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# On the hybrid fractional semi-linear evolution equations

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**Abstract**

In this manuscript, we study the existence of mild solutions to initial value problems for hybrid fractional semi-linear evolution equations. On the other hand, we prove four different types of Ulam-Hyers stability results for mild solutions. The existence of mild solutions is proved by the Dhage fixed point theorem. Finally, an example is given to illustrate our results.

Keywords: hybrid fractional evolution equation, mild solution, Ulam-Hyers stability, fixed point theorem
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## 1 Introduction

Fractional differential equations arise in the mathematical modeling of systems and processes occurring in many engineering and scientific disciplines such as physics, chemistry, economics, signal and image processing, biophysics, etc. [6, 12]. It is well known that many phenomena in various fields of science and engineering are best characterized by fractional differential equations. Among the branches of fractional calculus research is the theory of fractional evolution equations, which is based on the description of the equations that change over time. For more details on the theory of fractional evolution equations, we refer reader to [14, 15] and [4, 3]. The main motivation for this work comes from the above works. In this present paper, we investigate the existence and different types of Ulam stability results of mild solutions on a sub-interval $J = [0, b]$, $(b < \infty)$ for the following semi-linear fractional hybrid differential equations involving the Caputo fractional derivative of order $0 \le \alpha \le 1$

$$\begin{cases} {}^cD^{\alpha}_{0+}\left(\frac{u(t)}{f(t,u(t))}\right) = A\left(\frac{u(t)}{f(t,u(t))}\right) + g(t,u(t)), \quad t \in [0,b] \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $A$ is the infinitesimal generator of a $C_0$-semi-group $\{T(t)\}_{t\ge 0}$ on a Banach space X, $f \in C([0,b] \times X, X \setminus \{0\})$ and $g \in Car([0,b] \times X, X)$. ( $Car([0,b] \times X, X)$ is called the Caratheodory class of functions).

The plan of this paper is as follows: In Section 2, we present some preliminary results of fractional calculus and functional analysis, which will be employed throughout this paper. Section 3 is devoted to the study of the existence, of solution of the problem (1.1). In Section 4, we discuss different types of stability results of solutions to the problem (1.1). As an application of our main results, an illustrative example is given in the last section.

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## 2 Preliminaries

Let $J = [0, b]$ be a finite interval of the real line $\mathbb{R}$. $C(J, X)$ be the Banach space of all continuous functions with the norm $\|h\| = sup\{|h(t)| : t \in J\}$. We denote by $Car(J \times X, X)$ the class of functions $g : J \times X \longrightarrow X$ such that the map $t \longrightarrow g(t, u)$ is measurable for each $u \in X$ and the map $u \longrightarrow g(t, u)$ is continuous for each $t \in J$.

Now, let us recall some definitions and properties of fractional calculus.

**Definition 2.1.** [11]. Let $(n - 1 < \alpha < n)$ such that $n = [\alpha] + 1$. The Riemann-Liouville fractional integral of order $\alpha$ of a function $h : (0, b] \longrightarrow X$ is defined by

$$I^{\alpha}_{0^+} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} h(s) ds,$$

where $\Gamma(.)$ is the Euler gamma function defined by

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt, z > 0.$$

**Definition 2.2.** [11]. Let $(n - 1 < \alpha < n)$ such that $n = [\alpha] + 1$. The Caputo derivative of order $\alpha$ of a function $h \in C^n([0, b], X)$ is defined by

$${}^c D^{\alpha}_{0_+} h(t) = I^{n-\alpha} h^{(n)}(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} h^{(n)}(s) ds.$$

Let us now adopt the definition of a $C_0$-semi-group and some lemmas that we will make use of in this paper. For more details on a strongly continuous semi-group, we refer the reader to [5, 9] and [13].

**Definition 2.3.** A strongly continuous semi-group (i.e., $C_0$-semi-group) is a family of operators $\{T(t)\}_{t\geq 0}$ of bounded linear operators $T(t)$ on a Banach space X satisfying the following conditions:

(i) $T(0) = I$.

(ii) $T(t + s) = T(t)T(s)$, for all $t, s \geq 0$.

(iii) $\lim_{t\to 0^+} T(t)x = x$, for each $x \in X$.

**Lemma 2.4.** [5, 9] A linear operator A is the infinitesimal generator of a $C_0$-semi-group if and only if

(i) $\overline{D(A)} = X$.

(ii) The resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$ and, $\forall\lambda > 0$, we have,

$$\| R(\lambda, A) \| \leq \frac{1}{\lambda},$$

where $R(\lambda, A) := (\lambda^{\alpha} I - A)^{-1} x = \int_0^{\infty} e^{-\lambda^{\alpha} t} T(t) x dt$.

**Lemma 2.5.** [5, 9] Let $\{T(t)\}_{t\geq 0}$ be a $C_0$-semi-group, then there exist constants $\omega \geq 0$ and $M \geq 1$ such that $\| T(t) \| \leq M e^{\omega t}$ for all $t \geq 0$.

Throughout this manuscript we consider the case where $\omega = 0$, (i.e $\{T(t)\}_{t\geq 0}$ is uniformly bounded)

$$\| T(t) \| \leq M, \quad \forall t \geq 0.$$

**Definition 2.6.** The Laplace transform of a real valued function $u : [a, \infty[\longrightarrow \mathbb{R}$ is given by

$$Ł\{u(t)\}(s) = \int_a^\infty e^{-st}u(t)dt,$$

for all s.

