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On study the existing result for the time fractional equation using the topological degree method

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Abstract

In this article, we suggest introducing the Riemann-Liouville time fractional derivative to a fractional equation (FPDE) involving a fractional Laplacian. Our work is divided into two parts. In the first part, the existence and uniqueness of time fractional linear equations are demonstrated, and the Galerkin approach is proposed to deal with them. In the second part, we investigate the existence results of the time fractional semilinear equation. To solve This problem, the Leray-Schauder degree method has been used with some conditions on the semilinear term.

Keywords: Time fractional derivative, Galerkin method, Leray-Schauder degree, Riemann-Liouville derivative, fractional semilinear parabolic equation
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1 Introduction

In the last decade, we note that there has been a noticeable interest for researchers in the field of fractional differential equations because they are an effective tool for modelling many phenomena and important applications in various fields such as viscoelasticity, electrodynamics, physics, furthermore other fields of engineering and science (see [8, 9, 11, 12]) and the references therein. Consequently, during the 19^{th} and 20^{th} centuries, fractional calculus theory and applications greatly increased.

The main objective of our research is to study the existence and uniqueness of time-fractional that introduced fractional Laplacian. In the first part, we are interested in studying the following fractional linear problem

$$\begin{cases} \text{Find } \mu : [0,\mathcal{T}] \times \Omega \to \mathbb{R} \\ {}^{RL}\mathscr{D}^{\delta}_{0,t}\mu(t,x) + (-\Delta)^{\delta}\mu(t,x) = h(t,x) & \text{on } [0,\mathcal{T}] \times \Omega \\ \mu = 0 & \text{on } [0,\mathcal{T}] \times \mathbb{R}^n/\Omega \\ (g_{1-\delta} * \mu)(0) = w & \text{on } \mathbb{R}^n/\Omega, \end{cases} \tag{1.1}$$

and we suggest the Galerkin method to study the existence and uniqueness for this problem. In the second Part, we

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interested in studying the folowing fractinal semilinear problem

$$\begin{cases} \text{Find } \mu : [0,\mathcal{T}] \times \Omega \to \mathbb{R} \\ {}^{RL}\mathscr{D}^{\delta}_{0,t}\mu + (-\Delta)^{\delta}\mu + h(\mu) = 0 & \text{on } [0,\mathcal{T}] \times \Omega \\ \mu = 0 & \text{on } [0,\mathcal{T}] \times \mathbb{R}^n/\Omega \\ (g_{1-\delta} * \mu)(0,.) = w & \text{on } \mathbb{R}^n/\Omega. \end{cases} \tag{1.2}$$

In order to study the existence of a solution to this problem, we suggest the topological method to deal with it. Recently, the Leray-Schauder degree technique is well-known and worthy of consideration introduced by Leray and Schauder in the early 1930s, where this last needs weakly compact assumptions rather than strongly compact assumptions. The main arguments are to demonstrate a priori estimations, which are a frequent way to demonstrate existence results (see [1, 2, 14, 15, 17]. This method is a valuable topological tool for studying nonlinear partial differential equations, and it may also be used in fractional instances. It also has the advantage of providing information on the number of possible solutions, continuous families of solutions, and solution stability.

In fact, when we replace the nonlocal fractional Laplacian with local classical Laplacian, we can see our problems in [6] when the authors studied the existence and uniqueness of time-fractional diffusion problems. And in [13] the authors were interested in the time-fractional semilinear problem, to deal with it, they applied a Galerkin method. Basically, we can show that our problems are the generalizations of the problems mentioned above. Because in their problem they found the solution in a classical Sobolev space, and we found the solution in fractional Sobolev space. On the other hand, our problem is interesting by his nonlocal property.

Let us start by defining the term fractional Laplacian as an integral in the sens of the Cauchy principle value in the real space for all $z \in \mathscr{S}, \forall \delta \in (0,1)$, as follows

$$(-\Delta)^{\delta} z(x) = P(n,\delta) p.v. \int_{\mathbb{R}^n} \frac{z(x) - z(y)}{|x-y|^{n+2\delta}} dy, \quad x \in \mathbb{R}^n,$$

with $P(n,\delta) = \pi^{-(2\delta+n/2)} \dfrac{\Gamma(\delta + n/2)}{\Gamma(-\delta)}$, and $\mathscr{S}$ is the Schwartz space. This nonlocal operator have according to spectral theory an eigenvalues. These values a finite threat and form a diverging sequence (see[3])

$$0 < \lambda_1^{\delta}(\Omega) \leq \lambda_2^{\delta}(\Omega) \leq \lambda_3^{\delta}(\Omega) \leq \cdots \to +\infty.$$

In recent years, many authors have been working with this nonlocal operator of fractional Laplacian. We suggest some works, for example. In (2021), E. Abada et all [1] came up with the idea of their work from [10] and replaced the local operator with the nonlocal operator, we also refer to [2, 4, 7, 16], and the references therein. On the other hand, the authors in [6] wrote the Riemann-Liouville time fractional derivatives ${}^{RL}\mathscr{D}^{\delta}_{0,t}$ as follow

▷ For $z \in L^2(0,\mathcal{T};E)$, if $g_{1-\delta} * z \in H^1(0,\mathcal{T};E)$ then

$${}^{RL}\mathscr{D}^{\delta}_{0,t} z = \frac{d}{dt}\{g_{1-\delta} * z\},$$

▷ the adjoint of Riemann-Liouville derivatives denoted by ${}^{RL}\mathscr{D}^{\delta}_{t,\mathcal{T}}$ and define as follow

$${}^{RL}\mathscr{D}^{\delta}_{t,\mathcal{T}}\phi(t) = \int_t^{\mathcal{T}} g_{1-\delta}(y-t) \frac{d}{dt}\phi(y) dy, \quad \text{for all } t \in [0,\mathcal{T}].$$

with $g_{1-\delta}$ denoting the kernel of order $1-\delta$, and the convolution of $g_{1-\delta} * z$ defined by

$$g_{1-\delta} * z(t) = \int_0^t g_{1-\delta}(t-y) z(y) dy, \text{ a.e. } t \in [0,\mathcal{T}], \text{ and } g_{\delta}(t) = \frac{1}{\Gamma(\delta)} t^{\delta-1} \in L^1_{loc}([0,+\infty]).$$

The arrangement of our article is as follows; in the coming Section, we mention some characteristics and results that we will use in our research to reach our goal. In Section 03, we propose the Galerkin method to prove the existence and uniqueness of the linear equation of time-fractional, which we introduced on the fractional Laplacian. In Section 04, under some assumption, we prove the existence solution of the semilinear equation for time-fractional, and we use topological methods to deal with it. In conclusion, we end with an epilogue where we will mention some future works in it.

2 Preliminaries

In this section, we will define some characteristics and results which we will use to reach our goal in our work. Let $(E, \|.\|)$ be a real Banach space, and $\mathcal{T}$ be a positive number, and Ω subset of $\mathbb{R}^n$ with Lipschitz boundary.

