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Tsallis entropy of fuzzy σ-algebras

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Abstract

The Shannon entropy and the logical entropy of fuzzy σ-algebras are well-known instances of entropy. In this paper, we introduce and study the Tsallis entropy of order α of fuzzy $\sigma-$algebras on $F-$probability measure spaces, where $\alpha \in (0,1) \cup (1,\infty)$. Moreover, we study the conditional version of this entropy and examine its basic properties.

Keywords: Tsallis entropy, Entropy, Conditional Tsallis entropy, Fuzzy σ-algebras
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1 Introduction

Entropy is a tool that allows us to measure the amount of uncertainty in random events. The classical approach in information theory is based on the Shannon entropy [25]. The Shannon entropy of a probability distribution was studied in [15]. Kolmogorov and Sinai used the Shannon entropy to define the entropy of measurable partitions [26]. The Kolmogorov–Sinai entropy is a useful tool for the study of isomorphisms of dynamical systems.

Markechova [17, 18] studied the Shannon entropy of complete partitions and the entropy of an $F-$dynamical system. In [5], Dumitrescu used methods different from triangular norms and their resulting entropies to develop fuzzy partition theory (see [11]). In [14], Klement defined the notion of $F-$probability measure space by defining the notions of fuzzy $\sigma-$algebra and $F-$measure. M. Khare, A. Ebrahimzadeh and J. Jamalzadeh introduced the notions of Shannon entropy and logical entropy of fuzzy $\sigma-$algebras having finitely many atoms on an $F-$probability measure space, and obtained some results concerning these measures [6, 12, 13].

The Tsallis entropy plays a significant role in the non-extensive statistical mechanics of complex systems [31]. The number q is the so-called entropic index; it characterizes the degree of non-extensivity of the system. Some applications of the Tsallis entropy have been found in a wide range of phenomena in diverse disciplines such as chemistry, physics, geophysics, biology, economics, medicine, etc. We refer the reader to [1, 2, 4, 3, 7, 9, 10, 19, 22, 24, 29, 32, 34]. Also, the entropy has been applied to large domains in communication systems [16]; its applications in image processing through information theory can be found, for example, in [23]. For a full and regularly updated bibliography, see [20].

It is known that fuzzy logic outperforms the usual logic in the analysis of natural phenomena. As a result, logical algebraic structures have been considered to describe such phenomena. It is natural to define and study the concepts of dynamical structures, on fuzzy logical algebras. The notion of entropy is one of the most applicable concepts of dynamical systems, which can be used to measure the amount the system under consideration is chaotic. In this regard, M. Khare [12] and, A. Ebrahimzadeh and J. Jamalzadeh [6] defined and studied shannon entropy and logical

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entropy on fuzzy $\sigma-$algebras. In this paper, we introduce and study the Tsallis entropy of order α of fuzzy $\sigma-$algebras on $F-$probability measure spaces, where $\alpha \in (0,1) \cup (1,\infty)$, which is more general compared to Shannon entropy. Moreover, we study the conditional version of this entropy and examine its basic properties.

2 Preliminaries

Let X be a non-empty set, and I^X denote the set of all functions from X to the closed unit interval $I = [0,1]$. A *fuzzy σ-algebra* $\mathcal{M}$ on X [14] is a subfamily of I^X which satisfies the following conditions.

(i) $\mathbf{1} \in \mathcal{M}$.

(ii) If $\lambda \in \mathcal{M}$, then $\mathbf{1} - \lambda \in \mathcal{M}$.

(iii) If $\{\lambda\}_{i=1}^{\infty} \subseteq \mathcal{M}$, then $\bigvee_{i=1}^{\infty} \lambda_i = \sup_i \lambda_i \in \mathcal{M}$.

If $\mathcal{N}_1$ and $\mathcal{N}_2$ are fuzzy σ-algebras on X, then $\mathcal{N}_1 \vee \mathcal{N}_2$ is the smallest fuzzy σ-algebra on X that contains $\mathcal{N}_1 \cup \mathcal{N}_2$.

An *F-probability measure* m on $\mathcal{M}$ is a function $m : \mathcal{M} \to I$ which satisfies the following conditions.

(i) $m(\mathbf{1}) = \mathbf{1}$.

(ii) $m(\mathbf{1} - \lambda) = \mathbf{1} - m(\lambda),\ \lambda \in \mathcal{M}$.

(iii) $m(\lambda \vee \mu) + m(\lambda \wedge \mu) = m(\lambda) + m(\mu)$ for every $\lambda, \mu \in \mathcal{M}$.

(v) If $\{\lambda\}_{i=1}^{\infty} \subseteq \mathcal{M}$ and $\lambda_i \uparrow \lambda$, then $m(\lambda) = \sup_i m(\lambda_i)$.

The triple $(X, \mathcal{M}, m)$ is called an F-probability measure space.

Definition 2.1. [28] Let $(X, \mathcal{M}, m)$ be an $F-$probability measure space. We define a relation on $\mathcal{M}$, denoted by $= (\mod m)$, via

$$\lambda = \mu \ (\mod m) \Longleftrightarrow m(\lambda) = m(\mu) = m(\lambda \vee \mu).$$

The relation $= (\mod m)$ is an equivalence relation on $\mathcal{M}$. The set of all equivalence classes induced by this relation is denoted by $\tilde{\mathcal{M}}$, and $\tilde{\mu}$ denotes the equivalence class determined by μ.

We say that $\lambda, \mu \in \mathcal{M}$ are m-disjoint if $\lambda \wedge \mu = 0 \ (\mod m)$, that is, $m(\lambda \wedge \mu) = 0$.