**Lemma 2.7.** Let $\alpha > 0$ and u be a continuous function on the interval $[0, b]$. Then we have

$$Ł\{I_0^\alpha u(t)\}(s) = \frac{Ł(u(t))(s)}{s^\alpha}.$$

**Definition 2.8.** [7, 10] Let $0 < \alpha < 1$ and $z \in \mathbb{C}$. We define the Wright type function by

$$\begin{aligned}\Psi_\alpha(z) &= \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(-\alpha(k+1)+1)} \\ &= \sum_{k=0}^{\infty} \frac{(-z)^k\Gamma(\alpha(k+1))\sin(\pi\alpha(k+1))}{k!}.\end{aligned}$$

**Lemma 2.9.** [7, 10] Let $\Psi_\alpha$. The Wright function, then we have the following properties:

(i) $\Psi_\alpha(z) \geq 0$ for all $z \geq 0$ and $\int_0^\infty \Psi_\alpha(z)dz = 1$.

(ii) $\int_0^\infty \Psi_\alpha(z)z^r dz = \frac{\Gamma(1+r)}{\Gamma(1+\alpha r)}$ for $r > -1$.

**Theorem 2.10.** [2] Let S be a closed, bounded, and convex subset of the Banach algebra X. We consider the two operators $F : X \to X$ and $P : S \to X$ such that:

a) F is Lipschitzian with a Lipschitz constant $\alpha$.

b) P is completely continuous.

c) $u = FuPv \Rightarrow u \in S$, for all $v \in S$.

d) $\alpha N < 1$, Where $N = \|P(S)\|$ Then, the operator equation $u = FuPu$ has a solution in S.

## 3 Existence Results

In this partition, we prove the existence of a mild solution to the problem (1.1). We first present the following results through which we can prove our major results. According to Definition (2.2), it's easy to see the equivalence between problem (1.1) and the following integral equation

$$u(t) = f(t, u(t)) \left\{u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A\left(\frac{u(s)}{f(s,u(s))}\right) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, u(s))ds\right\}. \tag{3.1}$$

provided that the integrals in (3.1) exist.

**Lemma 3.1.** If (3.1) satisfied, then the following equation is a mild solution of the problem (1.1)

$$u(t) = f(t, u(t)) \left\{S_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1} T_\alpha(t, s)g(s, u(s))ds\right\}, \tag{3.2}$$

where, $S_\alpha(t)u = \int_0^\infty \Psi_\alpha(\theta)T(t^\alpha\theta)ud\theta$, and $T_\alpha(t, s)u = \alpha \int_0^\infty \theta\Psi_\alpha(\theta)T((t-s)^\alpha\theta)ud\theta$. The function $\Psi_\alpha(\theta)$ is a probability density defined on $(0, \infty)$.

**Proof .** Let $s>0$. By dividing (3.1) by $f(t,u(t))$ and applying the Laplace transform to it, we have

$$\begin{aligned}
Ł\left\{\frac{u(t)}{f(t,u(t))}\right\}(s) &= Ł\{u_0\}(s) + Ł_\alpha\left\{I^\alpha_{0^+}A\left(\frac{u(t)}{f(t,u(t))}\right)\right\} + Ł_\alpha\{I^\alpha_{0^+}g(t,u(t))\}(s)\\
&= \frac{u_0}{s} + \frac{1}{s^\alpha}AŁ\left\{\frac{u(t)}{f(t,u(t))}\right\}(s) + \frac{1}{s^\alpha}Ł\{g(t,u(t))\}(s)\\
&= s^{\alpha-1}(s^\alpha I - A)^{-1}u_0 + (s^\alpha I - A)^{-1}G(s),
\end{aligned}$$

where $G(s) = Ł\{g(t,u(t))\}(s)$, then

$$\begin{aligned}
Ł\left\{\frac{u(t)}{f(t,u(t))}\right\}(s) &= s^{\alpha-1}\int_0^\infty e^{-s^\alpha\tau}T(\tau)u_0 d\tau + \int_0^\infty e^{-s^\alpha\tau}T(\tau)G(s)d\tau\\
&= \int_0^\infty \alpha(st)^{\alpha-1}e^{-(st)^\alpha}T(t^\alpha)u_0 dt + \int_0^\infty \alpha t^{\alpha-1}e^{-(st)^\alpha}T(t^\alpha)G(s)dt.
\end{aligned}$$

We consider the one-sided stable probability density function defined as [8]

$$\rho_\alpha(\theta) = \frac{1}{\pi}\sum_{k=1}^\infty \frac{(-1)^{k-1}\theta^{-\alpha k-1}\sin(k\pi\alpha)\Gamma(\alpha k+1)}{k!}, \quad \theta\in(0,\infty),$$

and

$$\int_0^\infty e^{-s\theta}\rho_\alpha(\theta)d\theta = e^{-s^\alpha}, \quad \alpha\in(0,1).$$