Proposition 2.1. (see [1]) Let Ω be a Lipschitz bounded open subset of $\mathbb{R}^n$ and $0 < \delta < 1$ such that $n > 2\delta$. Let $\mu : \Omega \to \mathbb{R}$ be a measurable function compactly supported. Then, there exists a positive constant $C_{emb} > 0$ depending on n, δ and Ω such that

$$\|\mu\|_{L^2(\Omega)} \leq C_{emb}\|\mu\|_{H_0^\delta(\Omega)}. \tag{2.1}$$

Theorem 2.2. [6] If $p \in L^2(0, \mathcal{T}; E)$ and $q \in L^1(0, \mathcal{T})$ then

$$q * p \in L^2(0, \mathcal{T}; E) \quad \text{and } \|q * p\|_{L^2(0,\mathcal{T};E)} \leq \|q\|_{L^1(0,\mathcal{T})}\|p\|_{L^2(0,\mathcal{T};E)}. \tag{2.2}$$

Theorem 2.3. [6] Let $(H, (.,.))$ be real Hilbert space, $\mu \in L^2(0, \mathcal{T}; H)$ and $\delta \in (0, 1)$. Then

$$\int_0^{\mathcal{T}} (\mu(t), g_\delta * \mu(t))\, dt \geq 0.$$

Proposition 2.4. [6] Let $\delta \in (0, 1)$ and $v \in L^2(0, \delta; E)$. If v admits a derivative of order δ in $L^2(0, \mathcal{T}; E)$, then

$$v = (g_{1-\delta} * v)(0)g_\delta + g_\delta * {}^{RL}\mathscr{D}_{0,t}^\delta v \quad \text{in } L^1(0, \mathcal{T}; E). \tag{2.3}$$

Proposition 2.5. [6] Let $\delta \in (0, 1)$, $v \in L^2(0, \mathcal{T}; E)$ and $\phi \in H^1(0, \mathcal{T})$. Assume that v admits a derivative of order δ in $L^2(0, \mathcal{T}, E)$. Then

$$\int_0^{\mathcal{T}} {}^{RL}\mathscr{D}_{0,t}^\delta v(t)\phi(t)dt = -\int_0^{\mathcal{T}} v(t)^{RL}\mathscr{D}_{t,\mathcal{T}}^\delta \phi(t)dt + [g_{1-\delta} * v\phi]_0^{\mathcal{T}} \quad \text{in } E, \tag{2.4}$$

$\triangleright$ if $\phi \in C_c^\infty(0, \mathcal{T})$ then

$$\|\int_0^{\mathcal{T}} v(t)^{RL}\mathscr{D}_{t,\mathcal{T}}^\delta \phi(t)dt\| \leq \sqrt{\mathcal{T}} g_{2-\delta}(\mathcal{T})\|v\|_{L^2(0,\mathcal{T};E)}\|\phi'\|_{L^\infty(0,\mathcal{T})}. \tag{2.5}$$

Remark 2.6. We remark that from the Proposition 2.5, we can define the fractional derivative in the sense of distributions. and we sees the followinf linear map

$$\mathcal{D}(0, \mathcal{T}) \to E \tag{2.6}$$

$$\phi \mapsto -\int_0^{\mathcal{T}} v(t)^{RL}\mathscr{D}_{t,\mathcal{T}}^\delta \phi(t)dt. \tag{2.7}$$

The equation (2.7) is a distribution of order (at most) 1. Denote by $\mathcal{D}'(0, \mathcal{T}; E)$ the set of distributions with values in E.

The following definition defines the weak derivation

Definition 2.7. Let $\delta \in (0, 1)$ and $v \in L^2(0, \mathcal{T}; E)$. Then the *weak derivative* of order δ of v is the vactor valued distribution, denoted by ${}^{RL}\mathscr{D}_{0,t}^\delta$, and defined, for all $\phi \in \mathcal{D}(0, \mathcal{T})$, as follows

$$\left\langle {}^{RL}\mathscr{D}_{0,t}^\delta v, \phi \right\rangle = -\int_0^{\mathcal{T}} v(t)^{RL}\mathscr{D}_{t,\mathcal{T}}^\delta \phi(t)dt.$$

If we want to emphasize the duality occurring in the baracket above, we will write $\left\langle {}^{RL}\mathscr{D}_{0,t}^\delta v, \phi \right\rangle_{\mathcal{D}'(0,\mathcal{T};E),\mathcal{D}(0,\mathcal{T})}$, instead of $\left\langle {}^{RL}\mathscr{D}_{0,t}^\delta v, \phi \right\rangle$.

Proposition 2.8. [6] Let $\delta \in (0,1)$, E be a real Banach space and $\mu \in L^2(0,\mathcal{T},E^{'})$. We assume that μ admits a derivative of order δ in $L^2(0,\mathcal{T},E^{'})$. Then, for each w in E, $\langle \mu, w\rangle_{E',E}$ admits a derivative o order δ in $L^2(0,\mathcal{T})$ and

$$\left\langle {}^{RL}\mathscr{D}^{\delta}_{0,t}\mu, w\right\rangle_{E',E} = {}^{RL}\mathscr{D}^{\delta}_{0,t}\{\langle \mu, w\rangle_{E',E}\}, \quad \text{in} \quad L^2(0,\mathcal{T}). \tag{2.8}$$

Corollary 2.9. [13] For E be a real Banach space densely and continuously embedded into a real Hilbert space H. Assume that $v \in {}_0W^s_{2,2}(0,\mathcal{T};E,E^{'})$, then, for every $\tau \in [0,\mathcal{T}]$,

$$\frac{1}{2}g_{1-\delta} * \|v(.)\|^2_H(t) \leq \int_0^{\tau} \left\langle {}^{RL}\mathscr{D}^{\delta}_{0,t}v(x), v(x)\right\rangle dx. \tag{2.9}$$

Theorem 2.10. [13] Let E be a real Banach space densely and continuously embedded into a real Hilbert space H, $\delta \in (0,1)$ and $p \geq 2$ be such that $s > 1/p^{'}$. Assum $v \in W^s_{p,p'}(0,\mathcal{T},E,E^{'})$, and $(g_{1-\delta} * v)(0) \in E$. Then

$$\int_0^{\mathcal{T}} \left\langle {}^{RL}\mathscr{D}^{\delta}_{0,t}v(t), v(t) - (g_{1-\delta} * v)(0)g_{\delta}(t)\right\rangle_{E,E'} dt \geq 0.$$

Definition 2.11. Let $\mathcal{T} > 0$ and $\delta \in (0,1)$, we mention by

$$H^{\delta}(0,\mathcal{T};H^{\delta}_0(\Omega),H^{-\delta}(\Omega)) = \{\mu \in L^2(0,\mathcal{T};H^{\delta}_0(\Omega)) \text{ whose } {}^{RL}\mathscr{D}^{\delta}_{0,t}\mu \in L^2(0,\mathcal{T};H^{-\delta}(\Omega))\},$$

where ${}^{RL}\mathscr{D}^{\delta}_{0,t}\mu$ is weak fractional derivative.

▷ We put (just a notation)

$$H^{\delta}_0(\Omega) = Y \quad \text{and} \quad H^{-\delta}(\Omega) = Y^{'}.$$

▷ Throughout this research, we have assumed that $\delta > 1/2$ and $n > 2\delta$. ▷ Also in this work, we assume Ω be a Lipschitz bounded open subset of $\mathbb{R}^n$.

3 Galerkin method for Time-fractional linear equation

In this Section, we consider the following linear problem

$$\begin{cases} \text{Find } \mu \in H^{\delta}(0,\mathcal{T};Y,Y^{'}) \\ {}^{RL}\mathscr{D}^{\delta}_{0,t}\mu + (-\Delta)^{\delta}\mu = h \qquad \text{in } L^2(0,\mathcal{T};Y^{'}) \\ (g_{1-\delta} * \mu)(0) = w \qquad \text{in } L^2(\Omega), \end{cases} \tag{3.1}$$

and we use the Galerkin method to prove the existence and uniqueness of weak solotion for these problem.

