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Generalized weighted composition operators acting between Dirichlet-type spaces and Bloch-type spaces

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Abstract

Let $\mathbb{D} = \{\upsilon \in \mathbb{C} : |\upsilon| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and let $H(\mathbb{D})$ be the space of all holomorphic functions on $\mathbb{D}$. For a non-negative integer n and a function $f \in H(\mathbb{D})$, the $n^{th}-$ order differentiation operator is defined as $D^n f = f^{(n)}$. The weighted composition operator together with $n^{th}-$ order differentiation operator give rise to a new operator generally termed as generalized weighted composition operator denoted by $\mathcal{W}^n_{\phi,\xi}$ and is defined by

$$\mathcal{W}^n_{\phi,\xi} f(\upsilon) = \phi(\upsilon) f^{(n)}(\xi(\upsilon)), \quad f \in H(\mathbb{D}); \upsilon \in \mathbb{D},$$

where $\phi \in H(\mathbb{D})$ and ξ is a holomorphic self-map of $\mathbb{D}$. This operator is basically the combination of multiplication operator M_ϕ, composition operator C_ξ and $n^{th}-$ order differentiation operator D^n. We study the boundedness and compactness of this operator between Dirichlet-type spaces and Bloch-type spaces.

Keywords: Dirichlet-type space, Bloch-type spaces, generalized weighted composition operator, boundedness, compactness
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1 Introduction

Let $\mathbb{D} = \{\upsilon \in \mathbb{C} : |\upsilon| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. We denote the class of all analytic functions on $\mathbb{D}$ by $H(\mathbb{D})$ and the space of all analytic self-maps of $\mathbb{D}$ by $S(\mathbb{D})$. For each $\alpha > 0$, the space $\mathfrak{B}_\alpha$ termed as the *weighted Bloch space* consists of all those analytic functions f in $\mathbb{D}$ such that

$$\sup_{\upsilon \in \mathbb{D}} (1 - |\upsilon|^2)^\alpha |f'(\upsilon)| < \infty.$$

This space forms a Banach space with the natural norm defined by

$$\|f\|_{\mathfrak{B}_\alpha} = |f(0)| + \sup_{\upsilon \in \mathbb{D}} (1 - |\upsilon|^2)^\alpha |f'(\upsilon)|.$$

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For $\alpha = 1$, we obtain classical Bloch space $\mathfrak{B}$. A continuous function $\omega : \mathbb{D} \to (0, \infty)$ is termed as a *weight*. Weight ω is called to be a *standard weight*, if for $\upsilon \in \mathbb{D}$, we have $\lim_{|\upsilon| \to 1^-} \omega(\upsilon) = 0$. Further, for $\upsilon \in \mathbb{D}$, we call a weight ω to be *radial*, if $\omega(\upsilon) = \omega(|\upsilon|)$. The *Bloch-type spaces* $\mathfrak{B}_\omega$ for a weight function ω, is the class of all analytic function f in $\mathbb{D}$ such that $\sup_{\upsilon \in \mathbb{D}} \omega(\upsilon)|f'(\upsilon)| < \infty$. In the similar manner, the *little Bloch-type space* $\mathfrak{B}_{\omega,0}$ is the closure of the set of polynomials in $\mathfrak{B}_\omega$ and contains all those $f \in \mathfrak{B}_\omega$ such that $\lim_{|\upsilon| \to 1} \omega(\upsilon)|f'(\upsilon)| = 0$. The spaces $\mathfrak{B}_{\omega,0}$ and $\mathfrak{B}_\omega$ are Banach spaces under the following norm

$$\|f\|_{\mathfrak{B}_\omega} = |f(0)| + \sup_{\upsilon \in \mathbb{D}} \omega(\upsilon)|f'(\upsilon)|.$$

To know more about these spaces see [5, 6, 7, 8, 12, 14, 17, 20] and the references therein. Let $\xi \in S(\mathbb{D})$ and $\phi \in H(\mathbb{D})$. Then, the *composition, multiplication,* and *weighted composition operator* are respectively defined as

$$C_\xi f(\upsilon) = (f \circ \xi)(\upsilon) = f(\xi(\upsilon)), \;\; M_\phi f(\upsilon) = \phi(\upsilon) f(\upsilon) \text{ and } \mathcal{W}_{\phi,\xi} f(\upsilon) = (M_\phi C_\xi) f(\upsilon) = \phi(\upsilon) f(\xi(\upsilon)), \quad \upsilon \in \mathbb{D}; f \in H(\mathbb{D}).$$

Several classical operators are composition operators. For example, the space $l^p(\mathbb{N})$ of all functions from $\mathbb{N}$ to $\mathbb{C}$ whose p^{th} power is integrable with respect to counting measure. Let $x \in l^p$ as the function $x(k) = x_k$. If $\phi : \mathbb{N} \to \mathbb{N}$ is defined as $\phi(k) = k + 1$, then $(C_\phi x)k = x(\phi(k)) = x(k+1) = x_{k+1}$, that is, $C_\phi : (x_1, x_2, x_3, \dots) \longmapsto (x_2, x_3, x_4, \dots)$, so C_ϕ is the backward shift operator. In fact, backward shift operators of all multiplicities can be represented as composition operators. These natural type of operators have been appearing in explicit or implicit form in different areas of mathematical sciences such as classical mechanics, the ergodic theory, dynamical systems, Markov process, the theory of semigroups, isometries and homomorphism. The translations induced composition operators are very important for the study of different types of motions. Moreover, composition operators are deeply involved in the study of algebra homomorphisms on function algebras, further highlighting their versatility and importance in mathematical analysis. $\mathcal{W}_{\phi,\xi}$ is a product-type operator as $\mathcal{W}_{\phi,\xi} = M_\phi C_\xi$. More results on weighted composition operators on class of holomorphic functions can be found in [5, 7, 8, 9] and the references therein. Further, for $f \in H(\mathbb{D})$, the differentiation operator denoted by D is defined as $Df = f'$. The operators $\mathcal{W}_{\phi,\xi} D$ and $DW_{\phi,\xi}$ were respectively, considered in [11] and [12]. For $n \in \mathbb{N}_0$ and $f \in H(\mathbb{D})$, the $n^{th}-$ order differentiation operator is denoted and defined by $D^n f = f^{(n)}$. For $n = 0$, we obtain $D^0 f = f$. The weighted composition operator together with $n^{th}-$ order differentiation operator give rise to a new operator generally termed as generalized weighted composition operator denoted by $\mathcal{W}^n_{\phi,\xi}$ and is defined by

$$\mathcal{W}^n_{\phi,\xi} f(\upsilon) = \phi(\upsilon) f^{(n)}(\xi(\upsilon)), \quad f \in H(\mathbb{D}); \upsilon \in \mathbb{D},$$

where $\phi \in H(\mathbb{D})$ and ξ is a holomorphic self-map of $\mathbb{D}$. If $n = 0$, the operator $\mathcal{W}^n_{\phi,\xi}$ becomes the weighted composition operator $W_{\phi,\xi}$, which further for $\phi(\upsilon) \equiv 1$, get reduced to the composition operator C_ξ. If $n = 1$ and $\phi(\upsilon) = \xi'(\upsilon)$, then $\mathcal{W}^n_{\phi,\xi} = DC_\xi$. When $n = 1$, $\phi(\upsilon) \equiv 1$, then $\mathcal{W}^n_{\phi,\xi} = C_\xi D$. If $n = 1$, $\xi(\upsilon) = \upsilon$, then $\mathcal{W}^n_{\phi,\xi} = M_\phi D$, i.e. the product of differentiation operator and multiplication operator M_ϕ. For more about these operators one may refer [15, 16, 18, 22, 27] and the references therein.