Then, we have

$$\begin{aligned}
\int_0^\infty \alpha(st)^{\alpha-1}e^{-(st)^\alpha\tau}T(t^\alpha)u_0 dt &= \int_0^\infty -\frac{1}{s}\frac{d}{dt}(e^{-(st)^\alpha})T(t^\alpha)u_0 dt\\
&= \int_0^\infty -\frac{1}{s}\left(\frac{d}{dt}\int_0^\infty e^{-st\theta}\rho_\alpha(\theta)d\theta\right)T(t^\alpha)u_0 dt\\
&= \int_0^\infty\int_0^\infty \theta e^{-st\theta}\rho_\alpha(\theta)T(t^\alpha)u_0 d\theta dt\\
&= \int_0^\infty\int_0^\infty \theta e^{-s\lambda}\rho_\alpha(\theta)T\left(\frac{\lambda^\alpha}{\theta^\alpha}\right)u_0 d\theta d\lambda\\
&= \int_0^\infty e^{-s\lambda}\left(\int_0^\infty \rho_\alpha(\theta)T\left(\frac{\lambda^\alpha}{\theta^\alpha}\right)u_0 d\theta\right)d\lambda,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^\infty \alpha t^{\alpha-1}e^{-(st)^\alpha}T(t^\alpha)G(s)dt &= \int_0^\infty\int_0^\infty \alpha e^{-(st)^\alpha}t^{\alpha-1}T(t^\alpha)e^{-s\tau}g(\tau,u(\tau))d\tau dt\\
&= \int_0^\infty\int_0^\infty\int_0^\infty \alpha\rho_\alpha(\theta)e^{-st\theta}t^{\alpha-1}T(t^\alpha)e^{-s\tau}g(\tau,u(\tau))d\theta d\tau dt\\
&= \int_0^\infty\int_0^\infty\int_0^\infty \alpha\rho_\alpha(\theta)e^{-s\lambda}\frac{\lambda^{\alpha-1}}{\theta^\alpha}T\left(\frac{\lambda^\alpha}{\theta^\alpha}\right)e^{-s\tau}g(\tau,u(\tau))d\theta d\tau d\lambda\\
&= \int_0^\infty\int_0^\infty\int_0^\infty \alpha\rho_\alpha(\theta)e^{-s(\lambda+\tau)}\frac{\lambda^{\alpha-1}}{\theta^\alpha}T\left(\frac{\lambda^\alpha}{\theta^\alpha}\right)g(\tau,u(\tau))d\theta d\tau d\lambda\\
&= \int_0^\infty\int_\tau^\infty\int_0^\infty \alpha\rho_\alpha(\theta)e^{-s\gamma}\frac{(\gamma-\tau)^{\alpha-1}}{\theta^\alpha}T\left(\frac{(\gamma-\tau)^\alpha}{\theta^\alpha}\right)g(\tau,u(\tau))d\theta d\gamma d\tau\\
&= \int_0^\infty\int_0^\gamma\int_0^\infty \alpha\rho_\alpha(\theta)e^{-s\gamma}\frac{(\gamma-\tau)^{\alpha-1}}{\theta^\alpha}T\left(\frac{(\gamma-\tau)^\alpha}{\theta^\alpha}\right)g(\tau,u(\tau))d\theta d\tau d\gamma\\
&= \int_0^\infty e^{-s\gamma}\left(\int_0^\gamma\int_0^\infty \alpha\rho_\alpha(\theta)\frac{(\gamma-\tau)^{\alpha-1}}{\theta^\alpha}T\left(\frac{(\gamma-\tau)^\alpha}{\theta^\alpha}\right)g(\tau,u(\tau))d\theta d\tau\right)d\gamma.
\end{aligned}$$

Then, we have

$$Ł\left\{\frac{u(t)}{f(t,u(t))}\right\}(s)=\int_0^\infty e^{-st}\left(\int_0^\infty \rho_\alpha(\theta)T\left(\frac{t^\alpha}{\theta^\alpha}\right)u_0d\theta\right)dt$$
$$+\int_0^\infty e^{-s\gamma}\left(\int_0^\gamma\int_0^\infty \alpha\rho_\alpha(\theta)\frac{(\gamma-\tau)^{\alpha-1}}{\theta^\alpha}T\left(\frac{(\gamma-\tau)^\alpha}{\theta^\alpha}\right)g(\tau,u(\tau))d\theta d\tau\right)d\gamma.$$

We invert the Laplace transform, we get

$$\frac{u(t)}{f(t,u(t))}=\int_0^\infty \rho_\alpha(\theta)T\left(\frac{t^\alpha}{\theta^\alpha}\right)u_0d\theta+\int_0^t\int_0^\infty \alpha\rho_\alpha(\theta)\frac{(t-\tau)^{\alpha-1}}{\theta^\alpha}T\left(\frac{(t-\tau)^\alpha}{\theta^\alpha}\right)g(\tau,u(\tau))d\theta d\tau$$
$$=\int_0^\infty \Psi_\alpha(\theta)T(t^\alpha\theta)u_0d\theta+\int_0^t\int_0^\infty \alpha\Psi_\alpha(\theta)(t-\tau)^{\alpha-1}\theta T((t-\tau)^\alpha\theta)g(\tau,u(\tau))d\theta d\tau$$

where $\Psi_\alpha(\theta)=\frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\rho_\alpha(\theta^{-\frac{1}{\alpha}})$ is the probability density function defined on $(0,\infty)$. Therefore,

$$u(t)=f(t,u(t))\left\{S_\alpha(t)u_0+\int_0^t(t-s)^{\alpha-1}T_\alpha(t,s)g(s,u(s))ds\right\},$$

where $S_\alpha(t)u=\int_0^\infty \Psi_\alpha(\theta)T(t^\alpha\theta)ud\theta$ and $T_\alpha(t,s)u=\alpha\int_0^\infty \theta\Psi_\alpha(\theta)T((t-s)^\alpha\theta)ud\theta$. □

**Lemma 3.2.** The operators $S_\alpha$ and $T_\alpha$ have the following properties:

(i) For all $t\geq s\geq 0$, $u\in X$, $S_\alpha(t)$ and $T_\alpha(t,s)$ are bounded linear operators with

$$\| S_\alpha(t)(u)\|\leq M\|u\| \quad \textit{and} \quad \| T_\alpha(t,s)(u)\|\leq \frac{M}{\Gamma(\alpha)}\|u\|.$$

(ii) The two operators $S_\alpha(t)$ and $T_\alpha(t,s)$ are strongly continuous $\forall t\geq s\geq 0$.

