad x \in [a, b]. \end{aligned} \tag{2.5}$$

Solving (2.5) at t_k gives $y^{'}(b, t_k)$

2.2 Derivation of the Method

In order to derive our method, a power series of the form

$$y(x) = \sum_{i=0}^{v+m-1} a_i \left(\frac{x - x_n}{h}\right)^i \qquad x \in [x_n, x_{n+1}]. \tag{2.6}$$

is considered as approximation solution to (2.1) for aim of this research, where $n = 0, 1, 2, ..., N-1$, $m = 5$ denotes the number of collocation points, $v = 2$ represents the number of interpolation points, $h = x_n - x_{n-1}$ and $[a, b]$ is divided as following $a = x_0 < x_1 < ... < x_{N-1} < x_N = b$. Interpolating Equation (2.6) at points $x_{n+\frac{2}{8}}$, $x_{n+\frac{3}{8}}$ and collocating its second derivative at all points i.e x_n, $x_{n+\frac{1}{8}}$, $x_{n+\frac{2}{8}}$, $x_{n+\frac{3}{8}}$ and x_{n+1} produces the following equations

$$\begin{pmatrix} 1 & \frac{1}{4} & \frac{1}{16} & \frac{1}{64} & \frac{1}{256} & \frac{1}{1024} & \frac{1}{4096} \\ 1 & \frac{3}{8} & \frac{9}{64} & \frac{27}{512} & \frac{81}{4096} & \frac{243}{32768} & \frac{729}{262144} \\ 0 & 0 & \frac{2}{h^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{h^2} & \frac{3}{(4h^2)} & \frac{3}{(16h^2)} & \frac{5}{(128h^2)} & \frac{15}{(2048h^2)} \\ 0 & 0 & \frac{2}{h^2} & \frac{3}{(2*h^2)} & \frac{3}{(4h^2)} & \frac{5}{(16*h^2)} & \frac{15}{(128*h^2)} \\ 0 & 0 & \frac{2}{h^2} & \frac{9}{(4*h^2)} & \frac{27}{(16h^2)} & \frac{135}{(128h^2)} & \frac{1215}{(2048h^2)} \\ 0 & 0 & \frac{2}{h^2} & \frac{6}{h^2} & \frac{12}{h^2} & \frac{20}{h^2} & \frac{30}{h^2} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} y_{n+\frac{2}{8}} \\ y_{n+\frac{3}{8}} \\ f_n \\ f_{n+\frac{1}{8}} \\ f_{n+\frac{2}{8}} \\ f_{n+\frac{3}{8}} \\ f_{n+1} \end{pmatrix}. \tag{2.7}$$

Matlab software is employed for finding the values of $a_i{}'s$, $i = 0(1)6$ and are then substituted these values into equation (2.6) gives

$$y(x) = \sum_{i=\frac{2}{8},\frac{3}{8}} \alpha_i(x) y_{n+s_i} + \sum_{i=0}^{1} \beta_i(x) f_{n+i} + \sum_{i=\frac{1}{8},\frac{2}{8},\frac{3}{8}} \beta_i f_{n+i}. \tag{2.8}$$

The first derivative of equation (2.8) are given by

$$y^{'}(x) = \sum_{i=\frac{2}{8},\frac{3}{8}} \frac{d}{dx}\alpha_i(x) y_{n+s_i} + \sum_{i=0}^{1} \frac{d}{dx}\beta_i(x) f_{n+i} + \sum_{i=\frac{1}{8},\frac{2}{8},\frac{3}{8}} \frac{d}{dx}(x)\beta_i f_{n+i} \tag{2.9}$$