Lemma 3.1. The problem (3.1) has unique weak solution, $\mu \in H^{\delta}(0,\mathcal{T};Y,Y^{'})$.

Proof .

Part 01. ***Existence of a weak solution*** This Part is divided into fourth Step.

First, we take the space E_n the vector space generated by $\varphi_1,\dots,\varphi_n$, that is means $E_n = vect\{\varphi_1,\dots,\varphi_n\}$ and $(\varphi_k)_{k\leq 1}$ forms an Hilbertian basis of $L^2(\Omega)$.

▷ we remark that $((\lambda^{\delta}_k)^{-1/2}\varphi_k)$ is a Hilbertian basic of Y, where $\lambda^{\delta}_k \in (0,+\infty)$ is kth eigenvalues of the operator $(-\Delta)^{\delta}$, $k = 1,2,\dots$

Secondly, Let us decompose the initial condition w. Since Y is a Hilbert space the we writing w by

$$w = \sum_{k\geq 1} a_k\varphi_k \quad \text{in } Y, \tag{3.2}$$

and we have E_n a space of finite dimension then

$$w_n = \sum_{k=1}^{n} a_k\varphi_k. \quad \text{in } E_n, \tag{3.3}$$

the following property true that $w_n \to w$ in Y. Finally, we define our approximated problem For every integer $n \geq 1$, as the following form

$$\begin{cases} \text{Find } \mu_n \in L^2(0,\mathcal{T};E_n) \text{ such that } {}^{RL}\mathscr{D}^{\delta}_{0,t}\mu_n \in L^2(0,\mathcal{T};Y') \\ \langle {}^{RL}\mathscr{D}^{\delta}_{0,t}\mu_n, \varphi\rangle_{Y',Y} + P(n,\delta)\iint_{\mathbb{R}^{2n}} \dfrac{(\mu_n(x)-\mu_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2\delta}}dydx = \langle h,\varphi\rangle_{Y',Y} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } L^2(0,\mathcal{T}), \forall \varphi \in E_n \\ (g_{1-\delta} * \mu_n)(0) = w_n. \end{cases} \tag{3.4}$$

Step 01: ***Solvability of the approximated problem.*** We suppose that The following decomposition

$\mu_n(t) = \sum_{k=1}^n y_k(t)\varphi_k, h_k = \langle h(t),\varphi_k\rangle_{Y',Y}$, when we substitute in the problem 3.4, we get an equivalent equation for it and defined as follow

$$\begin{cases} {}^{RL}\mathscr{D}^{\delta}_{0,t}y_k(t) + \lambda^{\delta}_k y_k(t) = h_k \quad \text{in } L^2(0,\mathcal{T}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall k = 1,\dots,n \\ (g_{1-\delta} * y_k)(0) = a_k. \end{cases} \tag{3.5}$$

The local result for (3.5) in $L^2(0,\tau)$, for $\imath$ smal positive is solvable (see [5, chap 5]). We will now prove that there is a global result for (3.5), we know that if it is blow up then the global solution does not exist, and for that we prove by Contradiction. Let us assume that $\mathcal{T}_m$ is finite. Then, for every $\imath \in (0,\mathcal{T})$, we have by (3.5), Proposition 2.4 and (2.2)

$$y_k(t) + \lambda^{\delta}_k g_\delta * y_k(t) = a_k g_\delta + g_\delta * h_k \quad \text{in } L^2(0,\imath). \tag{3.6}$$

We multiply (3.6) by y_k and integrate on $0,\imath$. Since $\lambda^{\delta}_k \geq 0$ and $\delta > 1/2$, we get By Theorem 2.3

$$\int_0^{\imath} |y_k(t)|^2 dt \leq \int_0^{\imath} a_k g_\delta y_k(t)dt + \int_0^{\imath} g_\delta * h_k y_k(t)dt,$$

From Cauchy-Schwarz inequality, we arrived

$$\|y_k\|^2_{L^2(0,\imath)} \leq |a_k|\|g_\delta\|_{L^2(0,\mathcal{T}_m)}\|y_k\|_{L^2(0,\imath)} + \|g_\delta * h_k\|_{L^2(0,\mathcal{T}_m)}\|y_k\|_{L^2(0,\imath)}. \tag{3.7}$$

We deduced that y_k bounded in $L^2(0,\tau)$ as τ approaches $\mathcal{T}_m$. That contradiction with the condition of blow up, so that $\mathcal{T}_m = +\infty$. Our conclude here is for every time $\mathcal{T} \geq 0$ the (3.4) has only one solution.

Step 02: ***Priori estimates.*** In the present Step, we will prove that μ_n is bounded in $L^2(0,\mathcal{T};Y)$ and ${}^{RL}\mathscr{D}^{\delta}_{0,t}\mu_n$ is bounded in $L^2(0,\mathcal{T};Y')$. For that we use $g_\delta \in L^2(0,\mathcal{T})$, we have

$$\int_0^{\mathcal{T}} \langle {}^{RL}\mathscr{D}^{\delta}_{0,t}\mu_n, \mu_n - g_\delta w_n\rangle_{Y',Y} dt + P(n,\delta)\int_0^{\mathcal{T}}\iint_{\mathbb{R}^{2n}} \frac{(\mu_n(x)-\mu_n(y))\big((\mu_n(x)-\mu_n(y) - g_\delta(w_n(x)-w_n(y))\big)}{|x-y|^{n+2\delta}}dydxdt$$
$$= \int_0^{\mathcal{T}} \langle h, \mu_n - g_\delta w_n\rangle dxdt,$$

Hence, by the Theorem 2.10, we get

$$\begin{aligned} &P(n,\delta)\int_0^{\mathcal{T}}\iint_{\mathbb{R}^{2n}} \frac{|\mu_n(x)-\mu_n(y)|^2}{|x-y|^{n+2\delta}}dydxdt \\ &\quad \leq P(n,\delta)\int_0^{\mathcal{T}}\iint_{\mathbb{R}^{2n}} \frac{(\mu_n(x)-\mu_n(y))(w_n(x)-w_n(y))}{|x-y|^{n+2\delta}}g_\delta dydxdt + \int_0^{\mathcal{T}}\int_\Omega h\mu_n dxdt - \int_0^{\mathcal{T}}\int_\Omega h w_n g_\delta dxdt. \end{aligned}$$

Then, using Cauchy-Schwarz inequality, we get

$$\begin{aligned} P(n,\delta)\|\mu_n\|^2_{L^2(0,\mathcal{T};Y)} &\leq P(n,\delta)\int_0^{\mathcal{T}} \|\mu_n\|_Y\|w_n\|_Y g_\delta dt + \int_0^{\mathcal{T}} \|h\|_{Y'}\|\mu_n\|_y dt + \int_0^{\mathcal{T}} \|h\|_{Y'}\|w_n\|_Y g_\delta dt \\ &\leq \Big[P(n,\delta)\|w_n\|_Y\|g_\delta\|_{L^2(0,\mathcal{T})} + \|h\|_{L^2(0,\mathcal{T};Y')}\Big]\|\mu_n\|_{L^2(0,\mathcal{T};Y)} + \|w_n\|_Y\|g_\delta\|_{L^2(0,\mathcal{T})}\|h\|_{L^2(0,\mathcal{T};Y')}, \end{aligned}$$

we observe that $w_n \to w$ in Y, we obtain the desired estimate

$$\|\mu_n\|_{L^2(0,\mathcal{T};Y)} \leq K_1. \tag{3.8}$$

where the constant K_1 is independent of n.