**Proof .** See [14]. □

Next, we introduce the following hypotheses:

$(H_1)$ The operator $T(t)$ is compact $\forall t\in J$.

$(H_2)$ The function $f\in C(J\times X, X\setminus\{0\})$ is bounded and there exist constants $\delta>0$ and $L>0$ such that for all $p,q\in X$, and $t\in J$ we have

$$|f(t,p)-f(t,q)|\leq\delta|p-q|,\quad |f(t,p)|\leq L.$$

$(H_3)$ Let $g\in Car(J\times X, X)$, we consider the function $K\in C(J,X)$ such that

$$|g(t,p)|\leq K(t)\quad t\in J, p\in X.$$

We define the set S by

$$S=\{v\in X, \| v\|\leq R\},$$

where

$$R=LM\left(|u_0|+\frac{b^\alpha}{\Gamma(\alpha+1)}\| K\|\right).$$

Clearly, $S$ is a closed, bounded and convex subset of $X$. Define the operators $F:X\to X$ and $P:S\to X$ by

$$Fu(t)=f(t,u(t)),\quad \text{and}\quad Pu(t)=S_\alpha(t)u_0+\int_0^t(t-s)^{\alpha-1}T_\alpha(t,s)g(s,u(s))ds.$$

We consider the operator $T:S\to X$ defined by

$$Tu(t)=Fu(t)Pu(t).$$

**Theorem 3.3.** Suppose that the hypotheses $(H_1)-(H_3)$ are satisfied. Then, problem (1.1) has a mild solution $u \in X$ under the following condition

$$\delta M\left(|u_0| + \frac{b^\alpha}{\Gamma(\alpha+1)} \parallel K \parallel\right) < 1. \tag{3.3}$$

**Proof .** We prove that the operators $F$ and $P$ satisfy the conditions of Theorem 2.10. The proof is given in several steps:

**Step 1:** $F : X \to X$ is a Lipschitz operator. Using the hypothesis $(H_1)$, we get

$$\begin{aligned} |(Fu(t) - Fv(t))| &= |(f(t,u(t)) - f(t,v(t)))| \\ &\leq \delta|u(t) - v(t)| \\ &\leq \delta\|u - v\|. \end{aligned}$$

Then, $F$ is a Lipschitz operator, with a Lipschitz constant $\delta$.

**Step 2:** $P : S \to X$ is completely continuous.

i) $P : S \to X$ is continuous:

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of $S$ such that $u_n \to u$ as $n \to \infty$ in $S$. We prove that $Pu_n \to Pu$ as $n \to \infty$ in $S$. Then by using Lemma (3.2)($i$), we get

$$\begin{aligned} \|Pu_n - Pu\| &= \left\|\int_0^t (t-s)^{\alpha-1} T_\alpha(t,s) g(s,u_n(s))ds - \int_0^t (t-s)^{\alpha-1} T_\alpha(t,s) g(s,u(s))ds\right\| \\ &\leq \int_0^t (t-s)^{\alpha-1} \parallel T_\alpha(t,s) \parallel \parallel g(s,u_n(s)) - g(s,u(s)) \parallel ds \\ &\leq M I_{0^+}^\alpha \parallel g(s,u_n(s)) - g(s,u(s)) \parallel . \end{aligned}$$

Using Lebesgue dominated convergence theorem, we obtain $\|Pu_n - Pu\| \to 0$ as $n \to \infty$. This proves that $P : S \to X$ is continuous.

ii) $P(S) = \{Pu : u \in S\}$ is uniformly bounded. Using hypotheses $(H_1)$, $(H_3)$ and Lemma (3.2)($i$), for any $u \in S$ and $t \in J$, we get

$$\begin{aligned} \parallel Pu(t) \parallel &= \left\|S_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1} T_\alpha(t,s) g(s,u(s))ds\right\| \\ &\leq \parallel S_\alpha(t)u_0 \parallel + \int_0^t (t-s)^{\alpha-1} \parallel T_\alpha(t,s) g(s,u(s)) \parallel ds \\ &\leq M|u_0| + \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \parallel g(s,u(s)) \parallel ds \\ &\leq M|u_0| + \frac{M}{\Gamma(\alpha)} \parallel K \parallel \int_0^t (t-s)^{\alpha-1} ds \\ &\leq M|u_0| + \frac{Mb^\alpha}{\Gamma(\alpha+1)} \parallel K \parallel . \end{aligned}$$