Definition 2.2. [28] Let $(X, \mathcal{M}, m)$ be an $F-$probability measure space, and $\mathcal{N}$ be a fuzzy sub-σ-algebra of $\mathcal{M}$. An element $\tilde{\mu}$ of $\tilde{\mathcal{N}}$ is called an *atom* of $\mathcal{N}$ if $m(\mu) > 0$ and for any $\tilde{\lambda} \in \tilde{\mathcal{N}}$,

$$m(\lambda \wedge \mu) = m(\lambda) \neq m(\mu) \Rightarrow m(\lambda) = 0.$$

The set of all atoms of $\mathcal{N}$ is denoted by $\bar{\mathcal{N}}$. Also, $\mathcal{F}(\mathcal{M})$ denotes the collection of fuzzy sub-σ-algebras of $\mathcal{M}$ having finitely many atoms.

Definition 2.3. [28] Let $(X, \mathcal{M}, m)$ be an $F-$probability measure space, and $\mathcal{N}_1, \mathcal{N}_2$ be fuzzy sub-σ-algebras of $\mathcal{M}$. Then, $\mathcal{N}_2$ is called an $m-$refinement of $\mathcal{N}_1$, written as $\mathcal{N}_1 \leq_m \mathcal{N}_2$, if for $\mu \in \bar{\mathcal{N}}_2$ there exists $\lambda \in \bar{\mathcal{N}}_1$ such that $m(\lambda \wedge \mu) = m(\mu)$.

The fuzzy sub-σ-algebras $\mathcal{N}_1, \mathcal{N}_2$ are called $m-$equivalent, denoted by $\mathcal{N}_1 \cong_m \mathcal{N}_2$, if

$$m\left(\lambda \wedge \left(\bigvee\{\mu : \mu \in \bar{\mathcal{N}}_2\}\right)\right) = m(\lambda)$$

for each $\lambda \in \bar{\mathcal{N}}_1$, and

$$m\left(\mu \wedge \left(\bigvee\{\lambda : \lambda \in \bar{\mathcal{N}}_1\}\right)\right) = m(\mu)$$

for each $\mu \in \bar{\mathcal{N}}_2$.

The relation of $m-$equivalence is an equivalence relation on $\mathcal{F}(\mathcal{M})$, and $[\mathcal{N}]$ denotes the set of all $m-$equivalent fuzzy sub-σ-algebras in $\mathcal{F}(\mathcal{M})$.

Theorem 2.4. [28] Let $(X, \mathcal{M}, m)$ be an $F-$probability measure space and $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$ be elements of $\mathcal{F}(\mathcal{M})$. If $\mathcal{N}_1 \leq_m \mathcal{N}_2$, then $\mathcal{N}_1 \vee \mathcal{N}_3 \leq_m \mathcal{N}_2 \vee \mathcal{N}_3$.

Definition 2.5. [27] Define a relation $\sim$ on $\mathcal{F}(\mathcal{M})$ by

$$\mathcal{N}_1 \sim \mathcal{N}_2 \Longleftrightarrow \mathcal{N}_1 \leq_m \mathcal{N}_2 \text{ and } \mathcal{N}_2 \leq_m \mathcal{N}_1.$$

Then, $\sim$ is an equivalence relation on $\mathcal{F}(\mathcal{M})$. This relation is called *equivalence modulo* 0.

3 The Tsallis entropy and the conditional Tsallis entropy

In this section, we define the Tsallis entropy and the conditional Tsallis entropy of fuzzy σ-algebras on an F-probability measure space.

Definition 3.1. Let $(X, \mathcal{M}, m)$ be an $F-$probability measure space and $\mathcal{N} \in \mathcal{F}(\mathcal{M})$, where $\bar{\mathcal{N}} = \{\lambda_i : i = 1, \ldots, s\}$ and $\alpha \in (0,1) \cup (1, \infty)$. We define the Tsallis entropy $T_\alpha(\mathcal{N})$ by

$$T_\alpha(\mathcal{N}) = \frac{1}{\alpha - 1}(1 - \sum_{i=1}^{s} m(\lambda_i)^\alpha). \tag{3.1}$$

Definition 3.2. Let $(X, \mathcal{M}, m)$ be an $F-$probability measure space, $\mathcal{N}_1, \mathcal{N}_2 \in \mathcal{F}(\mathcal{M})$, and let $\bar{\mathcal{N}_1} = \{\lambda_i : i = 1, \ldots, s\}$, $\bar{\mathcal{N}_2} = \{\mu_j : j = 1, \ldots, t\}$. We define the conditional Tsallis entropy $T_\alpha(\mathcal{N}_1|\mathcal{N}_2)$ by

$$T_\alpha(\mathcal{N}_1|\mathcal{N}_2) = \frac{1}{\alpha - 1}\left(\sum_{j=1}^{t} m(\mu_j)^\alpha - \sum_{i=1}^{s}\sum_{j=1}^{t} m(\lambda_i \wedge \mu_j)^\alpha\right). \tag{3.2}$$

Remark 3.3. If $\mathcal{N}_1, \mathcal{N}_2 \in \mathcal{F}(\mathcal{M})$, then $T_\alpha(\mathcal{N}_1|\mathcal{N}_2) \geq 0$ and $T_\alpha(\mathcal{N}_1|\mathcal{N}_1) = 0$.

Theorem 3.4. Let $(X, \mathcal{M}, m)$ be an $F-$probability measure space and $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$ be elements of $\mathcal{F}(\mathcal{M})$. If $\mathcal{N}_1 \cong_m \mathcal{N}_2 \cong_m \mathcal{N}_3$, then the following statements are true.

(*i*) $T_\alpha(\mathcal{N}_1 \vee \mathcal{N}_2|\mathcal{N}_3) = T_\alpha(\mathcal{N}_1|\mathcal{N}_3) + T_\alpha(\mathcal{N}_2|\mathcal{N}_1 \vee \mathcal{N}_3)$.

(*ii*) $T_\alpha(\mathcal{N}_1 \vee \mathcal{N}_2) = T_\alpha(\mathcal{N}_1) + T_\alpha(\mathcal{N}_2|\mathcal{N}_1)$.

(*iii*) $T_\alpha(\mathcal{N}_1|\mathcal{N}_2) = T_\alpha(\mathcal{N}_1) + (1-\alpha)T_\alpha(\mathcal{N}_1)T_\alpha(\mathcal{N}_2)$.