$\triangleright$ Our next goal is to prove that ${}^{RL}\mathscr{D}^{\delta}_{0,t}\mu_n$ is bounded in $L^2(0,\mathcal{T};Y^{'})$. Indeed, we have

$$\left|\left\langle {}^{RL}\mathscr{D}^{\delta}_{0,t}\mu_n, \varphi\right\rangle_{Y',Y}\right| \leq P(n,\delta)\left|\iint_{\mathbb{R}^{2n}} \frac{(\mu_n(x)-\mu_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2\delta}}dydx\right| + \left|\langle h,\varphi\rangle_{Y',Y}\right| \quad \forall \varphi \in Y.$$

Moreover, from Cauchy-Schwarz inequality obtain

$$\left|\langle {}^{RL}\mathscr{D}^{\delta}_{0,t}\mu_n, \varphi\rangle_{Y',Y}\right| \leq P(n,\delta)\|\mu_n\|_Y\|\varphi\|_Y + \|h\|_{Y'}\|\varphi\|_Y,$$

then, we find

$$\int_0^{\mathcal{T}} \|{}^{RL}\mathscr{D}^{\delta}_{0,t}\mu_n\|^2_{Y'}dt \leq 2P^2(n,\delta)\int_0^{\mathcal{T}} \|\mu_n\|^2_Y dt + 2\int_0^{\mathcal{T}} \|h\|^2_{Y'}dt$$

In Finally, we get to

$$\|{}^{RL}\mathscr{D}^{\delta}_{0,t}\mu_n\|_{L^2((0,\mathcal{T};Y'))} \leq k_2, \tag{3.9}$$

where $k_2 = \Big(2P^2(n,\delta)k_1^2 + 2\|h\|^2_{L^2(0,\mathcal{T},Y')}\Big)^{1/2}$. Therefore, we have from (3.8) and (3.9) that there exists $\mu \in L^2(0,\mathcal{T};Y)$ such that

$$\mu_n \rightharpoonup \mu \quad \text{in } L^2(0,\mathcal{T};Y) - \text{weak}$$

and

$${}^{RL}\mathscr{D}^{\delta}_{0,t}\mu_n \rightharpoonup {}^{RL}\mathscr{D}^{\delta}_{0,t}\mu \quad \text{in } L^2(0,\mathcal{T};Y^{'}) - \text{weak}$$

Step 03: ***Passage to the limit.*** Our aim in this Step is to return from the approximation problem to the exact problem. we have

$${}^{RL}\mathscr{D}^{\delta}_{0,t}\mu_n + (-\Delta)^{\delta}\mu_n = h, \tag{3.10}$$

multiplying equation (3.10) by $\psi \in \mathcal{D}(0,\mathcal{T})$ and integre on 0, $\mathcal{T}$, and we multiply it again by φ_k and integre, for $k \geq 1$ be fixed and $n \geq k$, we get to

$$\left\langle \int_0^{\mathcal{T}} {}^{RL}\mathscr{D}^{\delta}_{0,t}\mu_n(t)\psi(t)dt, \varphi_k\right\rangle_{Y',Y} + \int_{\mathbb{R}^n}\int_0^{\mathcal{T}} (-\Delta)^{\delta}\mu_n\psi dt\varphi_k dx = \int_{\Omega}\int_0^{\mathcal{T}} h(t)\psi(t)dt\varphi_k dx$$

and thus, we derive from Proposition 2.5, and passing to the limit in n, we arrive to

$${}^{RL}\mathscr{D}^{\delta}_{0,t}\mu + (-\Delta)^{\delta}\mu - h = 0 \quad \text{in } \mathcal{D}^{'}(0,\mathcal{T};Y^{'}).$$

We observe that $\mu \in W^{\delta}_{2,2}(0,\mathcal{T};Y,Y^{'})$ and

$${}^{RL}\mathscr{D}^{\delta}_{0,t}\mu + (-\Delta)^{\delta}\mu - h = 0 \quad \text{in } L^2(0,\mathcal{T};Y^{'}).$$

This result we concluded because $(-\Delta)^{\delta}\mu$ and h belong to $L^2(0,\mathcal{T};Y^{'})$.

Step 04: ***Initial condition.*** Let $\psi \in H^{\delta}_0(0,\mathcal{T})$, $\quad \psi(\mathcal{T}) = 0$, we have

$$\int_0^{\mathcal{T}} \left\langle {}^{RL}\mathscr{D}^{\delta}_{0,t}\mu, \varphi_k\right\rangle \psi(t)dt + \int_0^{\mathcal{T}}\int_{\mathbb{R}^n} (-\Delta)^{\delta}\mu\varphi_k dx\psi(t)dt = \int_0^{\mathcal{T}} \langle h,\varphi_k\rangle \psi(t)dt.$$

Moreover, from the Proposition 2.8, and Proposition 2.5, we find

$$-\int_0^{\mathcal{T}} \langle \mu,\varphi_k\rangle\, {}^{RL}\mathscr{D}^{\delta}_{t,\mathcal{T}}\psi(t)dt + \langle (g_{1-\delta}*\mu)(0),\varphi_k\rangle \psi(0) + \int_0^{\mathcal{T}}\int_{\mathbb{R}^n} (-\Delta)^{\delta}\mu\varphi_k dx\psi(t)dt = \int_0^{\mathcal{T}} \langle h,\varphi_k\rangle \psi(t)dt. \tag{3.11}$$

In other to, we have

$$\int_0^{\mathcal{T}} \left\langle {}^{RL}\mathscr{D}_{0,t}^{\delta}\mu_n, \varphi_k \right\rangle \psi(t)dt + \int_0^{\mathcal{T}} \int_{\mathbb{R}^n} (-\Delta)^{\delta} \mu_n \varphi_k dx \psi(t)dt = \int_0^{\mathcal{T}} \langle h, \varphi_k \rangle \psi(t)dt,$$

then, thanks to Proposition 2.5 and Proposition 2.8, we get to

$$-\int_0^{\mathcal{T}} \langle \mu_n, \varphi_k \rangle {}^{RL}\mathscr{D}_{t,\mathcal{T}}^{\delta}\psi(t)dt + \langle w_n, \varphi_k \rangle \psi(0) + \int_0^{\mathcal{T}} \int_{\mathbb{R}^n} (-\Delta)^{\delta} \mu_n \varphi_k dx \psi(t)dt = \int_0^{\mathcal{T}} \langle h, \varphi_k \rangle \psi(t)dt.$$

Therefore, when passing to the limit, we find

$$-\int_0^{\mathcal{T}} \langle \mu, \varphi_k \rangle {}^{RL}\mathscr{D}_{t,\mathcal{T}}^{\delta}\psi(t)dt + \langle w, \varphi_k \rangle \psi(0) + \int_0^{\mathcal{T}} \int_{\mathbb{R}^n} (-\Delta)^{\delta} \mu \varphi_k dx \psi(t)dt = \int_0^{\mathcal{T}} \langle h, \varphi_k \rangle \psi(t)dt. \tag{3.12}$$

In conclusion, from the uniqueness of the limit, we obtain equation (3.11) equal equation (3.12). In the end , our result is

$$(g_{1-\delta} * \mu)(0) = w \quad a.e. \text{ in } \Omega.$$

Which implies that completes the proof of the existence result.