where

$$\begin{aligned}
\alpha_{\frac{3}{8}} &= \frac{(8x-8x_n)}{h} - 2 \\
\alpha_{\frac{2}{8}} &= -\frac{(8x-8x_n)}{h} - 3 \\
\beta_1 &= \frac{(x-x_n)^3}{210h} + \frac{11(x-x_n)^4}{315h^2} - \frac{16(x-x_n)^5}{175h^3} + \frac{128(x-x_n)^6}{1575h^4} + \frac{11h(x-x_n)}{268800} - \frac{h^2}{358400} \\
\beta_0 &= \frac{(x-x_n)^2}{2} - \frac{47(x-x_n)^3}{18h} + \frac{59(x-x_n)^4}{9h^2} - \frac{112(x-x_n)^5}{15h^3} + \frac{128(x-x_n)^6}{5h^4} - \frac{983h(x-x_n)}{23040} + \frac{37h^2}{30720} \\
\beta_{\frac{2}{8}} &= -\frac{8(x-x_n)^3}{3h} + \frac{140(x-x_n)^4}{9h^2} - \frac{128(x-x_n)^5}{5h^3} + \frac{512(x-x_n)^6}{45h^4} - \frac{61h(x-x_n)}{640} + \frac{69h^2}{2560} \\
\beta_{\frac{1}{8}} &= \frac{32(x-x_n)^3}{7h} - \frac{368(x-x_n)^4}{21h^2} + \frac{832(x-x_n)^5}{35h^3} - \frac{1024(x-x_n)^6}{105h^4} - \frac{1069h(x-x_n)}{6720} + \frac{143h^2}{8960} \\
\beta_{\frac{3}{8}} &= \frac{32(x-x_n)^3}{45h} - \frac{208(x-x_n)^4}{45h^2} + \frac{704(x-x_n)^5}{75h^3} - \frac{1024(x-x_n)^6}{225h^4} - \frac{223h(x-x_n)}{14400} + \frac{53h^2}{19200}.
\end{aligned}$$

Evaluating Equation(2.8) at the non-interpolating point i.e., $x_{n+\frac{1}{8}}$, x_{n+1} and Equation (2.9) at all points gives a discrete schemes and its derivatives. The discrete scheme and its derivatives are then combined in a matrix form as below

$$A^{[3]_2} Y_m^{[3]_2} = B^{[3]_2} R_1^{[3]_2} + h^2 D^{[3]_2} R_2^{[3]_2} + h^2 E^{[3]_2} R_3^{[3]_2} \tag{2.10}$$

where

$$A^{[3]_2} = \begin{pmatrix} 0 & -3 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & -6 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{8}{h} & \frac{-8}{h} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{8}{h} & \frac{-8}{h} & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{8}{h} & \frac{-8}{h} & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{8}{h} & \frac{-8}{h} & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{8}{h} & \frac{-8}{h} & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad Y_m^{[3]_2} = \begin{pmatrix} y_{n+\frac{1}{8}} \\ y_{n+\frac{2}{8}} \\ y_{n+\frac{3}{8}} \\ y_{n+1} \end{pmatrix}, \quad B^{[3]_2} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad R_1^{[3]_2} = \begin{bmatrix} y_n \\ y_n^{'} \end{bmatrix}$$

$$D^{[3]_2} = \begin{pmatrix} \frac{37h^2)}{30720} \\ \frac{-1}{30720} \\ \frac{-449)}{2048} \\ \frac{-983}{23040h} \\ \frac{59}{23040h} \\ \frac{-13}{7680h} \\ \frac{43}{23040h} \\ \frac{-7069}{7680h} \end{pmatrix}, \; R_2^{[3]_3} = \begin{bmatrix} f_n \end{bmatrix}, \quad E^{[3]_2} = \begin{pmatrix} \frac{143}{8960} & \frac{69}{2560} & \frac{53}{19200} & \frac{-1}{358400} \\ \frac{19}{13440} & \frac{33}{2560} & \frac{13}{9600} & \frac{-1}{1075200} \\ \frac{1655}{1792} & \frac{-2153}{1536} & \frac{1167}{1280} & \frac{4309}{215040} \\ \frac{-1069}{6720h} & \frac{-61}{640h} & \frac{-223}{14400h} & \frac{11}{268800h} \\ \frac{-61}{1120h} & \frac{-737}{5760h} & \frac{-11}{1440h} & \frac{-1}{161280h} \\ \frac{67}{6720h} & \frac{-65}{1152h} & \frac{-23}{1600h} & \frac{17}{806400h} \\ \frac{-1}{96h} & \frac{23}{640h} & \frac{253}{7200h} & \frac{-1}{38400h} \\ \frac{5111}{1344h} & \frac{-32293}{5760h} & \frac{5209}{1600h} & \frac{125729}{806400h} \end{pmatrix} \text{ and } R_3^{[3]_3} = \begin{pmatrix} f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{2}{3}} \\ f_{n+1} \end{pmatrix}.$$