(*iv*) $T_\alpha(\mathcal{N}_1|\mathcal{N}_2) \leq T_\alpha(\mathcal{N}_1)$.

(*v*) $T_\alpha(\mathcal{N}_1 \vee \mathcal{N}_2) \leq T_\alpha(\mathcal{N}_1) + T_\alpha(\mathcal{N}_2)$.

Proof . Let $\bar{\mathcal{N}_1} = \{\lambda_i : i = 1, \ldots, s\}$, $\bar{\mathcal{N}_2} = \{\mu_j : j = 1, \ldots, t\}$ and $\bar{\mathcal{N}_3} = \{\gamma_k : k = 1, \ldots, l\}$. In [12], it is proved that $\mathcal{N}_1 \vee \mathcal{N}_2 \cong \mathcal{N}_3$. Hence,

$$\begin{aligned}
\sum_{j=1}^{t} m(\lambda_i \wedge \mu_j \wedge \gamma_k) &= m\left(\bigvee_{j=1}^{t}(\lambda_i \wedge \mu_j \wedge \gamma_k)\right) \\
&= m\left((\lambda_i \wedge \gamma_k) \wedge \left(\bigvee_{j=1}^{t} \mu_j\right)\right) \\
&= m(\lambda_i \wedge \gamma_k).
\end{aligned}$$

Thus, using $\mathcal{N}_1 \cong \mathcal{N}_2$ we obtain

$$\begin{aligned}\sum_{j=1}^{t} m(\lambda_i \wedge \mu_j) &= m\left(\bigvee_{j=1}^{t}(\lambda_i \wedge \mu_j)\right) \\ &= m\left(\lambda_i \wedge \bigvee_{j=1}^{t} \mu_j\right) \\ &= m(\lambda_i).\end{aligned}$$

(i)

$$\begin{aligned}T_\alpha(\mathcal{N}_1 \vee \mathcal{N}_2|\mathcal{N}_3) &= \frac{1}{\alpha-1}\left(\sum_{k=1}^{l} m(\gamma_k)^\alpha - \sum_{i=1}^{s}\sum_{j=1}^{t}\sum_{k=1}^{l} m(\lambda_i \wedge \mu_j \wedge \gamma_k)^\alpha\right) \\ &= \frac{1}{\alpha-1}\left(\sum_{k=1}^{l} m(\gamma_k)^\alpha - \sum_{i=1}^{s}\sum_{k=1}^{l} m(\lambda_i \wedge \gamma_k)^\alpha\right) \\ &+ \frac{1}{\alpha-1}\left(\sum_{i=1}^{s}\sum_{k=1}^{l} m(\lambda_i \wedge \gamma_k)^\alpha - \sum_{i=1}^{s}\sum_{j=1}^{t}\sum_{k=1}^{l} m(\lambda_i \wedge \mu_j \wedge \gamma_k)^\alpha\right) \\ &= T_\alpha(\mathcal{N}_1|\mathcal{N}_3) + T_\alpha(\mathcal{N}_2|\mathcal{N}_1 \vee \mathcal{N}_3).\end{aligned}$$

(ii) We know that $\mathcal{N}_1 \vee \{\mathbf{1}\} = \mathcal{N}_1$. Consequently,

$$\begin{aligned}T_\alpha(\mathcal{N}_1|\{\mathbf{1}\}) &= \frac{1}{\alpha-1}\left(\sum_{j=1}^{t} m(\{\mathbf{1}\})^\alpha - \sum_{i=1}^{s}\sum_{j=1}^{t} m(\lambda_i \wedge \{\mathbf{1}\})^\alpha\right) \\ &= \frac{1}{\alpha-1}(1 - \sum_{i=1}^{s} m(\lambda_i)^\alpha) \\ &= T_\alpha(\mathcal{N}_1).\end{aligned}$$

Thus, according to the first part, we can write

$$\begin{aligned}T_\alpha(\mathcal{N}_1 \vee \mathcal{N}_2) &= T_\alpha(\mathcal{N}_1 \vee \mathcal{N}_2|\{\mathbf{1}\}) \\ &= T_\alpha(\mathcal{N}_1|\{\mathbf{1}\}) + T_\alpha(\mathcal{N}_2|\mathcal{N}_1 \vee \{\mathbf{1}\}) \\ &= T_\alpha(\mathcal{N}_1) + T_\alpha(\mathcal{N}_2|\mathcal{N}_1).\end{aligned}$$

(iii) By Definition 3.1,

$$(1-\alpha)T_\alpha(\mathcal{N}_1)T_\alpha(\mathcal{N}_2) = (1-\alpha)\frac{1}{\alpha-1}\left(1 - \sum_{i=1}^{s} m(\lambda_i)^\alpha\right)\frac{1}{\alpha-1}\left(1 - \sum_{j=1}^{t} m(\mu_j)^\alpha\right).$$

Thus, Definition 3.2 allows us to write

$$(1-\alpha)T_\alpha(\mathcal{N}_1)T_\alpha(\mathcal{N}_2) = T_\alpha(\mathcal{N}_1|\mathcal{N}_2) - T_\alpha(\mathcal{N}_1).$$

(iv) By Definition 3.2,

$$\begin{aligned}T_\alpha(\mathcal{N}_1|\mathcal{N}_2) &= \frac{1}{\alpha-1}\left(\sum_{j=1}^{t} m(\mu_j)^\alpha - \sum_{i=1}^{s}\sum_{j=1}^{t} m(\lambda_i\wedge\mu_j)^\alpha\right)\\ &= \frac{1}{\alpha-1}\left(\sum_{j=1}^{t} m(\mu_j)^\alpha - \sum_{i=1}^{s} m(\lambda_i)^\alpha\right)\\ &\leq \frac{1}{\alpha-1}\left(1 - \sum_{i=1}^{s} m(\lambda_i)^\alpha\right)\\ &= T_\alpha(\mathcal{N}_1).\end{aligned}$$