The *Dirichlet space* is the class of all those analytic functions in $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f'|^2 dA(\upsilon) < \infty,$$

where $dA(\upsilon)$ is the normalized Lebesgue area measure defined on $\mathbb{D}$. The space forms a Hilbert space under the following norm

$$\|f\|^2_{\mathcal{D}} = |f(0)|^2 + \int_{\mathbb{D}} |f'|^2 dA(\upsilon).$$

Let $K : [0, \infty) \to [0, \infty)$ be a function with the property that it is right continuous and increasing. These functions have been considered in various papers, some of which are [4, 24, 25]. By treating function K as a weight, we can obtain the space $\mathcal{D}_K$ termed as the *Dirichlet-type space* which consists of all those analytic functions in $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f'|^2 K\big(1 - |\upsilon|^2\big) dA(\upsilon) < \infty.$$

Further, we can check that the space $\mathcal{D}_K$ forms a Banach space under the norm $\|\cdot\|_{\mathcal{D}_K}$ given as follows

$$\|f\|^2_{\mathcal{D}_K} = |f(0)|^2 + \int_{\mathbb{D}} |f'|^2 K\big(1 - |\upsilon|^2\big) dA(\upsilon).$$

For $K(t) = t^p$, where $0 \leq p < \infty$, the space $\mathcal{D}_K$ gives the usual Dirichlet-type space $\mathcal{D}_p$. Further, by taking $p = 0$, we obtain the classical Dirichlet space $\mathcal{D}$ and for $p = 1$, we gain the Hardy space H^2. The study of these spaces is an important topic in complex analysis and is closely related to several other function spaces. These spaces finds applications in various areas, such as harmonic analysis, interpolation theory, geometric function theory, and operator theory. Understanding the properties and behavior of functions in these spaces is of significant interest in complex analysis and is an active area of research due to their connections with various mathematical disciplines. These spaces have been studied widely in various papers. For details one can see [1, 2, 3, 4, 6, 13, 19, 21, 23] and the references therein. In [7], we characterized the boundedness as well as compactness of weighted composition operator acting from $\mathcal{D}_K$ to Bloch and Bers-type spaces, and compute their essential norm in [8].

Motivated by the aforementioned works and the numerous applications of these spaces in various areas, such as harmonic analysis, geometric function theory, operator theory, etc., we have considered the operator $\mathcal{W}^n_{\phi,\xi}$ acting between $\mathcal{D}_K$ and Bloch-type spaces and we study the boundedness and compactness of this operator between Dirichlet-type space and Bloch-type spaces. The Bloch space is the largest space of analytic functions on $\mathbb{D}$ that is Möbius invariant, making it a welcoming environment to study composition and weighted composition operators. These techniques can be applied to study a larger class of spaces, allowing for a more comprehensive understanding of the relationships between different function spaces and the operators acting on them.

Throughout this paper, let weighted function K satisfies:

$$\int_0^1 \frac{\varphi_K(s)}{s} ds \quad < \infty \tag{1.1}$$

and

$$\int_1^\infty \frac{\varphi_K(s)}{s^2} ds \quad < \infty, \tag{1.2}$$

where

$$\varphi_K(s) = \sup_{0 \leq t \leq 1} \frac{K(st)}{K(t)}, \quad 0 < s < \infty.$$

From [10], one can see that if K satisfies (1.1), then

$$K_1(t) = \int_0^t K(s) \frac{ds}{s} \approx K(t), \ \ 0 < t < 1.$$

If K satisfies (1.2), then

$$K_2(t) = t \int_t^\infty K(s) \frac{ds}{s^2} \approx K(t), \ \ t > 0.$$

From condition (1.2), we get that $K(2t) \approx K(t)$ for $0 < t < 1$. Also there exist $C > 0$ sufficiently small for which $t^{-C} K_1(t)$ is increasing and $K_2(t) t^{C-1}$ is decreasing.
This paper is formulated in a systematic way. Introduction and literature part is kept in Section 1 and some auxiliary results which are used to derive the main results are considered in Section 2. In Section 3, we characterize the boundedness and compactness of operator $\mathcal{W}^n_{\phi,\xi} : \mathcal{D}_K \to \mathfrak{B}_\omega$. Throughout the text, we use certain notations. Let a and b be two positive quantities. Then, for some positive constant C, notations $a \gtrsim b$ implies that $a \geq Cb$. The value of C may vary from place to place. Further, if both $a \gtrsim b$ and $b \gtrsim a$ hold, then we simply write $a \approx b$.

2 Auxiliary Results

To arrive at the main results we use following lemmas. The first lemma can be easily obtained from the arguments in [4, 26, 28].

Lemma 2.1. Let K be a weight function. Then for any $\varsigma, \upsilon \in \mathbb{D}$ and $\rho > 0$, we have

$$f_\varsigma(\upsilon) = \frac{(1 - |\varsigma|^2)^{\rho/2}}{\sqrt{K(1 - |\varsigma|^2)}(1 - \bar{\varsigma}\upsilon)^{1+\rho/2}}$$

is in $\mathcal{D}_K$. Moreover,

$$\sup_{\varsigma\in\mathbb{D}} \|f_\varsigma\|_{\mathcal{D}_K} \lesssim 1,$$

and f_ς converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|\varsigma| \to 1^-$.