Part 02. ***Uniqueness of the solution*** Let μ and $\ddot{\mu}$ two solution the problem (3.1), then we have

$$\begin{cases} {}^{RL}\mathscr{D}_{0,t}^{\delta}\mu + (-\Delta)^{\delta}\mu = h \\ (g_{1-\delta} * \mu)(0) = w, \end{cases} \tag{3.13}$$

and

$$\begin{cases} {}^{RL}\mathscr{D}_{0,t}^{\delta}\hat{\mu} + (-\Delta)^{\delta}\hat{\mu} = h \\ (g_{1-\delta} * \hat{\mu})(0) = w. \end{cases} \tag{3.14}$$

Taking the deference between equations (3.13) and (3.14), we get to

$$\begin{cases} {}^{RL}\mathscr{D}_{0,t}^{\delta}(\mu - \hat{\mu}) + (-\Delta)^{\delta}(\mu - \hat{\mu}) = 0 \\ (g_{1-\delta} * (\mu - \hat{\mu}))(0) = 0. \end{cases} \tag{3.15}$$

Moreover, from the initial condition we have $(\mu - \hat{\mu}) \in {}_0W_{2,2}^{\delta}(0, \mathcal{T}; Y, Y^{'})$. Then multiplier equation (3.15) by $(\mu - \hat{\mu})$ and integre on $(0, s)$, $s \in (0, \mathcal{T}]$, we derive

$$\int_0^{s} \left\langle {}^{RL}\mathscr{D}_{0,t}^{\delta}(\mu - \hat{\mu}), (\mu - \hat{\mu}) \right\rangle dt + \int_0^{s} \int_{\mathbb{R}^n} (-\Delta)^{\delta}(\mu - \hat{\mu})(\mu - \hat{\mu})dxdt = 0. \tag{3.16}$$

Therefore, from to Corollary 2.9, we get

$$g_{1-\delta} * \|(\mu - \hat{\mu})(.)\|_{L^2(\Omega)}^2(s) + \int_0^{s} \|\mu - \hat{\mu}\|_Y^2 \leq 0,$$

and thus

$$\int_0^{s} g_{1-\delta}(s-t)\|(\mu - \hat{\mu})(t)\|_{L^2(\Omega)}^2 dt \leq 0.$$

But we have $g_{1-\delta}$ is decreasing map, then

$$g_{1-\delta}(\mathcal{T}) \int_0^{s} \|(\mu - \hat{\mu})(t)\|_{L^2(\Omega)}^2 dt \leq 0.$$

Finally, we conclude that $(\mu - \hat{\mu})(t) = 0$. So that the problem (3.1) has a unique solution. □

4 Topological degree for a time-fractional semilinear equation

In this section, we are interesting to prove the existence of weak solution the following fractional semilinear problem

$$\begin{cases} \text{Find } \mu \in H^{\delta}(0,\mathcal{T};Y,Y^{'}), \\ {}^{RL}\mathscr{D}^{\delta}_{0,t}\mu + (-\Delta)^{\delta}\mu + h(\mu) = 0 & \text{in } L^2(0,\mathcal{T};Y^{'}) \\ (g_{1-\delta} * \mu)(0,.) = w & \text{in } L^2(\Omega), \end{cases} \tag{4.1}$$

where $h : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous map satisfy, for some positive concant r , h satisfying the following assumptions

(c_1) Growth assumption:

$$|h(\mu)| \leq r + r|\mu|, \quad \forall \mu \in \mathbb{R}.$$

(c_2) Sing assumption:

$$h(\mu)\mu \geq -r$$

and we propose the Leray-Schauder degree theory to prove it.

▷ the following theorem give the existence of solution

Theorem 4.1. Thanks to hypothesis (c_1) and (c_2) the problem (4.1) has a weak solution $\mu \in H^{\delta}(0,\mathcal{T};Y,Y^{'})$.

4.1 New formulation of problem (4.1)

In this subsection, we will present a fixed point problem which equivalent to problem (4.1).
First, we will define the following homotopy H by

$$\begin{aligned} H : [0,1] \times L^2(0,\mathcal{T};L^2(\Omega)) &\to L^2(0,\mathcal{T};Y) \\ (\tau,\overline{\mu}) &\mapsto H(\tau,\overline{\mu}) = \mu, \end{aligned}$$

where μ is a weak solution to the following linear problem

$$\begin{cases} \text{Find } \mu : [0,\mathcal{T}] \times \Omega \to \mathbb{R} & \text{such that} \\ {}^{RL}\mathscr{D}^{\delta}_{0,t}\mu + (-\Delta)^{\delta}\mu + \tau h(\overline{\mu}) = 0 & \text{on } [0,\mathcal{T}] \times \Omega \\ \mu = 0 & \text{on} [0,\mathcal{T}] \times \mathbb{R}^n/\Omega \\ (g_{1-\beta} * \mu)(0,.) = \tau w & \text{on } \mathbb{R}^n/\Omega. \end{cases} \tag{4.2}$$

Lemma 4.2. We show in the Section 03 for the problem (4.2) is uniquely solution $\mu \in H^{\delta}(0,\mathcal{T};Y,Y^{'})$.

After that, we observe that problem (4.1) is equivalent to the fixed point problem

$$\begin{cases} \mu \in L^2(0,\mathcal{T};L^2(\Omega)), \\ H(1,\mu) = \mu. \end{cases} \tag{4.3}$$

Our goal here in this section is prove that and we using the Leray-Schauder degree to prove the problem (4.1) has weak solution.

4.2 several auxiliary Lemmas

In this subsection, we present several auxiliary Lemmas about the conditions of the Leray-Schauder degree method.

Lemma 4.3. (Priori estimate). Thanks to assumptions (c_1), (c_2) there exists $R > 0$, for every $\mu \in L^2(0,\mathcal{T},L^2(\Omega))$ such that

$$\begin{cases} H(\tau,\mu) = \mu \\ \tau \in [0,1], \mu \in L^2(0,\mathcal{T},L^2(\Omega)) \end{cases} \Rightarrow \|\mu\|_{L^2(0,\mathcal{T},L^2(\Omega))} < R.$$

Proof . Let $H(\tau,\mu)=\mu$, for all $\tau\in[0,1]$, that is means

$$\begin{cases}\text{Find } \mu\in L^2(0,\mathcal{T};Y) \text{ such that } {}^{RL}\mathscr{D}^{\delta}_{0,t}\mu\in L^2(0,\mathcal{T};Y^{'}) \\ \int_0^{\mathcal{T}}\left\langle {}^{RL}\mathscr{D}^{\delta}_{0,t}\mu,\varphi\right\rangle_{Y',Y}dt+P(n,\delta)\int_0^{\mathcal{T}}\iint_{\mathbb{R}^{2n}}\dfrac{(\mu(x)-\mu(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2\delta}}dydxdt+\tau\int_0^{\mathcal{T}}\int_\Omega h(\mu)\varphi dxdt=0, \quad \forall\varphi\in Y.\end{cases}$$