Therefore,

$$\parallel Pu(t) \parallel \leq M\left(|u_0| + \frac{b^\alpha}{\Gamma(\alpha+1)} \parallel K \parallel\right). \tag{3.4}$$

iii) $P(S)$ is equicontinuous. Let $t_1, t_2 \in J$ such that $t_1 < t_2$ and Let $u \in S$. Then using hypotheses $(H_1)$, $(H_3)$, and

Lemma (3.2)$(ii)$, we get,

$$
\begin{aligned}
\| Pu(t_2) - Pu(t_1) \| =& \| S_\alpha(t_2)u_0 + \int_0^{t_2} (t_2-s)^{\alpha-1} T_\alpha(t_2,s) g(s,u(s))ds \\
&- S_\alpha(t_1)u_0 - \int_0^{t_1} (t_1-s)^{\alpha-1} T_\alpha(t_1,s) g(s,u(s))ds \| \\
\leq& \| S_\alpha(t_2)u_0 - S_\alpha(t_1)u_0 \| \\
&+ \left\| \int_0^{t_2} (t_2-s)^{\alpha-1} \| T_\alpha(t_2,s) g(s,u(s)) \| ds - \int_0^{t_1} (t_1-s)^{\alpha-1} \| T_\alpha(t_1,s) g(s,u(s)) \| ds \right\| \\
\leq& \| S_\alpha(t_2)u_0 - S_\alpha(t_1)u_0 \| + \frac{M}{\Gamma(\alpha)} \| K \| \left\| \int_0^{t_2} (t_2-s)^{\alpha-1} ds - \int_0^{t_1} (t_1-s)^{\alpha-1} ds \right\| \\
\leq& \| S_\alpha(t_2)u_0 - S_\alpha(t_1)u_0 \| + \frac{M}{\Gamma(\alpha+1)} \| K \| (t_2^\alpha - t_1^\alpha).
\end{aligned}
$$

Then, $\| Pu(t_2) - Pu(t_1) \| \to 0$ as $|t_1 - t_2| \to 0$. Therefore, from (ii), (iii), and the Arzela-Ascoli theorem, deduce that the operator P(S) is relatively compact. Since $P : S \to X$ is a continuous operator, then it is completely continuous.

**Step 3:**

Let any $u \in X$ and $v \in S$ such that $u(t) = Fu(t)Pv(t)$, then by using the hypothesis $H_2$ for all $t \in J$ we have

$$
\begin{aligned}
\| u(t) \| =& \| Fu(t)Pv(t) \| \\
\leq& \| f(t,u(t)) \| \left\| S_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1} T_\alpha(t,s) g(s,v(s))ds \right\| \\
\leq& LM \left( |u_0| + \frac{b^\alpha}{\Gamma(\alpha+1)} \| K \| \right).
\end{aligned}
$$

Then, $\|u\| \leq R$. This implies that $u \in S$.

**Step 4:**

Let $N = \| P(S) \| = sup\{ \| Pu \| : u \in S\}$. According to inequalities (3.3) and (3.4), we have

$$
\begin{aligned}
\delta N \leq& \delta M \left( |u_0| + \frac{b^\alpha}{\Gamma(\alpha+1)} \| K \| \right) \\
<& 1.
\end{aligned}
$$

We notice that all the conditions of Theorem (2.10) are satisfied. Hence, operator T admits a fixed point in $S$. This implies that problem (1.1) has a mild solution in $C(J, X)$. □

## 4 Stability Analysis

In this section, we study the Ulam-Hyers (UH) stability, Ulam-Hyers-Rassias (UHR) stability, Generalized Ulam-Hyers (GUH) stability, and Generalized Ulam-Hyers (GUHR) stability of the mild solution to the problem (1.1).

**Definition 4.1.** The problem (1.1) is UH stable if there exists a real number $C > 0$ such that for all $\varepsilon > 0$ and for each solution $\nu \in C^1(J, X)$ of the following inequality:

$$
\left| {}^cD_{0^+}^\alpha \left( \frac{\nu(t)}{f(t,\nu(t))} \right) - A \left( \frac{\nu(t)}{f(t,\nu(t))} \right) - g(t,\nu(t)) \right| \leq \varepsilon, \tag{4.1}
$$

there exists a mild solution $u \in C(J, X)$ satisfying the problem (1.1) with

$$
\|u - \nu\| \leq C\varepsilon, \quad t \in [0, b].
$$

**Definition 4.2.** Let $\varepsilon > 0$ and $\nu \in C^1(J,X)$ is a solution to the inequality (4.1). problem (1.1) is GUH stable if there exist a function $\varphi \in C(\mathbb{R}^+,\mathbb{R}^+)$ with $\varphi(0)=0$ and a mild solution $u \in C(J,X)$ of problem (1.1) such that,

$$\|u-\nu\| \leq \lambda\varphi(\varepsilon), \quad t\in[0,b].$$

**Remark 4.3.** We say that a function $\nu \in C^1(J,\mathbb{R})$ is a solution of the inequality (4.1) if and only if there exists a function $\Upsilon \in C^1(J,X)$ such that,

i) $|\Upsilon(t)| \leq \varepsilon \quad$ for all $t\in J$.

ii) ${}^cD^\alpha_{0_+}\left(\frac{\nu(t)}{f(t,\nu(t))}\right) = A\left(\frac{\nu(t)}{f(t,\nu(t))}\right) + g(t,\nu(t)) + \Upsilon(t)$.