Lemma 2.2. Let K be a weight function. Then for every $f \in \mathcal{D}_K$ we have

$$|f(\upsilon)| \lesssim \frac{\|f\|_{\mathcal{D}_K}}{\sqrt{K(1-|\upsilon|^2)(1-|\upsilon|^2)}}, \quad \upsilon \in \mathbb{D}.$$

Proof . We know the fact that

$$|1-\overline{\upsilon}\varsigma| \approx 1-|\upsilon|^2 \approx 1-|\varsigma|^2, \varsigma \in D(\upsilon,r)$$

and

$$K(1-|\upsilon|^2) \approx K(1-|\varsigma|^2), \varsigma \in D(\upsilon,r),$$

where $D(\upsilon,r) = \{\varsigma : |\varphi_\upsilon(\varsigma) < r|\}$ .Using the sub-mean value property of $|f'^2|$, we can deduce that

$$\begin{aligned}
|f'^2| &\lesssim \frac{1}{(1-|\upsilon|^2)^2}\int_{D(\upsilon,r)} |f'^2| dA(\varsigma)\\
&\approx \frac{1}{K(1-|\upsilon|^2)(1-|\upsilon|^2)^2}\int_{D(\upsilon,r)} |f'^2| K(1-|\varsigma|^2) dA(\varsigma)\\
&\leq \frac{1}{K(1-|\upsilon|^2)(1-|\upsilon|^2)^2}\int_{\mathbb{D}} |f'^2| K(1-|\varsigma|^2) dA(\varsigma).
\end{aligned}$$

Thus,

$$|f'(\upsilon)| \lesssim \|f\|_{\mathcal{D}_K} \frac{1}{\sqrt{K(1-|\upsilon|^2)(1-|\upsilon|^2)^2}}.$$

Since

$$\begin{aligned}
|f(\upsilon)-f(0)| &= \left|\upsilon \int_0^1 f(\upsilon s) ds\right|\\
&\leq |\upsilon| \int_0^1 |f(\upsilon s)| ds,
\end{aligned}$$

we can easily get,

$$\begin{aligned}
|f(\upsilon)-f(0)| &\lesssim \int_0^1 |f(\upsilon s)| d(|\upsilon| s)|\\
&\lesssim \|f\|_{\mathcal{D}_K} \int_0^1 \frac{1}{\sqrt{K(1-|\upsilon| s)(1-|\upsilon| s)^2}} d(|\upsilon| s)\\
&\lesssim \|f\|_{\mathcal{D}_K} \int_0^{|\upsilon|} \frac{1}{\sqrt{K(1-t)(1-t)^2}} dt\\
&= \|f\|_{\mathcal{D}_K} \frac{1}{\sqrt{K(1-|\upsilon|)}} \int_0^{|\upsilon|} \sqrt{\frac{K(1-|\upsilon|)}{K(1-t)(1-t)^2}} dt.
\end{aligned}$$

Noted that K satisfies (1.2), thus there exists a small $C > 0$ such that $\varphi_K(t) \lesssim t^{1-C}$, $t \geq 1$. Hence, we obtain,

$$\begin{aligned}
|f(\upsilon)-f(0)| &\lesssim \|f\|_{\mathcal{D}_K} \frac{1}{\sqrt{K(1-|\upsilon|)}} \int_0^{|\upsilon|} \sqrt{\frac{K(1-|\upsilon|)}{K(1-t)(1-t)^2}} dt\\
&\lesssim \|f\|_{\mathcal{D}_K} \frac{1}{\sqrt{K(1-|\upsilon|)}} \int_0^{|\upsilon|} \sqrt{\left(\frac{1-|\upsilon|}{1-t}\right)^{1-C} \frac{1}{(1-t)^2}} dt\\
&\lesssim \|f\|_{\mathcal{D}_K} \frac{1}{\sqrt{K(1-|\upsilon|^2)(1-|\upsilon|^2)}}.
\end{aligned}$$

That is,

$$\begin{aligned}|f(\upsilon)| &\lesssim |f(0)| + \|f\|_{\mathcal{D}_K} \frac{1}{\sqrt{K(1-|\upsilon|^2)}(1-|\upsilon|^2)}\\ &\lesssim \|f\|_{\mathcal{D}_K} \frac{1}{\sqrt{K(1-|\upsilon|^2)}(1-|\upsilon|^2)}.\end{aligned}$$

This completes the proof. □

Lemma 2.3. Let K be a weight function and n be a positive integer. Then for every $f \in \mathcal{D}_K$ we have

$$|f^{(n)}(\upsilon)| \lesssim \frac{\|f\|_{\mathcal{D}_K}}{\sqrt{K(1-|\upsilon|^2)}(1-|\upsilon|^2)^{n+1}}, \quad \upsilon \in \mathbb{D}.$$

The following criterion characterize the compactness. Its proof can be easily follows from Proposition 3.11 in [6].

Lemma 2.4. Let ω be the standard weight and the operator $\mathcal{W}^n_{\phi,\xi} : \mathcal{D}_K \to \mathfrak{B}_\omega$ is bounded. Then, $\mathcal{W}^n_{\phi,\xi} : \mathcal{D}_K \to \mathfrak{B}_\omega$ is compact if and only if for every bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{D}_K$ where $f_n \to 0$ uniformly on every compact subsets of $\mathbb{D}$ as $n \to \infty$, we have $\lim_{n\to\infty} \|\mathcal{W}^n_{\phi,\xi} f_n\|_{\mathfrak{B}_\omega} = 0$.

3 Boundedness and Compactness of $\mathcal{W}^n_{\phi,\xi}$ between Dirichlet-type space and Bloch-type spaces

Theorem 3.1. Let ω and K be two weight functions, $\phi \in H(\mathbb{D})$ and $\xi \in S(\mathbb{D})$. Then $\mathcal{W}^n_{\phi,\xi}$ from $\mathcal{D}_K$ to $\mathfrak{B}_\omega$ is bounded if and only if the functions ξ and ϕ satisfy the following conditions :

(i) $L_1 = \sup_{\upsilon\in\mathbb{D}} \frac{\omega(\upsilon)|\phi'(\upsilon)|}{\sqrt{K(1-|\xi(\upsilon)|^2)}(1-|\xi(\upsilon)|^2)^{n+1}} < \infty$,

(ii) $L_2 = \sup_{\upsilon\in\mathbb{D}} \frac{\omega(\upsilon)|\phi(\upsilon)\xi'(\upsilon)|}{\sqrt{K(1-|\xi(\upsilon)|^2)}(1-|\xi(\upsilon)|^2)^{n+2}} < \infty$.