Multiplying Equation (2.10) by $(A^{[3]_2})^{-1}$ produces the following one step hybrid block method

$$I^{[3]_2} Y_m^{[3]_2} = \bar{B}^{[3]_3} R_1^{[3]_3} + +h^2 \bar{D}^{[3]_2} R_2^{[3]_2} + h^2 \bar{E}^{[3]_2} R_3^{[3]_2} \tag{2.11}$$

where

$$I^{[3]_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; \bar{B}^{[3]_2} = \begin{pmatrix} 1 & \frac{h}{8} \\ 1 & \frac{h}{4} \\ 1 & \frac{3h}{8} \\ 1 & h \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \; \bar{D}^{[3]_2} = \begin{pmatrix} \frac{151)}{36864} \\ \frac{109)}{11520} \\ \frac{303)}{20480} \\ \frac{-8)}{45} \\ \frac{521}{11520h} \\ \frac{59}{1440h} \\ \frac{57}{1280h} \\ \frac{-79}{90h} \end{pmatrix}, \quad \bar{E}^{[3]_2} = \begin{pmatrix} \frac{41h^2}{7680} & \frac{-11}{5120} & \frac{61}{115200} & \frac{-1}{307200} \\ \frac{1}{42} & \frac{-1}{320} & \frac{1}{900} & \frac{-1}{134400} \\ \frac{783}{17920} & \frac{9}{1024} & \frac{39}{12800} & \frac{-9}{716800} \\ \frac{16}{15} & \frac{-4}{3} & \frac{208}{225} & \frac{1}{50} \\ \frac{703}{6720h} & \frac{-47}{1440h} & \frac{113}{14400h} & \frac{-19}{403200h} \\ \frac{71}{420h} & \frac{7}{180h} & \frac{1}{900h} & \frac{-1}{50400h} \\ \frac{333}{2240h} & \frac{21}{160h} & \frac{81}{1600h} & \frac{-3}{44800h} \\ \frac{416}{105h} & \frac{-248}{45h} & \frac{736}{225h} & \frac{491}{3150h} \end{pmatrix}$$

which can be clearly written as

$$\begin{aligned}
y_{n+\frac{1}{8}} &= y_n + \frac{h}{8}y_n^{'} + \frac{151h^2}{36864}f_n - \frac{h^2}{307200}f_{n+1} + \frac{41h^2}{7680}f_{n+\frac{1}{8}} - \frac{11h^2}{5120}f_{n+\frac{2}{8}} + \frac{61h^2}{115200}f_{n+\frac{3}{8}} \\
y_{n+\frac{2}{8}} &= y_n + \frac{h}{4}y_n^{'} + \frac{109h^2}{11520}f_n - \frac{h^2}{134400}f_{n+1} + \frac{h^2}{42}f_{n+\frac{1}{8}} - \frac{h^2}{320}f_{n+\frac{2}{8}} + \frac{h^2}{900}f_{n+\frac{3}{8}} \\
y_{n+\frac{3}{8}} &= y_n + \frac{3h}{8}y_n^{'} + \frac{303h^2}{20480}f_n - \frac{9h^2}{716800}f_{n+1} + \frac{783h^2}{17920}f_{n+\frac{1}{8}} + \frac{9h^2}{1024}f_{n+\frac{2}{8}} + \frac{39h^2}{12800}f_{n+\frac{3}{8}} \\
y_{n+1} &= y_n + hy_n^{y} - \frac{8h^2}{45}f_n + \frac{h^2}{50}f_{n+1} + \frac{16h^2}{15}f_{n+\frac{1}{8}} - \frac{4h^2}{3}f_{n+\frac{2}{8}} + \frac{208h^2}{225}f_{n+\frac{3}{8}} \\
y_{n+\frac{1}{8}}^{'} &= y_n^{'} + \frac{521h}{11520}f_n - \frac{19h}{403200}f_{n+1} + \frac{703h}{6720}f_{n+\frac{1}{8}} - \frac{47h}{1440}f_{n+\frac{2}{8}} + \frac{113h}{14400}f_{n+\frac{3}{8}} \\
y_{n+\frac{2}{8}}^{'} &= y_n^{'} + \frac{59h}{1440}f_n - \frac{h}{50400}f_{n+1} + \frac{71h}{420}f_{n+\frac{1}{8}} + \frac{7h}{180}f_{n+\frac{2}{8}} + \frac{h}{900}f_{n+\frac{3}{8}} \\
y_{n+\frac{3}{8}}^{'} &= y_n^{'} + \frac{57h}{1280}f_n - \frac{3h}{44800}f_{n+1} + \frac{333h}{2240}f_{n+\frac{1}{8}} + \frac{21h}{160}f_{n+\frac{2}{8}} + \frac{81h}{1600}f_{n+\frac{3}{8}} \\
y_{n+1}^{'} &= y_n^{'} - \frac{79h}{90}f_n + \frac{491h}{3150}f_{n+1} + \frac{416h}{105}f_{n+\frac{1}{8}} - \frac{248h}{45}f_{n+\frac{2}{8}} + \frac{736h}{225}f_{n+\frac{3}{8}}
\end{aligned}$$