**Definition 4.4.** The problem (1.1) is UHR stable with respect to $\chi \in C(\mathbb{R}^+,\mathbb{R}^+)$, if there exists a real number $C>0$ such that for each $\varepsilon>0$ and for each solution $\nu\in C^1(J,X)$ of the following inequality:

$$\left|{}^cD^\alpha_{0_+}\left(\frac{\nu(t)}{f(t,\nu(t))}\right) - A\left(\frac{\nu(t)}{f(t,\nu(t))}\right) - g(t,\nu(t))\right| \leq \varepsilon\chi(t), \quad t\in J, \tag{4.2}$$

there exists a mild solution $u\in C(J,X)$ of problem (1.1) such that,

$$\|u-\nu\| \leq C\varepsilon\chi(t), \quad t\in J.$$

**Definition 4.5.** The problem (1.1) is GUHR stable with respect to $\chi \in C(\mathbb{R}^+,\mathbb{R}^+)$, if there exists a real number $C>0$ such that for each solution $\nu\in C^1(J,X)$ of the following inequality:

$$\left|{}^cD^\alpha_{0_+}\left(\frac{\nu(t)}{f(t,\nu(t))}\right) - A\left(\frac{\nu(t)}{f(t,\nu(t))}\right) - g(t,\nu(t))\right| \leq \chi(t), \quad t\in J, \tag{4.3}$$

there exists a mild solution $u\in C(J,X)$ of problem (1.1) such that,

$$\|u-\nu\| \leq C\chi(t), \quad t\in J.$$

**Remark 4.6.** A function $\nu\in C^1(J,X)$ is a solution of the inequality (4.2) if and only if there exist two functions $\Upsilon\in C^1(J,X)$ and $\chi\in C(\mathbb{R}^+,\mathbb{R}^+)$ such that,

i) $|\Upsilon(t)| \leq \varepsilon\chi(t)$, for all $t\in J$.

ii) ${}^cD^\alpha_{0_+}\left(\frac{\nu(t)}{f(t,\nu(t))}\right) = A\left(\frac{\nu(t)}{f(t,\nu(t))}\right) + g(t,\nu(t)) + \Upsilon(t)$.

**Lemma 4.7.** Let $\varepsilon>0$ and $\nu\in C^1(J,X)$ be a solution of the inequality (4.1), by mean of hypotheses $(H_1),(H_2)$, Lemma (3.2)$(ii)$ and Remark (4.3), we have

$$\left|\nu(t) - f(t,\nu(t))\left\{S_\alpha(t)\nu_0 + \int_0^t (t-s)^{\alpha-1}T_\alpha(t,s)g(s,\nu(s))ds\right\}\right| \leq \frac{Mb^\alpha}{\Gamma(\alpha+1)}L\varepsilon.$$

**Proof .** Indeed, by Remark (4.3), we have

$${}^cD^\alpha_{0_+}\left(\frac{\nu(t)}{f(t,\nu(t))}\right) = A\left(\frac{\nu(t)}{f(t,\nu(t))}\right) + g(t,\nu(t)) + \Upsilon(t), \quad t\in J.$$

From Theorem (3.3), we get

$$\nu(t) = f(t,\nu(t))\left\{S_\alpha(t)\nu_0 + \int_0^t (t-s)^{\alpha-1}T_\alpha(t,s)g(s,\nu(s))ds\right\} + f(t,\nu(t))\int_0^t (t-s)^{\alpha-1}T_\alpha(t,s)\Upsilon(s)ds.$$

Then from hypotheses $(H_1),(H_2)$, Lemma (3.2)$(ii)$ and remark (4.3) we have

$$\begin{aligned}\left|\nu(t) - f(t,\nu(t))\left\{S_\alpha(t)\nu_0 + \int_0^t (t-s)^{\alpha-1}T_\alpha(t,s)g(s,\nu(s))ds\right\}\right| &\leq |f(t,\nu(t))|\int_0^t (t-s)^{\alpha-1}|T_\alpha(t,s)\Upsilon(s)|ds \\ &\leq \frac{Mb^\alpha}{\Gamma(\alpha+1)}L\varepsilon.\end{aligned}$$

□

**Theorem 4.8.** Under the hypotheses of Theorem (3.3), if the inequality (4.1) is satisfied and

$$1-\delta M\left(|u_0|+\frac{b^\alpha}{\Gamma(\alpha+1)}\|K\|\right)>0,$$

then the problem (1.1) is UH stable in $C(J,X)$.

**Proof .** Let $\varepsilon>0$, $\nu\in C^1(J,X)$ be a solution of the inequality (4.1) and let $u\in C(J,X)$ be a mild solution of the following problem

$$\begin{cases} {}^cD^\alpha_{0_+}\left(\frac{u(t)}{f(t,u(t))}\right)=A\left(\frac{u(t)}{f(t,u(t))}\right)+g(t,u(t)), \quad t\in[0,b] \\ \\ u(0)=\nu(0)=u_0. \end{cases} \tag{4.4}$$