Moreover, if $\mathcal{W}^n_{\phi,\xi}$ from $\mathcal{D}_K$ to $\mathfrak{B}_\omega$ is non-zero and bounded, then

$$L_1 + L_2 \lesssim \|\mathcal{W}^n_{\phi,\xi}\|_{\mathcal{D}_K\to\mathfrak{B}_\omega} \lesssim L + L_1 + L_2,$$

where

$$L = \frac{|\phi(0)|}{\sqrt{K(1-|\xi(0)|^2)}(1-|\xi(0)|^2)^{n+1}}. \tag{3.1}$$

Proof . First assume that conditions (i) and (ii) hold. Since $\left(\mathcal{W}^n_{\phi,\xi} f\right)(\upsilon) = \phi(\upsilon)f^{(n)}(\xi(\upsilon))$. This implies that

$$\left(\mathcal{W}^n_{\phi,\xi} f\right)'(\upsilon) = \phi'^{(n)}(\xi(\upsilon)) + \phi(\upsilon)\xi'^{(n+1)}(\xi(\upsilon)) \quad \text{and} \quad \left(\mathcal{W}^n_{\phi,\xi} f\right)(0) = \phi(0)f^{(n)}(\xi(0)).$$

Thus, for all $f \in \mathcal{D}_K$, we get

$$\begin{aligned}\|\mathcal{W}^n_{\phi,\xi} f\|_{\mathcal{D}_K\to\mathfrak{B}_\omega} &= |(\mathcal{W}^n_{\phi,\xi} f)(0)| + \sup_{\upsilon\in\mathbb{D}} \omega(\upsilon)|(\mathcal{W}^n_{\phi,\xi} f)'(\upsilon)|\\ &\le |\phi(0)||f^{(n)}(\xi(0))| + \sup_{\upsilon\in\mathbb{D}} \omega(\upsilon)|\phi'(\upsilon)||f^{(n)}(\xi(\upsilon))| + \sup_{\upsilon\in\mathbb{D}} \omega(\upsilon)|\phi(\upsilon)\xi'^{(n+1)}(\xi(\upsilon))|\\ &\lesssim \Bigg(\frac{|\phi(0)|}{\sqrt{K(1-|\xi(0)|^2)}(1-|\xi(0)|^2)^{n+1}} + \sup_{\upsilon\in\mathbb{D}} \frac{\omega(\upsilon)|\phi'(\upsilon)|}{\sqrt{K(1-|\xi(\upsilon)|^2)}(1-|\xi(\upsilon)|^2)^{n+1}}\\ &\quad + \sup_{\upsilon\in\mathbb{D}} \frac{\omega(\upsilon)|\phi(\upsilon)\xi'(\upsilon)|}{\sqrt{K(1-|\xi(\upsilon)|^2)}(1-|\xi(\upsilon)|^2)^{n+2}}\Bigg)\|f\|_{\mathcal{D}_K}\\ &\lesssim \big(L + L_1 + L_2\big)\|f\|_{\mathcal{D}_K}.\end{aligned} \tag{3.2}$$

From (3.2), it follows that the operator $\mathcal{W}^n_{\phi,\xi}:\mathcal{D}_K\to\mathfrak{B}_\omega$ is bounded and

$$\|\mathcal{W}^n_{\phi,\xi}\|_{\mathcal{D}_K\to\mathfrak{B}_\omega}\lesssim L+L_1+L_2. \tag{3.3}$$

Conversely, suppose that the operator $\mathcal{W}^n_{\phi,\xi}:\mathcal{D}_K\to\mathfrak{B}_\omega$ is bounded. At first, we will prove that $L_1<\infty$. For this take the function $p_n(\upsilon)=\frac{\upsilon^n}{n!}$. Since the operator $\mathcal{W}^n_{\phi,\xi}:\mathcal{D}_K\to\mathfrak{B}_\omega$ is bounded, we get

$$\sup_{\upsilon\in\mathbb{D}}\omega(\upsilon)|\phi'(\upsilon)|\le\left\|(\mathcal{W}^n_{\phi,\xi}p_n)(\upsilon)\right\|_{\mathcal{D}_K\to\mathfrak{B}_\omega}\lesssim\|\mathcal{W}^n_{\phi,\xi}\|_{\mathcal{D}_K\to\mathfrak{B}_\omega}. \tag{3.4}$$

For $\varsigma\in\mathbb{D}$, consider a function

$$f_\varsigma(\upsilon)=a_1\frac{(1-|\xi(\varsigma)|^2)^{\frac{\rho}{2}}}{\sqrt{K(1-|\xi(\varsigma)|^2)}(1-\upsilon\overline{\xi(\varsigma)})^{\frac{\rho}{2}+1}}+b_1\frac{(1-|\xi(\varsigma)|^2)^{\frac{\rho}{2}+1}}{\sqrt{K(1-|\xi(\varsigma)|^2)}(1-\upsilon\overline{\xi(\varsigma)})^{\frac{\rho}{2}+2}},$$

where $a_1=\left(\frac{\rho}{2}+n+2\right)$ and $b_1=-\left(\frac{\rho}{2}+1\right)$. Using Lemma 2.1, it can be easily seen that $f_\varsigma\in\mathcal{D}_K$ and $\|f_\varsigma\|_{\mathcal{D}_K}\lesssim 1$. Further, we can check that

$$f_\varsigma^{(n)}(\upsilon)=a_1\prod_{r=1}^{n}\left(\frac{\rho}{2}+r\right)\frac{(1-|\xi(\varsigma)|^2)^{\frac{\rho}{2}}(\overline{\xi(\varsigma)})^n}{\sqrt{K(1-|\xi(\varsigma)|^2)}(1-\upsilon\overline{\xi(\varsigma)})^{\frac{\rho}{2}+n+1}}+b_1\prod_{r=1}^{n}\left(\frac{\rho}{2}+r+1\right)\frac{(1-|\xi(\varsigma)|^2)^{\frac{\rho}{2}+1}(\overline{\xi(\varsigma)})^n}{\sqrt{K(1-|\xi(\varsigma)|^2)}(1-\upsilon\overline{\xi(\varsigma)})^{\frac{\rho}{2}+n+2}}$$

and

$$f_\varsigma^{(n+1)}(\upsilon)=a_1\prod_{r=1}^{n+1}\left(\frac{\rho}{2}+r\right)\frac{(1-|\xi(\varsigma)|^2)^{\frac{\rho}{2}}(\overline{\xi(\varsigma)})^{n+1}}{\sqrt{K(1-|\xi(\varsigma)|^2)}(1-\upsilon\overline{\xi(\varsigma)})^{\frac{\rho}{2}+n+2}}+b_1\prod_{r=1}^{n+1}\left(\frac{\rho}{2}+r+1\right)\frac{(1-|\xi(\varsigma)|^2)^{\frac{\rho}{2}+1}(\overline{\xi(\varsigma)})^{n+1}}{\sqrt{K(1-|\xi(\varsigma)|^2)}(1-\upsilon\overline{\xi(\varsigma)})^{\frac{\rho}{2}+n+3}}.$$

Hence,

$$f_\varsigma^{(n+1)}(\xi(\varsigma))=0 \tag{3.5}$$

and

$$f_\varsigma^{(n)}(\xi(\varsigma))=\prod_{r=1}^{n}\left(\frac{\rho}{2}+r\right)\frac{(\overline{\xi(\varsigma)})^n}{\sqrt{K(1-|\xi(\varsigma)|^2)}(1-|\xi(\varsigma)|^2)^{n+1}}. \tag{3.6}$$