Taking $\varphi=\mu-\tau wg_\delta$, we derive

$$\int_0^{\mathcal{T}}\left\langle {}^{RL}\mathscr{D}^{\delta}_{0,t}\mu,\mu-\tau wg_\delta\right\rangle_{Y',Y}dt+P(n,\delta)\int_0^{\mathcal{T}}\iint_{\mathbb{R}^{2n}}\frac{(\mu(x)-\mu(y))\left((\mu(x)-\mu(y))-\tau g_\delta(w(x)-w(y))\right)}{|x-y|^{n+2\delta}}dydxdt$$
$$+\tau\int_0^{\mathcal{T}}\int_\Omega h(\mu)(\mu-\tau wg_\delta)dxdt=0.$$

In fact, from the Theorem 2.10, we have the first integral above is positive. Hence, we get

$$P(n,\delta)\int_0^{\mathcal{T}}\iint_{\mathbb{R}^{2n}}\frac{|\mu(x)-\mu(y)|^2}{|x-y|^{n+2\delta}}dxdt\leq\left|\int_0^{\mathcal{T}}\int_\Omega -h(\mu)\mu dxdt\right|+\left|\int_0^{\mathcal{T}}\int_\Omega h(\mu)wg_\delta dxdt\right|$$
$$+\left|P(n,\delta)\int_0^{\mathcal{T}}\iint_{\mathbb{R}^{2n}}\frac{(\mu(x)-\mu(y))(w(x)-w(y))}{|x-y|^{n+2\delta}}g_\delta dydxdt\right|.$$

Under the assumption (c_2), we obtain

$$P(n,\delta)\int_0^{\mathcal{T}}\iint_{\mathbb{R}^{2n}}\frac{|\mu(x)-\mu(y)|^2}{|x-y|^{n+2\delta}}dxdt$$
$$\leq r|\Omega|\mathcal{T}+\left|\int_0^{\mathcal{T}}\int_\Omega h(\mu)wg_\delta dxdt\right|+P(n,\delta)\left|\int_0^{\mathcal{T}}\iint_{\mathbb{R}^{2n}}\frac{(\mu(x)-\mu(y))(w(x)-w(y))}{|x-y|^{n+2\delta}}g_\delta dydxdt\right|.$$

Then

$$\|\mu\|^2_{L^2(0,\mathcal{T};H^{1=0}(\Omega))}\leq r|\Omega|\mathcal{T}+I_1+I_2. \tag{4.4}$$

when

$$I_1=\left|\int_0^{\mathcal{T}}\int_\Omega h(\mu)wg_\delta dxdt\right|,$$

and

$$I_2=P(n,\delta)\left|\int_0^{\mathcal{T}}\iint_{\mathbb{R}^{2n}}\frac{(\mu(x)-\mu(y))(w(x)-w(y))}{|x-y|^{n+2\delta}}g_\delta dydxdt\right|.$$

We get from the assumption (c_1), and using the Cauchy-Schwarz inequality and Proposition 2.1, of

- $$I_1\leq r\int_0^{\mathcal{T}}\int_\Omega |w||g_\delta|dxdt+r\int_0^{\mathcal{T}}\int_\Omega |\mu||w||g_\delta|dxdt$$
  $$\leq rC_{emb}\sqrt{\mathcal{T}|\Omega|}\|w\|_Y\|g_\delta\|_{L^2(0,\mathcal{T})}+rC^2_{emb}\|w\|_Y\|g_\delta\|_{L^2(0,\mathcal{T})}\|\mu\|_{L^2(0,\mathcal{T};Y)}.$$

Hence, using Hölder inequality on I_2, we get to

$$I_2\leq P(n,\delta)\|w\|_Y\|g_\delta\|_{L^2(0,\mathcal{T})}\|\mu\|_{L^2(0,\mathcal{T};Y)}.$$

Finally, when we return to (4.4), we get

$$P(n,\delta)\|\mu\|^2_{L^2(0,\mathcal{T};Y)}\leq\left[\left(rC^2_{emb}+C(n,\delta)\right)\|w\|_Y\|g_\delta\|_{L^2(0,\mathcal{T})}\right]\|\mu\|_{L^2(0,\mathcal{T};Y)}+rC_{emb}\sqrt{\mathcal{T}|\Omega|}\|g_\delta\|_{L^2(0,\mathcal{T})}\|w\|_Y+r|\Omega|\mathcal{T},$$

The solvability of above inequality, which is solved by solving a second-degree equation, we get to

$$\|\mu\|_{L^2(0,\mathcal{T};L^2(\Omega))}\leq L=R,$$

hence, we get

$$\|\mu\|_{L^2(0,\mathcal{T};L^2(\Omega))}<R+1. \tag{4.5}$$

from (4.5) our result is that, there are no solution of the equation $H(\tau,\mu)=\mu$ on the edge of $B_{R+1}=\{\mu\in L^2(0,\mathcal{T};L^2(\Omega)):\|\mu\|_{L^2(0,\mathcal{T};L^2(\Omega))}<R+1\}$, and this is for each $\tau\in[0,1]$. □

Lemma 4.4. Thanks to assumption (c_1), the homotopy $\{H(\tau,\overline{\mu});\tau \in [0,1], \overline{\mu} \in B_{R+1}\}$ is relatively compact in $L^2(0,\mathcal{T};L^2(\Omega))$.

Proof . Let $(\tau_n,\overline{\mu}_n)_{n\in\mathbb{N}} \subset [0,1]\times \overline{B}(0,R+1)$, we have

$$\int_0^{\mathcal{T}} \langle {}^{RL}\mathscr{D}^{\delta}_{0,t}\mu_n, \mu_n - \tau_n w g_\delta\rangle_{Y',Y} dt + P(n,\delta)\int_0^{\mathcal{T}}\iint_{\mathbb{R}^{2n}} \frac{(\mu_n(x)-\mu_n(y))\big((\mu_n(x)-\mu_n(y)) - \tau_n g_\delta(w(x)-w(y))\big)}{|x-y|^{n+2\delta}} dydxdt$$
$$+\tau_n\int_0^{\mathcal{T}}\int_\Omega h(\overline{\mu}_n)(\mu_n-\tau_n w g_\delta)dxdt = 0.$$

By Theorem 2.10 , we get

$$P(n,\delta)\int_0^{\mathcal{T}}\iint_{\mathbb{R}^{2n}} \frac{|\mu_n(x)-\mu_n(y)|^2}{|x-y|^{n+2\delta}}dydxdt \le \left|P(n,\delta)\int_0^{\mathcal{T}}\iint_{\mathbb{R}^{2n}} \frac{(\mu_n(x)-\mu_n(y))(w(x)-w(y))}{|x-y|^{n+2\delta}} g_\delta dydxdt\right| + \left|\int_0^{\mathcal{T}}\int_\Omega h(\overline{\mu}_n)\mu_n dxdt\right|$$
$$+\left|\int_0^{\mathcal{T}}\int_\Omega h(\overline{\mu}_n) w g_\delta dxdt\right|.$$

Then, we put

$$\|\mu\|^2_{L^2(0,\mathcal{T};Y)} \le I_1^{'} + I_2^{'} + I_3^{'}. \tag{4.6}$$

When

$$I_1^{'} = P(n,\delta)\left|\int_0^{\mathcal{T}}\iint_{\mathbb{R}^{2n}} \frac{(\mu_n(x)-\mu_n(y))(w(x)-w(y))}{|x-y|^{n+2\delta}} g_\delta dydxdt\right|,$$

and $I_2^{'} = \left|\int_0^{\mathcal{T}}\int_\Omega h(\overline{\mu}_n)\mu_n dxdt\right|$, with $I_3^{'} = \left|\int_0^{\mathcal{T}}\int_\Omega h(\overline{\mu}_n) w g_\delta dxdt\right|$. Thanks to hypothesis (c_1) and Cauchy-Schwarz inequality and Proposition 2.1, we obtain