(v) Letting $N_3=\{\mathbf{1}\}$ in (i) we obtain

$$T_\alpha(\mathcal{N}_1\vee\mathcal{N}_2|\{\mathbf{1}\}) = T_\alpha(\mathcal{N}_1|\{\mathbf{1}\}) + T_\alpha(\mathcal{N}_2|\mathcal{N}_1\vee\{\mathbf{1}\}).$$

Therefore,

$$T_\alpha(\mathcal{N}_1\vee\mathcal{N}_2) = T_\alpha(\mathcal{N}_1) + T_\alpha(\mathcal{N}_2|\mathcal{N}_1).$$

Finally, by (iv),

$$T_\alpha(\mathcal{N}_1\vee\mathcal{N}_2) \leq T_\alpha(\mathcal{N}_1) + T_\alpha(\mathcal{N}_2).$$

□

Theorem 3.5. Let $(X,\mathcal{M},m)$ be an $F-$probability measure space and $\mathcal{N}_1,\mathcal{N}_2,\mathcal{N}_3$ be elements of $\mathcal{F}(\mathcal{M})$. If $\mathcal{N}_1\cong_m\mathcal{N}_2\cong_m\mathcal{N}_3$, then the following statements are true.

(i) $T_\alpha(\mathcal{N}_1|\mathcal{N}_2)=0 \Longleftrightarrow \mathcal{N}_1\leq_m\mathcal{N}_2$.

(ii) $T_\alpha(\mathcal{N}_1\vee\mathcal{N}_2)=T_\alpha(\mathcal{N}_1) \Longleftrightarrow \mathcal{N}_1\leq_m\mathcal{N}_2$.

(iii) If $\mathcal{N}_1\sim\mathcal{N}_2$, then $T_\alpha(\mathcal{N}_1)=T_\alpha(\mathcal{N}_2)$.

(iv) If $\mathcal{N}_1\sim\mathcal{N}_2$, then $T_\alpha(\mathcal{N}_1|\mathcal{N}_3)=T_\alpha(\mathcal{N}_2|\mathcal{N}_3)$.

(v) If $\mathcal{N}_2\sim\mathcal{N}_3$, then $T_\alpha(\mathcal{N}_1|\mathcal{N}_2)=T_\alpha(\mathcal{N}_1|\mathcal{N}_3)$.

Proof . Let $\bar{\mathcal{N}_1}=\{\lambda_i: i=1,\ldots,s\}$, $\bar{\mathcal{N}_2}=\{\mu_j: j=1,\ldots,t\}$ and $\bar{\mathcal{N}_3}=\{\gamma_k: k=1,\ldots,l\}$.

(i) Suppose that $\mathcal{N}_1\leq_m\mathcal{N}_2$. Then, for any $\mu_j\in\mathcal{N}_2$ there exists $\lambda_i\in\mathcal{N}_1$ such that $m(\lambda_i\wedge\mu_j)=m(\mu_j)$.

$$\begin{aligned}T_\alpha(\mathcal{N}_1|\mathcal{N}_2) &= \frac{1}{\alpha-1}\left(\sum_{j=1}^{t} m(\mu_j)^\alpha - \sum_{i=1}^{s}\sum_{j=1}^{t} m(\lambda_i\wedge\mu_j)^\alpha\right)\\ &= \frac{1}{\alpha-1}\left(\sum_{j=1}^{t} m(\mu_j)^\alpha - \sum_{i=1}^{s}\sum_{j=1}^{t} m(\mu_j)^\alpha\right)\\ &= 0.\end{aligned}$$

(ii) Suppose that $\mathcal{N}_1\leq_m\mathcal{N}_2$. Then,

$$\begin{aligned}T_\alpha(\mathcal{N}_1\vee\mathcal{N}_2) &= \frac{1}{\alpha-1}(1-\sum_{i=1}^{s}\sum_{j=1}^{t} m(\lambda_i\wedge\mu_j)^\alpha)\\ &= \frac{1}{\alpha-1}(1-\sum_{i=1}^{s}\sum_{j=1}^{t} m(\mu_j)^\alpha)\\ &= T_\alpha(\mathcal{N}_2).\end{aligned}$$

(iii) By Definition 2.5, $\mathcal{N}_1 \leq_m \mathcal{N}_2$ and $\mathcal{N}_2 \leq_m \mathcal{N}_1$. Since $\mathcal{N}_1 \leq_m \mathcal{N}_2$, we obtain $T_\alpha(\mathcal{N}_1|\mathcal{N}_2) = 0$ and $T_\alpha(\mathcal{N}_1 \vee \mathcal{N}_2) = T_\alpha(\mathcal{N}_2) + T_\alpha(\mathcal{N}_1|\mathcal{N}_2)$. Thus, $T_\alpha(\mathcal{N}_2 \vee \mathcal{N}_1) = T_\alpha(\mathcal{N}_2)$.

Similarly, from $\mathcal{N}_2 \leq_m \mathcal{N}_1$ we deduce that $T_\alpha(\mathcal{N}_1|\mathcal{N}_2) = 0$ and $T_\alpha(\mathcal{N}_1 \vee \mathcal{N}_2) = T_\alpha(\mathcal{N}_1) + T_\alpha(\mathcal{N}_2|\mathcal{N}_1)$. Thus, $T_\alpha(\mathcal{N}_1 \vee \mathcal{N}_2) = T_\alpha(\mathcal{N}_1)$ and $T_\alpha(\mathcal{N}_1) = T_\alpha(\mathcal{N}_2)$.