Since the operator $\mathcal{W}^n_{\phi,\xi}:\mathcal{D}_K\to\mathfrak{B}_\omega$ is bounded, thus we get

$$\begin{aligned}\|\mathcal{W}^n_{\phi,\xi}\|_{\mathcal{D}_K\to\mathfrak{B}_\omega}&\gtrsim\|\mathcal{W}^n_{\phi,\xi}f_\varsigma\|_{\mathfrak{B}_\omega}\\&\gtrsim\omega(\varsigma)\left|\phi_\varsigma'^{(n)}(\xi(\varsigma))+\phi(\varsigma)\xi_\varsigma'^{(n+1)}(\xi(\varsigma))\right|\\&=\prod_{r=1}^{n+1}\left(\frac{\rho}{2}+r\right)\frac{\omega(\varsigma)|\phi'^{n}|}{\sqrt{K(1-|\xi(\varsigma)|^2)}(1-|\xi(\varsigma)|^2)^{n+1}}.\end{aligned} \tag{3.7}$$

For fixed $\eta\in(0,1)$, inequalities (3.4) and (3.7) implies that

$$\begin{aligned}\sup_{\varsigma\in\mathbb{D}}\frac{\omega(\varsigma)|\phi'(\varsigma)|}{\sqrt{K(1-|\xi(\varsigma)|^2)}(1-|\xi(\varsigma)|^2)^{n+1}}&\le\sup_{|\xi(\varsigma)|\le\eta}\frac{\omega(\varsigma)|\phi'(\varsigma)|}{\sqrt{K(1-|\xi(\varsigma)|^2)}(1-|\xi(\varsigma)|^2)^{n+1}}+\sup_{|\xi(\varsigma)|>\eta}\frac{\omega(\varsigma)|\phi'(\varsigma)|}{\sqrt{K(1-|\xi(\varsigma)|^2)}(1-|\xi(\varsigma)|^2)^{n+1}}\\&\le\frac{1}{(1-\eta^2)^{n+1}}\sup_{|\xi(\varsigma)|\le\eta}\frac{\omega(\varsigma)|\phi'(\varsigma)|}{\sqrt{K(1-\eta^2)}}+\frac{1}{\eta^n}\sup_{|\xi(\varsigma)|>\eta}\frac{\omega(\varsigma)|\phi'^{n}|}{\sqrt{K(1-|\xi(\varsigma)|^2)}(1-|\xi(\varsigma)|^2)^{n+1}}\\&\lesssim\left(\frac{1}{(1-\eta^2)^{n+1}}+\frac{1}{\eta^n}\right)\|\mathcal{W}^n_{\phi,\xi}\|_{\mathcal{D}_K\to\mathfrak{B}_\omega}.\end{aligned}$$

This implies that (i) holds and

$$L_1\lesssim\|\mathcal{W}^n_{\phi,\xi}\|_{\mathcal{D}_K\to\mathfrak{B}_\omega}. \tag{3.8}$$

Next, we will show that $L_2 < \infty$. Taking $p_{n+1}(\upsilon) = \frac{\upsilon^{n+1}}{(n+1)!}$, we have

$$\sup_{\upsilon\in\mathbb{D}} \omega(\upsilon)|\phi'(\upsilon)\xi(\upsilon) + \phi(\upsilon)\xi'(\upsilon)| \leq \left\|(\mathcal{W}^n_{\phi,\xi} p_{n+1})(\upsilon)\right\|_{\mathcal{D}_K\to\mathfrak{B}_\omega} \lesssim \|\mathcal{W}^n_{\phi,\xi}\|_{\mathcal{D}_K\to\mathfrak{B}_\omega}. \tag{3.9}$$

Using (3.4) with the fact that $|\xi(\upsilon)| < 1$, from (3.9) we get

$$\sup_{\upsilon\in\mathbb{D}} \omega(\upsilon)|\phi(\upsilon)\xi'(\upsilon)| \lesssim \|\mathcal{W}^n_{\phi,\xi}\|_{\mathcal{D}_K\to\mathfrak{B}_\omega}. \tag{3.10}$$

For $\varsigma \in \mathbb{D}$, consider a function

$$g_\varsigma(\upsilon) = a_2 \frac{(1-|\xi(\varsigma)|^2)^{\frac{\rho}{2}}}{\sqrt{K(1-|\xi(\varsigma)|^2)}(1-\upsilon\overline{\xi(\varsigma)})^{\frac{\rho}{2}+1}} + b_2 \frac{(1-|\xi(\varsigma)|^2)^{\frac{\rho}{2}+1}}{\sqrt{K(1-|\xi(\varsigma)|^2)}(1-\upsilon\overline{\xi(\varsigma)})^{\frac{\rho}{2}+2}},$$

where $a_2 = \left(\frac{\rho}{2}+n+1\right)$ and $b_2 = -\left(\frac{\rho}{2}+1\right)$. Using Lemma 2.1, it can be easily seen that $g_\varsigma \in \mathcal{D}_K$ and $\|g_\varsigma\|_{\mathcal{D}_K} \lesssim 1$. Further, we can check that

$$g^{(n)}_\varsigma(\upsilon) = a_2 \prod_{r=1}^{n}\left(\frac{\rho}{2}+r\right) \frac{(1-|\xi(\varsigma)|^2)^{\frac{\rho}{2}}(\overline{\xi(\varsigma)})^n}{\sqrt{K(1-|\xi(\varsigma)|^2)}(1-\upsilon\overline{\xi(\varsigma)})^{\frac{\rho}{2}+n+1}} + b_2 \prod_{r=1}^{n}\left(\frac{\rho}{2}+r+1\right) \frac{(1-|\xi(\varsigma)|^2)^{\frac{\rho}{2}+1}(\overline{\xi(\varsigma)})^n}{\sqrt{K(1-|\xi(\varsigma)|^2)}(1-\upsilon\overline{\xi(\varsigma)})^{\frac{\rho}{2}+n+2}}$$

and

$$g^{(n+1)}_\varsigma(\upsilon) = a_2 \prod_{r=1}^{n+1}\left(\frac{\rho}{2}+r\right) \frac{(1-|\xi(\varsigma)|^2)^{\frac{\rho}{2}}(\overline{\xi(\varsigma)})^{n+1}}{\sqrt{K(1-|\xi(\varsigma)|^2)}(1-\upsilon\overline{\xi(\varsigma)})^{\frac{\rho}{2}+n+2}} + b_2 \prod_{r=1}^{n+1}\left(\frac{\rho}{2}+r+1\right) \frac{(1-|\xi(\varsigma)|^2)^{\frac{\rho}{2}+1}(\overline{\xi(\varsigma)})^{n+1}}{\sqrt{K(1-|\xi(\varsigma)|^2)}(1-\upsilon\overline{\xi(\varsigma)})^{\frac{\rho}{2}+n+3}}.$$