- $$I_2^{'} \le r\int_0^{\mathcal{T}}\int_\Omega |\mu_n|dxdt + r\int_0^{\mathcal{T}}\int_\Omega |\overline{\mu}_n||\mu_n|dxdt \le rC_{emb}\left[\sqrt{\mathcal{T}|\Omega|} + \|\overline{\mu}_n\|_{L^2(0,\mathcal{T};L^2(\Omega))}\right]\|\mu_n\|_{L^2(0,\mathcal{T};Y)},$$

and

- $$I_3^{'} \le r\int_0^{\mathcal{T}}\int_\Omega |w_n||g_\delta|dxdt + r\int_0^{\mathcal{T}}\int_\Omega |\overline{\mu}_n||w||g_\delta|dxdt \le rC_{emb}\left[\sqrt{\mathcal{T}|\Omega|} + \|\overline{\mu}_n\|_{L^2(0,\mathcal{T};L^2(\Omega))}\right]\|w\|_Y\|g_\delta\|_{L^2(0,\mathcal{T}}.$$

Hence, applying the Hölder inequality to $I_1^{'}$, we find

- $$I_1^{'} \le P(n,\delta)\|w\|_Y\|g_\delta\|_{L^2(0,\delta)}\|\mu_n\|_{L^2(0,\mathcal{T};Y)}.$$

Then, when we return to (4.2), we derive

$$P(n,\delta)\|\mu_n\|^2_{L^2(0,\mathcal{T};Y)} \le \Big[rC_{emb}\big(\sqrt{\mathcal{T}|\Omega|} + \|\overline{\mu}_n\|_{L^2(0,\mathcal{T};L^2(\Omega))}\big) + P(n,\delta)\|w\|_Y\|g_\delta\|_{L^2(0,\mathcal{T})}\Big]\|\mu_n\|_{L^2(0,\mathcal{T};Y)}$$
$$+ rC_{emb}\Big(\sqrt{\mathcal{T}|\Omega|} + \|\overline{\mu}_n\|_{L^2(0,\mathcal{T};L^2(\Omega))}\Big)\|w\|_Y\|g_\delta\|_{L^2(\Omega)}.$$

Finally, we get to

$$\|\mu_n\|_{L^2(0,\mathcal{T};Y)} \le \hat{M}_1,$$

where the constant $\hat{M}_1$ is independent of n.

▷ We proceed now to prove that ${}^{RL}\mathscr{D}^{\delta}_{0,t}\mu_n$ is bounded in $L^2(0,\mathcal{T};Y^{'})$. Indeed, we have

$$\left|\left\langle {}^{RL}\mathscr{D}^{\delta}_{0,t}\mu_n, \varphi\right\rangle_{Y',Y}\right| \le P(n,\delta)\left|\iint_{\mathbb{R}^{2n}} \frac{(\mu_n(x)-\mu_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2\delta}}dydx\right| + \int_\Omega |h(\overline{\mu}_n)||\varphi|dx \quad \forall\varphi\in Y.$$

Moreover, from Cauchy-Schwarz inequality, Proposition 2.1 and assumption (c_1), we obtain

$$\left| \left\langle {}^{RL}\mathscr{D}^{\delta}_{0,t}\mu_n, \varphi \right\rangle_{Y',Y} \right| \leq P(n,\delta)\|\mu_n\|_Y\|\varphi\|_Y + rC_{emb}\sqrt{\Omega}\|\varphi\|_Y + rC_{emb}\|\overline{\mu}_n\|_{L^2(\Omega)}\|\varphi\|_Y.$$

Then, we find

$$\int_0^{\mathcal{T}} \|{}^{RL}\mathscr{D}^{\delta}_{0,t}\mu_n\|^2_{Y'}dt \leq 2P^2(n,\delta)\int_0^{\mathcal{T}}\|\mu_n\|^2_Y dt + 4r^2C^2_{emb}\left(|\Omega|\mathcal{T} + \int_0^{\mathcal{T}}\|\overline{\mu}_n\|^2_{L^2(\Omega)}dt\right)$$

In Finally, we get to

$$\|{}^{RL}\mathscr{D}^{\delta}_{0,t}\mu_n\|_{L^2((0,\mathcal{T};Y'))} \leq \hat{M}_2, \tag{4.7}$$

where $\hat{M}_2 = \left(2P^2(n,\delta)\hat{M}_1^2 + 4r^2C^2_{emb}\left(\mathcal{T}|\Omega| + (R+1)^2\right)\right)^{1/2}$. In conclusion, we get to $(\mu_n)_{n\in\mathbb{N}}$ is bounded in $L^2(0,\mathcal{T};Y)$ and $({}^{RL}\mathscr{D}^{\delta}_{0,t}\mu_n)_{n\in\mathbb{N}}$ is bounded in $L^2(0,\mathcal{T};Y^{'})$. Therefore, according to Aubin-Simon theorem we conclude that the homotopy $\{H(\tau,\overline{\mu}); \tau\in[0,1], \overline{\mu}\in B_{R+1}\}$ is relatively compact in $L^2(0,\mathcal{T};L^2(\Omega))$. □

Lemma 4.5. Thanks to hypothesis (c_1), $H : [0,1]\times L^2(0,\mathcal{T};L^2(\Omega)) \to L^2(0,\mathcal{T};L^2(\Omega))$ is continuous.

Proof . Let $\{(\tau_n,\overline{\mu}_n)\}_{n\in\mathbb{N}} \subset [0,1]\times L^2(0,\mathcal{T};L^2(\Omega))$ which converge to $(\tau,\overline{\mu})$ in $[0,1]\times L^2(0,\mathcal{T};L^2(\Omega))$ when $n\to+\infty$. Our goal will to prove that $H(\tau_n,\overline{\mu}_n)\to H(\tau,\overline{\mu})$ in $L^2(0,\mathcal{T};L^2(\Omega))$, we pose for every $n\in\mathbb{N}$ that $H(\tau_n,\overline{\mu}_n)=\mu_n$ and $H(\tau_n,\overline{\mu}_n)=\mu_n$, we have

$$\int_0^{\mathcal{T}}\left\langle {}^{RL}\mathscr{D}^{\delta}_{0,t}\mu_n,\varphi\right\rangle_{Y',Y}dt + P(n,\delta)\int_0^{\mathcal{T}}\iint_{\mathbb{R}^{2n}}\frac{(\mu_n(x)-\mu_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2\delta}}dydxdt + \tau_n\int_0^{\mathcal{T}}\int_\Omega h(\overline{\mu}_n)\varphi dxdt = 0 \quad \forall\varphi\in Y, \tag{4.8}$$

and

$$\int_0^{\mathcal{T}}\left\langle {}^{RL}\mathscr{D}^{\delta}_{0,t}\mu,\varphi\right\rangle_{Y',Y}dt + P(n,\delta)\int_0^{\mathcal{T}}\iint_{\mathbb{R}^{2n}}\frac{(\mu(x)-\mu(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2\delta}}dydxdt + \tau\int_0^{\mathcal{T}}\int_\Omega h(\overline{\mu})\varphi dxdt = 0 \quad \forall\varphi\in Y. \tag{4.9}$$