(iv) Since $\mathcal{N}_1 \sim \mathcal{N}_2$, by Theorem 2.4, $\mathcal{N}_1 \vee \mathcal{N}_3 \sim \mathcal{N}_2 \vee \mathcal{N}_3$. By Theorem 3.4 and (ii) we obtain

$$\begin{aligned} T_\alpha(\mathcal{N}_1|\mathcal{N}_3) &= T_\alpha(\mathcal{N}_1 \vee \mathcal{N}_3) - T_\alpha(\mathcal{N}_3) \\ &= T_\alpha(\mathcal{N}_2 \vee \mathcal{N}_3) - T_\alpha(\mathcal{N}_3) \\ &= T_\alpha(\mathcal{N}_2|\mathcal{N}_3). \end{aligned}$$

(v) Since $\mathcal{N}_2 \sim \mathcal{N}_3$, by Theorem 2.4, $\mathcal{N}_1 \vee \mathcal{N}_2 \sim \mathcal{N}_1 \vee \mathcal{N}_3$. By Theorem 3.4 and (ii),

$$\begin{aligned} T_\alpha(\mathcal{N}_1|\mathcal{N}_2) &= T_\alpha(\mathcal{N}_1 \vee \mathcal{N}_2) - T_\alpha(\mathcal{N}_2) \\ &= T_\alpha(\mathcal{N}_1 \vee \mathcal{N}_3) - T_\alpha(\mathcal{N}_3) \\ &= T_\alpha(\mathcal{N}_1|\mathcal{N}_3). \end{aligned}$$

□

In the following example, we discuss some results of Theorem 3.4.

Example 3.6. Let $X = [0,1]$, $\mathcal{N}_1 = \{\lambda_1, \lambda_2\}$ and $\mathcal{N}_2 = \{\mu_1, \mu_2\}$, where $\lambda_i : [0,1] \to [0,1]$ and $\mu_j : [0,1] \to [0,1]$ are defined by $\lambda_1 = x,\ \lambda_2 = 1 - x$ and $\mu_1 = x^2,\ \mu_2 = 1 - x^2$.

It is clear that $\mathcal{N}_1$ and $\mathcal{N}_2$ are fuzzy $\sigma-$algebras on X. Define $m_1 : \mathcal{N}_1 \to [0,1]$ by $m(\lambda_i) = \int_0^1 \lambda_i(x)dx$ and $m_2 : \mathcal{N}_2 \to [0,1]$ by $m(\mu_j) = \int_0^1 \mu_j(x)dx$.

Also, let $\vee$ and $\wedge$ denote the supremum and infimum, respectively. It is easy to see that m_1, m_2 are $F-$probability measure spaces on $\mathcal{N}_1$ and $\mathcal{N}_2$. So, $(X, \mathcal{N}_1, m_1)$ and $(X, \mathcal{N}_2, m_2)$ are $F-$probability measure spaces.

Since $m(\lambda_1) = m(\lambda_2) = \frac{1}{2}$, $m(\mu_1) = \frac{1}{3}$ and $m(\mu_2) = \frac{2}{3}$, by Definition 3.1,

$$T_2(\mathcal{N}_1) = 1 - (m(\lambda_1)^2 + m(\lambda_2)^2) = \frac{1}{2},$$

$$T_2(\mathcal{N}_2) = 1 - (m(\mu_1)^2 + m(\mu_2)^2) = \frac{4}{9},$$

$$T_3(\mathcal{N}_1) = \frac{1}{2}(1 - (m(\lambda_1)^3 + m(\lambda_2)^3)) = \frac{3}{8},$$

and

$$T_3(\mathcal{N}_2) = \frac{1}{2}(1 - (m(\mu_1)^3 + m(\mu_2)^3)) = \frac{1}{3}.$$

Thus, by using

$$T_\alpha(\mathcal{N}_1 \vee \mathcal{N}_2) = \frac{1}{\alpha - 1}\left(1 - \sum_{i=1}^{s}\sum_{j=1}^{t} m(\lambda_i \wedge \mu_j)^\alpha\right)$$

we obtain

$$\mathcal{N}_1 \vee \mathcal{N}_2 = \{\lambda_1 \wedge \mu_1, \lambda_1 \wedge \mu_2, \lambda_2 \wedge \mu_1, \lambda_2 \wedge \mu_2\}$$

and

$$m(\lambda_1 \wedge \mu_1) = m(\lambda_1 \wedge \mu_2) = \frac{1}{4}\ ,\ m(\lambda_2 \wedge \mu_1) = \frac{1}{12}\ ,\ m(\lambda_2 \wedge \mu_2) = \frac{5}{12}.$$

Therefore,

$$T_2(\mathcal{N}_1 \vee \mathcal{N}_2) = 1 - \sum_{i=1}^{2}\sum_{j=1}^{2} m(\lambda_i \wedge \mu_j)^2 = \frac{25}{36}$$

and

$$T_3(\mathcal{N}_1 \vee \mathcal{N}_2) = \frac{1}{2}(1 - \sum_{i=1}^{2}\sum_{j=1}^{2} m(\lambda_i \wedge \mu_j)^3) = \frac{43}{96}.$$

Thus, we conclude that $T_2(\mathcal{N}_1 \vee \mathcal{N}_2) \leq T_2(\mathcal{N}_1) + T_2(\mathcal{N}_2)$ and $T_3(\mathcal{N}_1 \vee \mathcal{N}_2) \leq T_3(\mathcal{N}_1) + T_3(\mathcal{N}_2)$, which are consistent with assertion (v) of Theorem 3.4.