Hence,

$$g^{(n)}_\varsigma(\xi(\varsigma)) = 0 \quad \text{and} \tag{3.11}$$

$$g^{(n+1)}_\varsigma(\xi(\varsigma)) = -\prod_{r=1}^{n+1}\left(\frac{\rho}{2}+r\right) \frac{(\overline{\xi(\varsigma)})^{n+1}}{\sqrt{K(1-|\xi(\varsigma)|^2)}(1-|\xi(\varsigma)|^2)^{n+2}}. \tag{3.12}$$

Since the operator $\mathcal{W}^n_{\phi,\xi} : \mathcal{D}_K \to \mathfrak{B}_\omega$ is bounded, thus we get

$$\begin{aligned}
\|\mathcal{W}^n_{\phi,\xi}\|_{\mathcal{D}_K\to\mathfrak{B}_\omega} &\gtrsim \|\mathcal{W}^n_{\phi,\xi} g_\varsigma\|_{\mathfrak{B}_\omega} \\
&\gtrsim \omega(\varsigma)\left|\phi'^{(n)}_\varsigma(\xi(\varsigma)) + \phi(\varsigma)\xi'^{(n+1)}_\varsigma(\xi(\varsigma))\right| \\
&= \prod_{r=1}^{n+1}\left(\frac{\rho}{2}+r\right) \frac{\omega(\varsigma)|\phi(\varsigma)\xi'^{(n+1)}|}{\sqrt{K(1-|\xi(\varsigma)|^2)}(1-|\xi(\varsigma)|^2)^{n+2}}.
\end{aligned} \tag{3.13}$$

For fixed $\eta \in (0,1)$, inequalities (3.10) and (3.13) implies that

$$\begin{aligned}
\sup_{\varsigma\in\mathbb{D}} \frac{\omega(\varsigma)|\phi(\varsigma)\xi'(\varsigma)|}{\sqrt{K(1-|\xi(\varsigma)|^2)}(1-|\xi(\varsigma)|^2)^{n+2}} &\leq \sup_{|\xi(\varsigma)|\leq\eta} \frac{\omega(\varsigma)|\phi(\varsigma)\xi'(\varsigma)|}{\sqrt{K(1-|\xi(\varsigma)|^2)}(1-|\xi(\varsigma)|^2)^{n+2}} + \sup_{|\xi(\varsigma)|>\eta} \frac{\omega(\varsigma)|\phi(\varsigma)\xi'(\varsigma)|}{\sqrt{K(1-|\xi(\varsigma)|^2)}(1-|\xi(\varsigma)|^2)^{n+2}} \\
&\leq \frac{1}{(1-\eta^2)^{n+2}} \sup_{|\xi(\varsigma)|\leq\eta} \frac{\omega(\varsigma)|\phi(\varsigma)\xi'(\varsigma)|}{\sqrt{K(1-\eta^2)}} + \frac{1}{\eta^{n+1}} \sup_{|\xi(\varsigma)|>\eta} \frac{\omega(\varsigma)|\phi(\varsigma)\xi'^{n+1}|}{\sqrt{K(1-|\xi(\varsigma)|^2)}(1-|\xi(\varsigma)|^2)^{n+2}} \\
&\lesssim \left(\frac{1}{(1-\eta^2)^{n+2}} + \frac{1}{\eta^{n+1}}\right)\|\mathcal{W}^n_{\phi,\xi}\|_{\mathcal{D}_K\to\mathfrak{B}_\omega}.
\end{aligned}$$

This implies that (ii) holds and

$$L_2 \lesssim \|\mathcal{W}^n_{\phi,\xi}\|_{\mathcal{D}_K\to\mathfrak{B}_\omega}. \tag{3.14}$$

Combining inequalities (3.8) and (3.14), we get that

$$L_1 + L_2 \lesssim \|\mathcal{W}^n_{\phi,\xi}\|_{\mathcal{D}_K\to\mathfrak{B}_\omega}. \tag{3.15}$$

Thus, from (3.3) and (3.15), it follows that

$$L_1 + L_2 \lesssim \|\mathcal{W}^n_{\phi,\xi}\|_{\mathcal{D}_K \to \mathfrak{B}_\omega} \lesssim L + L_1 + L_2. \tag{3.16}$$

Hence the theorem. □

Theorem 3.2. Let ω and K be two weight functions, $\phi \in H(\mathbb{D})$ and $\xi \in S(\mathbb{D})$. Then, the following statements are equivalent:

(i) The operator $\mathcal{W}^n_{\phi,\xi} : \mathcal{D}_K \to \mathfrak{B}_\omega$ is compact.

(ii) Functions ϕ and ξ are such that

$$l_1 = \sup_{\upsilon \in \mathbb{D}} \omega(\upsilon)|\phi'(\upsilon)| < \infty, \qquad l_2 = \sup_{\upsilon \in \mathbb{D}} \omega(\upsilon)|\phi(\upsilon)\xi'(\upsilon)| < \infty,$$

$$\lim_{|\xi(\upsilon)| \to 1} \frac{\omega(\upsilon)|\phi'(\upsilon)|}{\sqrt{K(1-|\xi(\upsilon)|^2)}(1-|\xi(\upsilon)|^2)^{n+1}} = 0, \quad \lim_{|\xi(\upsilon)| \to 1} \frac{\omega(\upsilon)|\phi(\upsilon)\xi'(\upsilon)|}{\sqrt{K(1-|\xi(\upsilon)|^2)}(1-|\xi(\upsilon)|^2)^{n+2}} = 0.$$