3 Numerical Results

3.1 Examples

In order to confirm the performance of the new one step hybrid block method, the following two non-linear second Dirichlet BVPS were solved and compared with existing methods in [5], [6] and [8] as demonstrated in **tables 2,4**. The notation below are used in the tables:

HA : Method proposed by [5].
2PDAM5 : Method proposed by [8].
DMS : Method proposed by [6].

OSHBT : Implementation of the one step hybrid block method with three off-step points.

Problem 1: $y^{''}(x) = y^3 - yy^{'}$, $y(1) = \dfrac{1}{2}$, $y(2) = \dfrac{1}{3}$, $x \in [1, 2]$.

Exact solution : $y(x) = \dfrac{1}{x+1}$

Table 1: The iteration of guess in Problem 1

$t_0 = \frac{\beta-\alpha}{b-a}$
-0.166666666666667
-0.250000000171601

Table 2: Showing of the approximated errors for solving Problem 1

	$t_0 = 4.0$			$t_0 = \frac{\beta-\alpha}{b-a}$	
x	Error HA	Error 2PDAM5	Error OSHBT	Error 2PDAM5	Error OSHBT
1.1	$5.50e^{-5}$	$7.16e^{-9}$	$1.50e^{-11}$	$7.16e^{-9}$	$1.51e^{-11}$
1.2	$9.10e^{-5}$	$1.17e^{-8}$	$2.29e^{-11}$	$1.17e^{-8}$	$2.30e^{-11}$
1.3	$1.11e^{-4}$	$1.19e^{-8}$	$2.60e^{-11}$	$1.19e^{-8}$	$2.61e^{-11}$
1.4	$1.18e^{-4}$	$1.18e^{-8}$	$2.59e^{-11}$	$1.18e^{-8}$	$2.61e^{-11}$
1.5	$2.90e^{-7}$	$1.05e^{-6}$	$2.38e^{-11}$	$1.08e^{-8}$	$2.40e^{-11}$
1.6	$1.05e^{-4}$	$9.33e^{-9}$	$2.02e^{-11}$	$9.30e^{-9}$	$2.05e^{-11}$
1.7	$8.80e^{-5}$	$7.37e^{-9}$	$1.58e^{-11}$	$7.33e^{-9}$	$1.61e^{-11}$
1.8	$6.50e^{-5}$	$5.12e^{-9}$	$1.07e^{-11}$	$5.08e^{-9}$	$1.11e^{-11}$
1.9	$3.70e^{-5}$	$2.66e^{-9}$	$5.29e^{-11}$	$2.61e^{-9}$	$5.70e^{-12}$
2	$6.00e^{-6}$	$5.72e^{-11}$	$4.63e^{-11}$	$1.92e^{-13}$	$7.10e^{-15}$

Problem 2: $y^{''}(x) = \dfrac{3}{2}y^2$, $y(0) = 4$, $y(1) = 1$, $x \in [0, 1]$.

Exact solution : $y(x) = \dfrac{4}{(x+1)^2}$

Table 3: The iteration of guess in Problem 2

$t_0 = 5$	$t_0 = \frac{\beta-\alpha}{b-a}$
0.5000000000	-3.000000000000000
-6.735149691323533	-7.923358204435266
-7.993577850946418	-7.999999953573528
-7.999996778595551	

3.2 Application of the Developed Method in Troesch problem

According to [7], Troesch problem as stated below

$$y^{''} = \lambda \sinh(\lambda y), \quad y(0) = 0, \quad y(1) = 1. \tag{3.1}$$

is a nonlinear two-point BVP occurred in an investigation of the confinement of a plasma column by radiation pressure. Equation above had been studied by Troesch in order to explain the setbacks faced in the numerical solution of a two-point system of nonlinear BVP of ODEs.