Taking the subtraction between the two equations (4.8) and (4.9), we find

$$\begin{aligned}&\int_0^{\mathcal{T}}\left\langle {}^{RL}\mathscr{D}^{\delta}_{0,t}(\mu_n-\mu),\varphi\right\rangle_{Y',Y}dtP(n,\delta)\int_0^{\mathcal{T}}\iint_{\mathbb{R}^{2n}}\frac{\big((\mu_n(x)-\mu_n(y))-(\mu(x)-\mu(y))\big)(\varphi(x)-\varphi(y))}{|x-y|^{n+2\delta}}dydxdt\\&+\int_0^{\mathcal{T}}\int_\Omega(\tau h(\overline{\mu})-\tau_n h(\overline{\mu}_n))\varphi dxdt = 0 \quad \forall\varphi\in Y,\end{aligned}$$

Then, taking $\varphi = (\mu_n-\mu)-(\tau_n-\tau)wg_\delta$ and by the Theorem 2.10, we obtain

$$\begin{aligned}P(n,\delta)\int_0^{\mathcal{T}}\|\mu_n-\mu\|^2_Y dt \leq& P(n,\delta)\int_0^{\mathcal{T}}\iint_{\mathbb{R}^{2n}}\frac{\big((\mu_n(x)-\mu_n(y))-(\mu(x)-\mu(y))\big)(\tau_n-\tau)(w(x)-w(y))\big)}{|x-y|^{n+2\delta}}g_\delta dydxdt\\&+\int_0^{\mathcal{T}}\int_\Omega(\tau h(\overline{\mu})-\tau_n h(\overline{\mu}_n))(\mu_n-\mu)dxdt + \int_0^{\mathcal{T}}\int_\Omega(\tau h(\overline{\mu})-\tau_n h(\overline{\mu}_n))(\tau_n-\tau)wg_\delta dxdt.\end{aligned}$$

Using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}P(n,\delta)\|\mu_n-\mu\|^2_{L^2(0,\mathcal{T};Y)} \leq& \Big[P(n,\delta)\|g_\delta\|_{L^2(0,\mathcal{T})}|\tau_n-\tau|\|w\|_Y + \|\tau h(\overline{\mu})-\tau_n h(\overline{\mu}_n)\|_{L^2(0,\mathcal{T};Y')}\Big]\|\mu_n-\mu\|_{L^2(0,\mathcal{T};Y)}\\&+\|\tau h(\overline{\mu})-\tau_n h(\overline{\mu}_n)\|_{L^2(0,\mathcal{T};Y')}|\tau_n-\tau|\|w\|_Y\|g_\delta\|_{L^2(0,\mathcal{T})}.\end{aligned}$$

From h Lipschitz map, produces that $h(\overline{\mu}_n)\to h(\overline{\mu})$ in $L^2(0,\mathcal{T};L^2(\Omega))$ and we have $\tau_n\to\tau$ in $[0,1]$ when $n\to+\infty$, and by Poincaré inequality, we get

$$\|\mu_n-\mu\|_{L^2(0,\mathcal{T};L^2(\Omega))}\to 0 \quad \text{when } n\to+\infty.$$

So, H is continuous from $[0,1]\times L^2(0,\mathcal{T};L^2(\Omega))$ into $L^2(0,\mathcal{T};L^2(\Omega))$. □

Proof .[**Proof of Theorem 4.1**] We have from the previous lemmas 4.3, 4.4 and 4.5 the degree $d(I_d-H(\tau,.),B(0,R+1),0)$ is well define, and from the homotopy invariance property, we have $d(I_d-H(1,.),B(0,R+1),0) = d(I_d-H(0,.),B(0,R+1),0) = d(I_d,B(0,R+1),0) = 1\neq 0$, therefore $\mu - H(1,\mu)=0 \Rightarrow \mu = H(1,\mu)$. Our goal is that we prove the existence of weak solution to problem (4.1), $\mu\in H^\delta(0,\mathcal{T};Y,Y^{'})$. □

Conclusion

In this research, we studied theoretical results for linear and semilinear time fractional equations. Some future research, we will take another time fractional problem and we will replace the non-local operator with a new class of fractional derivative and include the numerical study.

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References

[1] E. Abada, H. Lakhal, and M. Maouni, *Topological, degree method for fractional Laplacian system*, Bull. Math. Anal. Appl. **13** (2021), 10–19.

[2] E. Abada, H. Lakhal, and M. Maouni, *Existence and uniqueness of solution for a nonlinear fractional problem involving the distributional Riesz derivative*, Math. Methods Appl. Sci. **45** (2022), 6181–6193.

[3] L. Abatangelo, V. Felli, and B. Noris, *On simple eigenvalues of the fractional Laplacian under removal of small fractional capacity sets*, Commun. Contemp. Math. **22** (2020), no. 8, 1950071.

[4] B. Barrios, E. Colorado, A. de Pablo, and U. Sánchez, *On some critical problems for the fractional Laplacian operator,* J. Differ. Equ. **252** (2012), 6133–6162.

[5] K. Diethelm and N.J. Ford, *The Analysis of Fractional Differential Equations,* Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2010.

[6] L. Djilali and A. Rougirel, *Galerkin method for time fractional diffusion equations*, J. Elliptic Parabol. Equ. **4** (2018), 349–368.

[7] S. Dob, H. Lakhal, and M. Maouni, *Existence and Uniqueness of solutions for a nonlinear fractional Elliptic system*, Malays. J. Math. Sci., **15** (2021), 347–356.

[8] R. Hilfer, *Applications of Fractional Calculus in Physics,* World Scientific, Singapore, 2000.

[9] A.-A. Kilbas, H.-M. Srivastava, and J.-J. Trujillo, *Theory and Applications of Fractional Differential Equation,* Elsevier, Amsterdam, 2006.

[10] H. Lakhal, B. Khodja, and W. Gharbi, *Existence results of nontrivial solutions for a semi linear elliptic system at resonance*, J. Adv. Res. Dyn. Control Syst. **5** (2013), 1–12.

[11] A.-M. Nakhushev, *Fractional Calculus and its Application,* FIZMATLIT, Moscow, 2003. [In Russian]

[12] K.-B. Oldham and J. Spanier, *The Fractional Calculus,* Academic Press, New York, 1974.

[13] Y. Ouedjedi, A. Rougirel, and K. Benmeriem, *Galerkin method for time fractional semilinear equations*, Fract. Calc. Appl. Anal. **24** (2021),755–774.

[14] C. Saadi, H. Lakhal, K. Slimani, and S. Dob, *Existence and uniqueness of distributional solution for semilinear fractional elliptic equation involving new operator and some numerical results,* Math. Methods Appl. Sci. **45** (2022), 3843–3854.

[15] C. Saadi, H. Lakhal, K. Slimani, and S. Dob, *Distributional solution for semilinear system involving fractional gradient and a numerical example*, Int. J. Nonlinear Anal. Appl. **14** (2023), no. 1, 1175–1185.

[16] R. Servadei and E. Valdinoci, *The Brezis-Nirenberg result for the fractional Laplacian*, Trans. Amer. Math. Soc. **367** (2015), 67–102.

[17] K. Slimani, C. Saadi, and H. Lakhal, *Existence results for convection-reaction fractional problem involving the distributional Riesz derivative,* Math. Methods Appl. Sci. **45** (2022), 10247–10255.