Proof . First, suppose that the condition (i) holds, that is, operator $\mathcal{W}^n_{\phi,\xi} : \mathcal{D}_K \to \mathfrak{B}_\omega$ is compact. This implies that $\mathcal{W}^n_{\phi,\xi} : \mathcal{D}_K \to \mathfrak{B}_\omega$ is bounded. Thus, from Theorem 3.1, we obtain that l_1, and $l_2 < \infty$. Consider a sequence $(u_m)_{m\in\mathbb{N}} \in \mathbb{D}$ such that $|\xi(u_m)| \to 1$ as $m \to \infty$. Conditions (ii) holds obviously if such a sequence does not exists. By making use of $(u_m)_{m\in\mathbb{N}}$, define

$$f_m(\upsilon) = a_1 \frac{(1-|\xi(u_m)|^2)^{\frac{\varrho}{2}}}{\sqrt{K(1-|\xi(u_m)|^2)}(1-\upsilon\overline{\xi(u_m)})^{\frac{\varrho}{2}+1}} + b_1 \frac{(1-|\xi(u_m)|^2)^{\frac{\varrho}{2}+1}}{\sqrt{K(1-|\xi(u_m)|^2)}(1-\upsilon\overline{\xi(u_m)})^{\frac{\varrho}{2}+2}}$$

and

$$g_m(\upsilon) = a_2 \frac{(1-|\xi(u_m)|^2)^{\frac{\varrho}{2}}}{\sqrt{K(1-|\xi(u_m)|^2)}(1-\upsilon\overline{\xi(u_m)})^{\frac{\varrho}{2}+1}} + b_2 \frac{(1-|\xi(u_m)|^2)^{\frac{\varrho}{2}+1}}{\sqrt{K(1-|\xi(u_m)|^2)}(1-\upsilon\overline{\xi(u_m)})^{\frac{\varrho}{2}+2}},$$

where a_1, b_1, a_2, b_2 are defined in Theorem 3.1. From Theorem 3.1, it can be seen that the sequences (f_m), and (g_m) are norm bounded in $\mathcal{D}_K$ and on compact subsets of $\mathbb{D}$ uniformly converge to zero as $m \to \infty$. Thus, by Lemma 2.4, we get

$$\lim_{m\to\infty} \|\mathcal{W}^n_{\phi,\xi} f_m\|_{\mathfrak{B}_\omega} = 0 \quad \text{and} \quad \lim_{m\to\infty} \|\mathcal{W}^n_{\phi,\xi} g_m\|_{\mathfrak{B}_\omega} = 0. \tag{3.17}$$

Thus, by inequalities (3.5), (3.6) and (3.17), we have

$$\lim_{m\to\infty} \frac{\omega(u_m)|\phi'(u_m)|}{\sqrt{K(1-|\xi(u_m)|^2)}(1-|\xi(u_m)|^2)^{n+1}} = 0. \tag{3.18}$$

Further, from (3.11), (3.12) and (3.17), it follows that

$$\lim_{m\to\infty} \frac{\omega(u_m)|\phi(u_m)\xi'(u_m)|}{\sqrt{K(1-|\xi(u_m)|^2)}(1-|\xi(u_m)|^2)^{n+2}} = 0. \tag{3.19}$$

Hence, from (3.18), (3.19), and the boundedness of $\mathcal{W}^n_{\phi,\xi}$, we get the desired results.

Conversely, suppose that condition (ii) holds. To prove the compactness of $\mathcal{W}^n_{\phi,\xi}$ we first show that $\mathcal{W}^n_{\phi,\xi}$ is bounded. Using condition (ii), we see that for every $\varepsilon > 0$, there is an $\eta \in (0,1)$ such that

$$M_1(\upsilon) = \frac{\omega(\upsilon)|\phi'(\upsilon)|}{\sqrt{K(1-|\xi(\upsilon)|^2)}(1-|\xi(\upsilon)|^2)^{n+1}} < \varepsilon \tag{3.20}$$

and

$$M_2(\upsilon) = \frac{\omega(\upsilon)|\phi(\upsilon)\xi'(\upsilon)|}{\sqrt{K(1-|\xi(\upsilon)|^2)}(1-|\xi(\upsilon)|^2)^{n+2}} < \varepsilon, \tag{3.21}$$

for any $\upsilon \in A = \{\upsilon \in \mathbb{D} : |\xi(\upsilon)| > \eta\}$.Now, by (3.20) and condition $l_1 < \infty$, we get

$$L_1 = \sup_{\upsilon\in\mathbb{D}} M_1(\upsilon) \leq \sup_{\upsilon\in\mathbb{D}\backslash A} M_1(\upsilon) + \sup_{\upsilon\in A} M_1(\upsilon) \leq \frac{l_1}{\sqrt{K(1-\eta^2)}(1-\eta^2)^{n+1}} + \varepsilon.$$

This implies that $L_1 < \infty$. Again by (3.21) and $l_2 < \infty$, we get

$$L_2 = \sup_{\upsilon\in\mathbb{D}} M_2(\upsilon) \leq \sup_{\upsilon\in\mathbb{D}\backslash A} M_2(\upsilon) + \sup_{\upsilon\in A} M_2(\upsilon) \leq \frac{l_2}{\sqrt{K(1-\eta^2)}(1-\eta^2)^{n+2}} + \varepsilon.$$

which implies that $L_2 < \infty$. Thus, we obtain that $L_1 < \infty$ and $L_2 < \infty$. Therefore, by Theorem 3.1, we have that the operator $\mathcal{W}^n_{\phi,\xi} : \mathcal{D}_K \to \mathfrak{B}_\omega$ is bounded. Now, we prove that $\mathcal{W}^n_{\phi,\xi} : \mathcal{D}_K \to \mathfrak{B}_\omega$ is compact. Consider a sequence $(f_m)_{m\in\mathbb{N}} \in \mathcal{D}_K$ such that for $m \to \infty$, $f_m \to 0$ uniformly on compact subsets of $\mathbb{D}$ and $\|f_m\|_{\mathcal{D}_K} \lesssim 1$. Then, $f^{(n)}_m$ and $f^{(n+1)}_m$ uniformly converges to zero on compact subsets of $\mathbb{D}$ as $m \to \infty$. By using Lemma 2.2, Lemma 2.3, (3.1) and condition (ii), for every $\varepsilon > 0$, and η, we have

$$\begin{aligned}
\|\mathcal{W}^n_{\phi,\xi} f_m\|_{\mathcal{D}_K\to\mathfrak{B}_\omega} &= |(\mathcal{W}^n_{\phi,\xi} f_m)(0)| + \sup_{\upsilon\in\mathbb{D}} \omega(\upsilon)|(\mathcal{W}^n_{\phi,\xi} f_m)'(\upsilon)| \\
&\lesssim L + \sup_{\upsilon\in\mathbb{D}} \omega(\upsilon)|\phi'(\upsilon)||f^{(n)}_m(\xi(\upsilon))| + \sup_{\upsilon\in\mathbb{D}} \omega(\upsilon)|\phi(\upsilon)\xi'(\upsilon)||f^{(n+1)}_m(\xi(\upsilon))| \\
&\leq L + \sup_{\upsilon\in\mathbb{D}\backslash A} \omega(\upsilon)|\phi'(\upsilon)||f^{(n)}_m(\xi(\upsilon))| + \sup_{\upsilon\in A} \omega(\upsilon)|\phi'(\upsilon)||f^{(n)}_m(\xi(\upsilon))| \\
&\quad + \sup_{\upsilon\in\mathbb{D}\backslash A} \omega(\upsilon)|\phi(\upsilon)\xi'(\upsilon)||f^{(n+1)}_m(\xi(\upsilon))| + \sup_{\upsilon\in A} \omega(\upsilon)|\phi(\upsilon)\xi'(\upsilon)||f^{(n+1)}_m(\xi(\upsilon))| \\
&\leq L + A_m + C\sup_{\upsilon\in A} M_1(\upsilon) + C\sup_{\upsilon\in A} M_2(\upsilon) \\
&\lesssim L + A_m + 2\varepsilon, \qquad (3.22)
\end{aligned}$$