By Definition 3.2 we obtain the conditional Tsallis entropies

$$T_2(\mathcal{N}_1|\mathcal{N}_2) = \sum_{j=1}^{2} m(\mu_j)^2 - \sum_{i=1}^{2}\sum_{j=1}^{2} m(\lambda_i \wedge \mu_j)^2 = \frac{1}{4}$$

and

$$T_2(\mathcal{N}_2|\mathcal{N}_1) = \sum_{i=1}^{2} m(\lambda_i)^2 - \sum_{i=1}^{2}\sum_{j=1}^{2} m(\lambda_i \wedge \mu_j)^2 = \frac{7}{36}.$$

Similarly, we get the conditional Tsallis entropies

$$T_3(\mathcal{N}_1|\mathcal{N}_2) = \frac{1}{2}\left(\sum_{j=1}^{2} m(\mu_j)^3 - \sum_{i=1}^{2}\sum_{j=1}^{2} m(\lambda_i \wedge \mu_j)^3\right) = \frac{11}{96}$$

and

$$T_3(\mathcal{N}_2|\mathcal{N}_1) = \frac{1}{2}\left(\sum_{i=1}^{2} m(\lambda_i)^3 - \sum_{i=1}^{2}\sum_{j=1}^{2} m(\lambda_i \wedge \mu_j)^3\right) = \frac{7}{96}.$$

Therefore, it can be seen that $T_2(\mathcal{N}_1|\mathcal{N}_2) \leq T_2(\mathcal{N}_1)$ and $T_2(\mathcal{N}_2|\mathcal{N}_1) \leq T_2(\mathcal{N}_2)$, and we conclude $T_3(\mathcal{N}_1|\mathcal{N}_2) \leq T_3(\mathcal{N}_1)$ and $T_3(\mathcal{N}_2|\mathcal{N}_1) \leq T_3(\mathcal{N}_2)$.

Commuting $\mathcal{N}_1$ and $\mathcal{N}_2$, and using elementary calculations, we find that parts (ii) and (iii) of Theorem 3.4 can be proved.

Definition 3.7. Let $(X, \mathcal{M}, s)$ be a fuzzy probability space. If $a, b \in \mathcal{M}$, then we define:

$$s(a/b) = \begin{cases} \frac{s(a \cap b)}{s(b)} & \text{if } s(b) > 0 \\ 0 & \text{if } s(b) = 0. \end{cases} \tag{3.3}$$

Let $s : \mathcal{M} \to [0, 1]$ be a fuzzy $\mathcal{P}$-measure, and let $b \in \mathcal{M}$ be such that $s(b) > 0$. Then, the map $s(./b) : \mathcal{M} \to [0, 1]$ defined by Equation 3.3 is a fuzzy P-measure. It plays the role of a conditional probability measure on the family $\mathcal{M}$ of fuzzy events. The following definition of a fuzzy partition was introduced in [21].

Definition 3.8. [21] A fuzzy partition of a fuzzy probability space $(X, \mathcal{M}, s)$ is a family $\alpha = \{a_1, a_2, ..., a_n\}$ of pairwise $\mathcal{W}$-separated fuzzy sets from $\mathcal{M}$ with the property $s(\cup_{i=1}^{n} a_i) = 1$.

In the class of all fuzzy partitions of a fuzzy probability space $(X, \mathcal{M}, s)$, we define the refinement partial order as follows. If $\alpha = \{a_1, a_2, \ldots, a_k\}$ and $\beta = \{b_1, b_2, \ldots, b_l\}$ are two fuzzy partitions of $(X, \mathcal{M}, s)$, then we say that β

is a refinement of α (and we write $\alpha \prec \beta$), if there exists a partition $\{I_1, I_2, \ldots, I_k\}$ of the set $\{1, 2, \ldots, l\}$ such that $a_i = \bigcup_{j \in l_i} b_j$, for $i = 1, 2, \ldots, k$. Further, we set $\alpha \vee \beta = \{a_i \cap b_j; i = 1, 2, \ldots, k, j = 1, 2, \ldots, l\}$. One can easily verify that the family $\alpha \vee \beta$ is a family of pairwise $\mathcal{W}$-separated fuzzy sets from $\mathcal{M}$; moreover,

$$s\left(\cup_{i=1}^{k} \cup_{j=1}^{l} (a_i \cap b_j)\right) = s\left((\cup_{i=1}^{k} a_i) \cap (\cup_{j=1}^{l} b_j)\right) = s\left(\cup_{i=1}^{k} a_i\right) = 1.$$

This means that $\alpha \vee \beta$ is a fuzzy partition of $(X, \mathcal{M}, s)$; it represents a combined experiment consisting of a realization of the experiments α and β. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are fuzzy paritions of $(X, \mathcal{M}, s)$, then we put $\bigvee_{i=1}^{n} \alpha_i = \alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n$.

In the following example, we study the Tsallis entropy of some fuzzy partitions.

Example 3.9. Consider the $\mathcal{P}$-partitions $\mathcal{P}_1 = \{[0, \frac{1}{3}), [\frac{1}{3}, \frac{2}{3}), [\frac{2}{3}, 1]\}$, $\mathcal{P}_2 = \{[0, \frac{1}{4}), [\frac{1}{4}, \frac{1}{2}), [\frac{1}{2}, \frac{4}{5}), [\frac{4}{5}, 1]\}$, $\mathcal{P}_3 = \{[0, \frac{1}{5})$ $, [\frac{1}{5}, \frac{1}{3}), [\frac{1}{3}, \frac{3}{4}), [\frac{3}{4}, 1]\}$, and $\mathcal{P}_4 = \{[0, \frac{1}{8}), [\frac{1}{8}, \frac{5}{8}), [\frac{5}{8}, \frac{3}{4}), [\frac{3}{4}, \frac{7}{8}), [\frac{1}{8}, 1]\}$. Then,

$$\mathcal{P}_1 \vee \mathcal{P}_2 = \left\{[0, \frac{1}{4}), [\frac{1}{4}, \frac{1}{3}), [\frac{1}{3}, \frac{1}{2}), [\frac{1}{2}, \frac{2}{3}), [\frac{2}{3}, \frac{4}{5}), [\frac{4}{5}, 1]\right\},$$

$$\mathcal{P}_2 \vee \mathcal{P}_3 = \left\{[0, \frac{1}{5}), [\frac{1}{5}, \frac{1}{4}), [\frac{1}{4}, \frac{1}{3}), [\frac{1}{3}, \frac{1}{2}), [\frac{1}{2}, \frac{3}{4}), [\frac{3}{4}, \frac{4}{5}), [\frac{4}{5}, 1]\right\}$$