Since the function $\nu\in C^1(J,X)$ satisfies the inequality (4.1). From our hypotheses, we have

$$\begin{aligned} |(\nu(t)-u(t))| &= \left|\nu(t)-f(t,u(t))\left\{S_\alpha(t)u_0+\int_0^t(t-s)^{\alpha-1}T_\alpha(t,s)g(s,u(s))ds\right\}\right| \\ &\le \left|\nu(t)-f(t,\nu(t))\left\{S_\alpha(t)u_0+\int_0^t(t-s)^{\alpha-1}T_\alpha(t,s)g(s,\nu(s))ds\right\}\right| \\ &+\left|f(t,\nu(t))\left\{S_\alpha(t)u_0+\int_0^t(t-s)^{\alpha-1}T_\alpha(t,s)g(s,\nu(s))ds\right\}\right. \\ &\left.-\ f(t,u(t))\left\{S_\alpha(t)u_0+\int_0^t(t-s)^{\alpha-1}T_\alpha(t,s)g(s,u(s))ds\right\}\right| \\ &\le \frac{Mb^\alpha}{\Gamma(\alpha+1)}L\varepsilon+|f(t,u(t))-f(t,\nu(t))|M\left(|u_0|+\frac{b^\alpha}{\Gamma(\alpha+1)}\|K\|\right) \\ &\le \frac{Mb^\alpha}{\Gamma(\alpha+1)}L\varepsilon+\delta|u(t)-\nu(t)|M\left(|u_0|+\frac{b^\alpha}{\Gamma(\alpha+1)}\|K\|\right). \end{aligned}$$

Hence,

$$|\nu(t)-u(t)|\le\frac{Mb^\alpha L}{\Gamma(\alpha+1)\left(1-\delta M\left(|u_0|+\frac{b^\alpha}{\Gamma(\alpha+1)}\|K\|\right)\right)}\varepsilon,$$

this implies that the problem (1.1) is UH stable. □

**Theorem 4.9.** We assume that the hypotheses of Theorem (4.8) are satisfied. If there exists $\varphi\in C(\mathbb{R}^+,\mathbb{R}^+)$ with $\varphi(0)=0$. Then, the problem (1.1) is GUH stable.

**Proof .** Putting $\varphi(\varepsilon)=\varepsilon$ with $\varphi(0)=0$ and

$$C=\frac{Mb^\alpha L}{\Gamma(\alpha+1)\left(1-\delta M\left(|u_0|+\frac{b^\alpha}{\Gamma(\alpha+1)}\|K\|\right)\right)},$$

we get

$$|\nu(t)-u(t)|\le C\varphi(\varepsilon),$$

this proves that the problem (1.1) is GUH stable. □

**Lemma 4.10.** Let $\varepsilon>0$, $\nu\in C^1(J,X)$ be a solution of the inequality (4.2) and

$$\int_0^t(t-s)^{\alpha-1}|\chi(s)|ds\le\mu_\chi\chi(t), \quad t\in J,$$

from our hypotheses and Remark (4.6), we have

$$\left|\nu(t)-f(t,\nu(t))\left\{S_\alpha(t)\nu_0+\int_0^t(t-s)^{\alpha-1}T_\alpha(t,s)g(s,\nu(s))ds\right\}\right|\le\frac{LM}{\Gamma(\alpha)}\varepsilon\mu_\chi\chi(t).$$

**Proof .** Thanks to Remark (4.6), we have

$$ {}^cD^{\alpha}_{0_+}\left(\frac{\nu(t)}{f(t,\nu(t))}\right) = A\left(\frac{\nu(t)}{f(t,\nu(t))}\right) + g(t,\nu(t)) + \Upsilon(t), \quad t\in J. $$

Using the theorem (3.3), we get

$$ \begin{aligned} \nu(t) &= f(t,\nu(t))\left\{S_\alpha(t)\nu_0 + \int_0^t (t-s)^{\alpha-1}T_\alpha(t,s)g(s,\nu(s))ds\right\} \\ &+ f(t,\nu(t))\int_0^t (t-s)^{\alpha-1}T_\alpha(t,s)\Upsilon(s)ds. \end{aligned} $$

Again from Remark (4.6), we have

$$ \begin{aligned} \left|\nu(t) - f(t,\nu(t))\left\{S_\alpha(t)\nu_0 + \int_0^t (t-s)^{\alpha-1}T_\alpha(t,s)g(s,\nu(s))ds\right\}\right| &\le |f(t,\nu(t))|\int_0^t (t-s)^{\alpha-1} \parallel T_\alpha(t,s)\parallel |\Upsilon(s)|ds \\ &\le \frac{LM}{\Gamma(\alpha)}\varepsilon\int_0^t (t-s)^{\alpha-1}|\chi(s)|ds \\ &\le \frac{LM}{\Gamma(\alpha)}\varepsilon\mu_\chi\chi(t). \end{aligned} $$

□

**Theorem 4.11.** Under the hypotheses of Theorem (3.3) and Lemma (4.10), together with the condition

$$ 1-\delta M\left(|u_0| + \frac{b^\alpha}{\Gamma(\alpha+1)}\|K\|\right) > 0, $$

the mild solution of problem (1.1) is UHR stable and consequently is GUHR stable.