where $A_m = l_1 \sup_{\{\upsilon:|\upsilon|\leq\eta\}} |f^{(n)}_m(\upsilon)| + l_2 \sup_{\{\upsilon:|\upsilon|\leq\eta\}} |f^{(n+1)}_m(\upsilon)|$. We know that if $(f_m)_{m\in\mathbb{N}}$ converges to zero uniformly on any compact subset of $\mathbb{D}$ then $(f^{(n)}_m)_{m\in\mathbb{N}}$ and $(f^{(n+1)}_m)_{m\in\mathbb{N}}$ do the same as $m \to \infty$. Thus $A_m \to 0$ as $m \to \infty$. Also, $\{\xi(0)\}$ and $\{\upsilon : |\upsilon| \leq \eta\}$ are compact subsets of $\mathbb{D}$, so by taking $m \to \infty$ in (3.22), we obtain

$$\lim_{m\to\infty} \|\mathcal{W}^n_{\phi,\xi} f_m\|_{\mathcal{D}_K\to\mathfrak{B}_\omega} = 0.$$

Hence, the operator $\mathcal{W}^n_{\phi,\xi} : \mathcal{D}_K \to \mathfrak{B}_\omega$ is compact. □

References

[1] A. Aleman, *Hilbert spaces of analytic functions between the Hardy space and the Dirichlet space*, Proc. Amer. Math. Soc. **115** (1992), 97–104.

[2] H.A. Alsaker, *Multipliers of the Dirichlet space*, Master's thesis, Department of Mathematics, The University of Bergen, 2009.

[3] N. Arcozzi, R. Rochberg, E.T. Sawyer, and B.D. Wick, *The Dirichlet space: a survey*, New York J. Math. **17A** (2011), 45–86.

[4] G. Bao, Z. Lou, R. Qian, and H. Wulan, *On multipliers of Dirichlet type spaces*, Complex Anal. Oper. Theory **9** (2015), 1701–1732.

[5] F. Colonna and S. Li, *Weighted composition operators from the minimal Möbius invariant space into the Bloch space*, Mediter. J. Math. **10** (2013), 395–409.

[6] C.C. Cowen and B.D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, FL, 1995.

[7] M. Devi, A.K. Sharma, and K. Raj, *Weighted composition operators from Dirichlet type spaces to some weighted-type spaces*, J. Comput. Anal. Appl. **28** (2020), 127–135.

[8] M. Devi, A.K. Sharma and K. Raj, *Inequalities involving essential norm estimates of product-type operators*, J. Math. **2021** (2021), Article ID 8811309, 9 pages.

[9] K. Esmaeili and M. Lindström, *Weighted composition operators between Zygmund type spaces and their essential norms*, Integral Equ. Oper. Theory **75** (2013), 473–490.

[10] M. Essen, H. Wulan, and J. Xiao, *Several function-theoretic characterizations of Möbius invariant $\mathcal{Q}_K$ spaces*, J. Funct. Anal. **230** (2006), 78–115.

[11] Z. J. Jiang, *On a class of operators from weighted Bergman spaces to some spaces of analytic functions*, Taiwan. J. Math. **15** (2011), 2095–2121.

[12] Z. J. Jiang, *On a product-type operator from weighted Bergman-Orlicz space to some weighted-type spaces*, Appl. Math. Comput. **256** (2015), 37–51.

[13] R. Kerman and E. Sawyer, *Carleson measures and multipliers of Dirichlet-type spaces*, Trans. Amer. Math. Soc. **309** (1988), 87–98.

[14] H. Li and X. Fu, *A new characterization of generalized weighted composition operators from the Bloch space into the Zygmund space*, J. Funct. Spaces Appl. **2013** (2013), Article ID 925901, 6 pages.

[15] Y. Liu and Y. Yu, *Weighted differentiation composition operators from mixed-norm to Zygmund spaces*, Numer. Funct. Anal. Optim. **31(8)** (2010), 936–954.

[16] Y. Liu and Y. Yu, *Composition followed by differentiation between H^∞ and Zygmund spaces*, Complex. Anal. Oper. Theory, **6** (2012), no. 1, 121–137.

[17] S. Ohno, K. Stroethoff, and R. Zhao, *Weighted composition operators between Bloch-type spaces*, Rocky Mt. J. Math. **33** (2003), 191–215.

[18] H. Qu, Y. Liu, and S. Cheng, *Weighted differentiation composition operator from logarithmic Bloch spaces to Zygmund-type spaces*, Abstr. Appl. Anal. **2014** (2014), Article ID 832713, 14 pages.

[19] R. Rochberg and Z. Wu, *A new characterization of Dirichlet type spaces and applications*, Illinois J. Math. **37** (1993), 101–122.

[20] A.H. Sanatpour and M. Hassanlou, *Essential norms of weighted differentiation composition operators between Zygmund-type spaces and Bloch type spaces*, Filomat **31** (2017), 2877–2889.

[21] D. Stegenga, *Multipliers of the Dirichlet space*, Illinois J. Math. **24** (1980), 113–139.

[22] S. Stević, *Weighted differentiation composition operators from H^∞ and Bloch spaces to $n^{th}-$ weighted-type spaces on the unit disk*, Appl. Math. Comput. **216** (2010), 3634–3641.

[23] G. Taylor, *Multipliers on $\mathcal{D}_\alpha$*, Trans. Amer. Math. Soc. **123** (1966), 229–240.

[24] H. Wulan and J. Zhou, *$\mathcal{Q}_K$ and Morrey type spaces*, Ann. Acad. Sci. Fenn. Math. **38** (2013), 193–207.

[25] H. Wulan and K. Zhu, *Lacunary Series in $\mathcal{Q}_K$ Spaces*, Stud. Math. **173** (2007), 217–230.

[26] L. Yang, *Integral operator acting on weighted Dirichlet spaces to Morrey type spaces*, Filomat **33** (2019), 3723–3736.

[27] Y. Yu and Y. Liu, *Weighted differentiation composition operators from H^∞ to Zygmund spaces*, Integral Transforms Spec. Funct. **22** (2011), 507–520.

[28] J.Z. Zhou and Y.T. Wu, *Decomposition theorems and conjugate pair in D_K spaces*, Acta Math. Sinica English Ser. **30** (2014), 1513–1525.