and

$$\mathcal{P}_1 \vee \mathcal{P}_3 = \left\{[0, \frac{1}{5}), [\frac{1}{5}, \frac{1}{3}), [\frac{1}{3}, \frac{2}{3}), [\frac{2}{3}, \frac{3}{4}), [\frac{3}{4}, 1]\right\}$$

are common refinements of $\mathcal{P}_1$ and $\mathcal{P}_2$, $\mathcal{P}_2$ and $\mathcal{P}_3$, $\mathcal{P}_1$ and $\mathcal{P}_3$, respectively; common refinement of the (finite) combinations of $\mathcal{P}$-partitions may be obtained similarly. The Tsallis entropies of these $\mathcal{P}$-partitions are

$$T_2(\mathcal{P}_1) = 1 - \left((\frac{1}{3})^2 + (\frac{1}{3})^2 + (\frac{1}{3})^2\right) = \frac{2}{3},$$

$$T_2(\mathcal{P}_2) = 1 - \left((\frac{1}{4})^2 + (\frac{1}{4})^2 + (\frac{3}{10})^2 + (\frac{1}{5})^2\right) = \frac{149}{200},$$

$$T_2(\mathcal{P}_3) = 1 - \left((\frac{1}{5})^2 + (\frac{2}{15})^2 + (\frac{5}{12})^2 + (\frac{1}{4})^2\right) = \frac{1271}{1800},$$

$$T_3(\mathcal{P}_1) = \frac{1}{2}\left(1 - \left((\frac{1}{3})^3 + (\frac{1}{3})^3 + (\frac{1}{3})^3\right)\right) = \frac{4}{9},$$

$$T_3(\mathcal{P}_2) = \frac{1}{2}\left(1 - \left((\frac{1}{4})^3 + (\frac{1}{4})^3 + (\frac{3}{10})^3 + (\frac{1}{5})^3\right)\right) = \frac{747}{1600},$$

$$T_3(\mathcal{P}_3) = \frac{1}{2}\left(1 - \left((\frac{1}{5})^3 + (\frac{2}{15})^3 + (\frac{5}{12})^3 + (\frac{1}{4})^3\right)\right) = \frac{541}{1200}$$

Therefore,

$$T_2(\mathcal{P}_1 \vee \mathcal{P}_2) = \frac{1}{2-1}\left(1 - \left((\frac{1}{4})^2 + (\frac{1}{12})^2 + (\frac{1}{6})^2 + (\frac{1}{6})^2 + (\frac{2}{15})^2 + (\frac{1}{5})^2\right)\right) = \frac{1471}{1800},$$

$$T_2(\mathcal{P}_1 \vee \mathcal{P}_3) = 1 - \left((\frac{1}{5})^2 + (\frac{2}{15})^2 + (\frac{1}{3})^2 + (\frac{1}{12})^2 + (\frac{1}{4})^2\right) = \frac{457}{600},$$

$$T_2(\mathcal{P}_2 \vee \mathcal{P}_3) = 1 - \left((\frac{1}{5})^2 + (\frac{1}{20})^2 + (\frac{1}{12})^2 + (\frac{1}{6})^2 + (\frac{1}{4})^2 + (\frac{1}{20})^2 + (\frac{1}{5})^2\right) = \frac{184}{225}$$

and,

$$T_3(\mathcal{P}_1 \vee \mathcal{P}_2) = \frac{1}{2}\left(1 - \left((\frac{1}{4})^3 + (\frac{1}{12})^3 + (\frac{1}{6})^3 + (\frac{1}{6})^3 + (\frac{2}{15})^3 + (\frac{1}{5})^3\right)\right) = \frac{1157}{2400},$$

$$T_3(\mathcal{P}_1 \vee \mathcal{P}_3) = \frac{1}{2}\left(1 - \left((\frac{1}{5})^3 + (\frac{1}{15})^3 + (\frac{1}{3})^3 + (\frac{1}{12})^3 + (\frac{1}{4})^3\right)\right) = \frac{3371}{7200},$$

$$T_3(\mathcal{P}_2 \vee \mathcal{P}_3) = \frac{1}{2}\left(1 - \left((\frac{1}{5})^3 + (\frac{1}{20})^3 + (\frac{1}{12})^3 + (\frac{1}{6})^3 + (\frac{1}{4})^3 + (\frac{1}{20})^3) + (\frac{1}{5})^3\right)\right) = \frac{2311}{4800}$$

Now, we obtain the conditional tsallis entropies

$$\begin{aligned} T_2(\mathcal{P}_1|\mathcal{P}_2) &= \left((\frac{1}{4})^2 + (\frac{1}{4})^2 + (\frac{3}{10})^2 + (\frac{1}{5})^2\right) - \left((\frac{1}{4})^2 + (\frac{1}{12})^2 + (\frac{1}{6})^2 + (\frac{1}{6})^2 + (\frac{2}{15})^2 + (\frac{1}{5})^2\right) \\ &= \frac{13}{180}, \end{aligned}$$

$$\begin{aligned} T_2(\mathcal{P}_2|\mathcal{P}_1) &= \left((\frac{1}{3})^2 + (\frac{1}{3})^2 + (\frac{1}{3})^2\right) - \left((\frac{1}{4})^2 + (\frac{1}{12})^2 + (\frac{1}{6})^2 + (\frac{1}{6})^2 + (\frac{2}{15})^2 + (\frac{1}{5})^2\right) \\ &= \frac{271}{1800}, \end{aligned}$$

$$\begin{aligned} T_2(\mathcal{P}_1|\mathcal{P}_3) &= \left((\frac{1}{5})^2 + (\frac{2}{15})^2 + (\frac{5}{12})^2 + (\frac{1}{4})^2\right) - \left((\frac{1}{5})^2 + (\frac{2}{15})^2 + (\frac{1}{3})^2 + (\frac{1}{12})^2 + (\frac{1}{4})^2\right) \\ &= \frac{100}{1800}, \end{aligned}$$