Table 4: Showing Errors for solving Problem 2

	$t_0 = -3.0$			$t_0 = \frac{\beta-\alpha}{b-a}$		
x	Error DMS	Error 2PDAM5	Error OSHBT	Error Ha	Error DMS	Error OSHBT
0.1	$3.19e^{-5}$	$1.45e^{-6}$	$6.90e^{-9}$	$3.0e^{-6}$	$3.19e^{-4}$	$3.16e^{-7}$
0.2	$2.62e^{-5}$	$9.97e^{-7}$	$3.75e^{-8}$	$4.0e^{-6}$	$2.62e^{-4}$	$6.40e^{-7}$
0.3	$2.11e^{-4}$	$8.48e^{-7}$	$8.14e^{-8}$	$4.0e^{-6}$	$2.11e^{-4}$	$1.00e^{-7}$
0.4	$1.69e^{-4}$	$6.13e^{-7}$	$1.35e^{-7}$	$4.0e^{-6}$	$1.68e^{-4}$	$1.44e^{6}$
0.5	$1.32e^{-4}$	$4.53e^{-7}$	$2.00e^{-7}$	$4.0e^{-6}$	$1.32e^{-4}$	$1.96e^{-6}$

Since then, Troesch problem has widely been considered as a test case for numerical methods of solving unstable two-point BVP due to its difficulties. Roberts and Shipmanhas introduced closed form analytical solution of this problem as

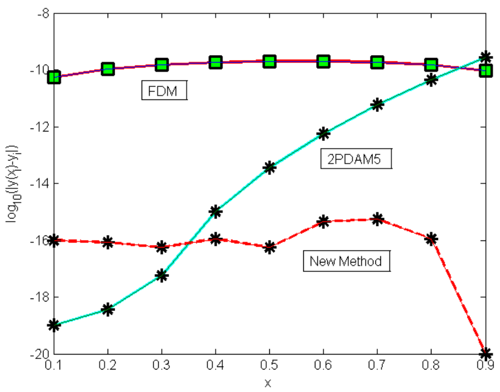
$$y(x) = \frac{\lambda}{2}\sinh^{-1}\left[\frac{y^{'}(0)}{2} sc\left(\lambda x, 1 - \frac{1}{4}(y^{'}(0))^2\right)\right] \tag{3.2}$$

where $y^{'}(0) = 2(1-m)^{\frac{1}{2}}$ and m is the solution of $\frac{\sinh(\frac{\lambda}{2})}{(1-m)^{\frac{1}{2}}} = sc(\lambda, m)$. The exact closed solution of this problem when $\lambda = 0.5$ has been found using wolframalpha engine software and then compared with numerical resulted generated by the developed method when $h = 0.01$

Table 3: Numerical results of the new method for solving Troesch Problem where $\lambda = 0.5$ and $h = 0.01$.

x	EXACT SOLUTION	COMPUTED SOLUTION	ERROR
0.1	0.095944349292286718	0.095944349292286818	$9.71445146e^{-17}$
0.2	0.192128747660289180	0.192128747660289260	$8.32667268e^{-17}$
0.3	0.288794400893448531	0.288794400893448490	$5.55111512e^{-17}$
0.4	0.386184846362337290	0.386184846362337410	$1.11022302e^{-16}$
0.5	0.484547164744892490	0.484547164744892570	$5.55111512e^{-17}$
0.6	0.584133248445574153	0.584133248445574570	$4.44089209e^{-16}$
0.7	0.685201148301847298	0.685201148301847860	$5.55111512e^{-16}$
0.8	0.788016522649566625	0.788016522649566790	$1.11022302e^{-16}$
0.9	0.892854216136313670	0.892854216136313660	0

Figure 1: Comparison with [8] and [9] for solving Troesch Problem where $h = 0.01$; $\lambda = 0.5$



4 Conclusion

High accurate one step hybrid block method for solving second order linear Boundary value problem directly has been successfully developed. The new method outperformed the existing methods when solving the same BVPs of second order ODEs directly.Furthermore, the developed method can be extended to solve a system of Boundary value problem of higher ODEs directly.

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