**Proof .** Let $\varepsilon > 0$, $\nu\in C^1(J,X)$ be a solution to the inequality (4.2), and let $u\in C(J,X)$ be a mild solution of the problem (4.4). Then, from our hypotheses and Lemma (4.10), we obtain

$$ \begin{aligned} |(\nu(t)-u(t))| &= \left|\nu(t) - f(t,u(t))\left\{S_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}T_\alpha(t,s)g(s,u(s))ds\right\}\right| \\ &\le \left|\nu(t) - f(t,\nu(t))\left\{S_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}T_\alpha(t,s)g(s,\nu(s))ds\right\}\right| \\ &+ \left|f(t,\nu(t))\left\{S_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}T_\alpha(t,s)g(s,\nu(s))ds\right\}\right. \\ &- \left. f(t,u(t))\left\{S_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}T_\alpha(t,s)g(s,u(s))ds\right\}\right| \\ &\le \frac{LM}{\Gamma(\alpha)}\varepsilon\int_0^t (t-s)^{\alpha-1}|\chi(s)|ds + |f(t,u(t)) - f(t,\nu(t))|M\left(|u_0| + \frac{b^\alpha}{\Gamma(\alpha+1)}\|K\|\right) \\ &\le \frac{LM}{\Gamma(\alpha)}\varepsilon\mu_\chi\chi(t) + \delta|u(t)-\nu(t)|M\left(|u_0| + \frac{b^\alpha}{\Gamma(\alpha+1)}\|K\|\right), \end{aligned} $$

therefore,

$$ |\nu(t)-u(t)| \le \frac{M\mu_\chi L}{\Gamma(\alpha)\left(1-\delta M\left(|u_0| + \frac{b^\alpha}{\Gamma(\alpha+1)}\|K\|\right)\right)}\chi(t)\varepsilon. $$

Then, for $C = \dfrac{M\mu_\chi L}{\Gamma(\alpha)\left(1-\delta M\left(|u_0| + \frac{b^\alpha}{\Gamma(\alpha+1)}\|K\|\right)\right)}$, we have

$$ |\nu(t)-u(t)| \le C\chi(t)\varepsilon \quad t\in J. $$

This implies that problem (1.1) is UHR stable. Then, in the same way above, we pose $\varepsilon = 1$, and we find that

$$|\nu(t) - u(t)| \leq C\chi(t), \quad t \in J.$$

This proves that, the problem (1.1) is GUHR stable. □

## 5 Example

In this section, we give an appropriate example to illustrate our results. In particular, our results can be reduced to the example in [1]. Let $X = L^2([0,1],\mathbb{R})$. We denote $H^2(0,1)$ the completion of the space $C^2(0,1)$ with respect to the norm

$$\| u \|_{H^2(0,1)} = \left( \int_0^1 \sum_{|k| \leq 2} |D^k u(x)|^2 \right)^{\frac{1}{2}},$$

and $H^1(0,1)$ is the completion of the space $C^1(0,1)$ with respect to the norm $\| u \|_{H^1(0,1)}$. We Consider the following hybrid fractional partial differential equation involving Caputo fractional derivative.

$$\begin{cases} {}^C\partial_t^{\frac{1}{2}} \left( \frac{u(t,x)}{1+\frac{cos(t)}{4}|u(t,x)|} \right) = \Delta \left( \frac{u(t,x)}{1+\frac{cos(t)}{4}|u(t,x)|} \right) + \frac{|u(t,x)|}{2+|u(t,x)|}, & t \in [0,1], x \in [0,1] \\ u(0,t) = u(1,t) = 0, & t \in [0,1] \\ u(x,0) = u_0(x) = 0, & x \in [0,1], \end{cases} \tag{5.1}$$

where ${}^C\partial_t^{\frac{1}{2}}$ is the Caputo fractional partial derivative of order $\frac{1}{2}$, and $A : D(A) \subset X \longrightarrow X$ be an operator defined by

$$D(A) := H^2(0,1) \cap H^1(0,1) = \{w \in H^2(0,1) : w(0) = w(1) = 0\} \quad and \quad Au = \Delta u.$$

Comparing the system (5.1) with the problem (1.1). Then $f(t,u) = 1 + \frac{cos(t)}{4}|u(t,x)|$, $g(t,u) = \frac{|u(t,x)|}{2+|u(t,x)|}$, $\alpha = \frac{1}{2}$, $u_0 = 0$, and $J = [0,1]$. It is clear that
$|f(t,u(t)) - f(t,v(t))| \leq \frac{1}{4}|u - v|$ and $|g(t,u(t))| \leq 1$. We take $K(t) = 1$, $L = 1$, $M = 1$ and $\delta = \frac{1}{4}$.
Now we check for condition (3.3). Further, consider

$$\begin{aligned} \delta M \left( |u_0| + \frac{b^\alpha}{\Gamma(\alpha+1)} \parallel K \parallel \right) &= \frac{1}{4} \left\{ 0 + \frac{1^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} \right\} \\ &\approx 0,370 \\ &< 1 \end{aligned}$$

We observe that all the conditions of Theorem (3.3) are satisfied. Therefore, problem (5.1) has a mild solution in $C([0,1],X)$. Further, to derive the conditions for UH, GUH, UHR, and GUHR, we have

$$1 - \delta M \left( |u_0| + \frac{b^\alpha}{\Gamma(\alpha+1)} \|K\| \right) \approx 0,63 > 0.$$

By using Theorem (4.8), the mild solution of the problem (5.1) is UH stable, and consequently is GUH stable. Also, by taking $\chi(t) = t$ which is an increasing function on $[0,1]$, by using theorem (4.11), the mild solution to the problem (5.1) is UHR stable and consequently is GUHR stable.

## 6 Conclusion

In this paper, we have established the existence theory for mild solutions to initial value problems for hybrid fractional semi-linear evolution equations introduced by the Caputo fractional derivative. The technique used is based on the Dhage fixed point theorem. On the other hand, we have studied the four different types of Ulam-Hyers stability of solutions to a given problem. Also, we presented an example to illustrate our main results.

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