$$\begin{aligned} T_2(\mathcal{P}_3|\mathcal{P}_1) &= \left((\frac{1}{3})^2 + (\frac{1}{3})^2 + (\frac{1}{3})^2\right) - \left((\frac{1}{5})^2 + (\frac{2}{15})^2 + (\frac{1}{3})^2 + (\frac{1}{12})^2 + (\frac{1}{4})^2\right) \\ &= \frac{219}{1800} \end{aligned}$$

$$\begin{aligned} T_2(\mathcal{P}_2|\mathcal{P}_3) &= \left((\frac{1}{5})^2 + (\frac{2}{15})^2 + (\frac{5}{12})^2 + (\frac{1}{4})^2\right) - \left((\frac{1}{5})^2 + (\frac{1}{20})^2 + (\frac{1}{12})^2 + (\frac{1}{6})^2 + (\frac{1}{4})^2 + (\frac{1}{20})^2 + (\frac{1}{5})^2\right) \\ &= \frac{201}{1800}, \end{aligned}$$

$$\begin{aligned} T_2(\mathcal{P}_3|\mathcal{P}_2) &= \left((\frac{1}{4})^2 + (\frac{1}{4})^2 + (\frac{3}{10})^2 + (\frac{1}{5})^2\right) - \left((\frac{1}{5})^2 + (\frac{1}{20})^2 + (\frac{1}{12})^2 + (\frac{1}{6})^2 + (\frac{1}{4})^2 + (\frac{1}{20})^2 + (\frac{1}{5})^2\right) \\ &= \frac{131}{1800}, \end{aligned}$$

$$\begin{aligned} T_3(\mathcal{P}_1|\mathcal{P}_2) &= \frac{1}{2}\left\{\left((\frac{1}{4})^3 + (\frac{1}{4})^3 + (\frac{3}{10})^3 + (\frac{1}{5})^3\right) - \left((\frac{1}{4})^3 + (\frac{1}{12})^3 + (\frac{1}{6})^3 + (\frac{1}{6})^3 + (\frac{2}{15})^3 + (\frac{1}{5})^3\right)\right\} \\ &= \frac{73}{4800}, \end{aligned}$$

$$\begin{aligned} T_3(\mathcal{P}_2|\mathcal{P}_1) &= \frac{1}{2}\left(\left((\frac{1}{3})^3 + (\frac{1}{3})^3 + (\frac{1}{3})^3\right) - \left((\frac{1}{4})^3 + (\frac{1}{12})^3 + (\frac{1}{6})^3 + (\frac{1}{6})^3 + (\frac{2}{15})^3 + (\frac{1}{5})^3\right)\right) \\ &= \frac{271}{7200}, \end{aligned}$$

$$
\begin{aligned}
T_3(\mathcal{P}_1|\mathcal{P}_3) &= \frac{1}{2}\left(\left((\frac{1}{5})^3+(\frac{2}{15})^3+(\frac{5}{12})^3+(\frac{1}{4})^3\right)-\left((\frac{1}{5})^3+(\frac{2}{15})^3+(\frac{1}{3})^3+(\frac{1}{12})^3+(\frac{1}{4})^3\right)\right)\\
&= \frac{5}{288},
\end{aligned}
$$

$$
\begin{aligned}
T_3(\mathcal{P}_3|\mathcal{P}_1) &= \frac{1}{2}\left(\left((\frac{1}{3})^3+(\frac{1}{3})^3+(\frac{1}{3})^3\right)-\left((\frac{1}{5})^3+(\frac{2}{15})^3+(\frac{1}{3})^3+(\frac{1}{12})^3+(\frac{1}{4})^3\right)\right)\\
&= \frac{109}{432}
\end{aligned}
$$

$$
\begin{aligned}
T_3(\mathcal{P}_2|\mathcal{P}_3) &= \frac{1}{2}\left\{\left((\frac{1}{5})^3+(\frac{2}{15})^3+(\frac{5}{12})^3+(\frac{1}{4})^3\right)-\left((\frac{1}{5})^3+(\frac{1}{20})^3+(\frac{1}{12})^3+(\frac{1}{6})^3+(\frac{1}{4})^3+(\frac{1}{20})^3+(\frac{1}{5})^3\right)\right\}\\
&= \frac{49}{1600},
\end{aligned}
$$

$$
\begin{aligned}
T_3(\mathcal{P}_3|\mathcal{P}_2) &= \frac{1}{2}\left\{\left((\frac{1}{4})^3+(\frac{1}{4})^3+(\frac{3}{10})^3+(\frac{1}{5})^3\right)-\left((\frac{1}{5})^3+(\frac{1}{20})^3+(\frac{1}{12})^3+(\frac{1}{6})^3+(\frac{1}{4})^3+(\frac{1}{20})^3+(\frac{1}{5})^3\right)\right\}\\
&= \frac{7}{240}.
\end{aligned}
$$

Conclusion

Fuzzy sets provide a mathematical models for random experiments whose outcomes are unclear, inaccurately defined events. M. Khare [12] and, A. Ebrahimzadeh and J. Jamalzadeh [6] introduced and studied the Shannon entropy and the logical entropy of fuzzy σ-algebras on $F-$probability measure spaces. The Tsallis entropy plays a significant role in the non-extensive statistical mechanics of complex systems. Some applications of the Tsallis entropy have been found in a wide range of phenomena in diverse disciplines such as chemistry, physics, geophysics, biology, economics,a medicine [1, 2, 4, 3, 7, 9, 10, 19, 22, 24, 29, 32, 34]. Also, the entropy has been applied to large domains in communication systems. In this paper, we introduced and investigate the Tsallis entropy of order α of fuzzy σ-algebras on $F-$probability measure spaces, where $\alpha \in (0,1) \cup (1,\infty)$. Also, we introduced the conditional version of this entropy and examine its properties. In future studies, we aim to investigated Renyi entropy on fuzzy σ-algebras. Also, we will define and examine Renyi and Tsallis entropies of functions on fuzzy dynamical systems as a applicable system.

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