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# Some fixed point theorems satisfying generalized contraction conditions in dislocated quasi metric space

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**Abstract**

The goal of this research article is to study some fixed point results in dislocated quasi metric spaces. These results are proved for certain generalized contraction conditions involving linear and rational expressions. Our results extend and generalize some of the well-known fixed point results of the literature available in fixed point theory. Suitable examples of the established results are also given.

Keywords: Fixed point,; contraction mapping, complete metric space, dislocated quasi metric space, Cauchy sequence
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## 1 Introduction

Fixed point theory is one of the most progressive and fascinating research areas in nonlinear analysis. Several researchers have put forward many results related to fixed point theory in different spaces. Fixed point theory deals with existence and uniqueness of fixed point. The first crucial result was given by Banach in 1922 for a contraction mapping in a complete metric space known as Banach contraction principle. It has numerous applications in different branches of mathematics such as differential and integral equation, numerical analysis etc. Dass and Gupta[3] gave a generalization of Banach contraction principle in a metric space for some rational type contraction conditions.

Hitzler and Seda [4] introduced the notion of dislocated metric space, a generalization of metrics where the distance of a point from itself need not be zero. This concept was not new as it was studied in the context of domain theory [9] under the name of metric domains. Zeyada et al. [13] presented the notion of dislocated quasi metric space and generalized the results of Hitzler and Seda [4] in this space. These metrics play an important role in topology and in other branches of science particularly in logic programming and electronic engineering. Recently, different types of contraction mappings and generalization of Banach contraction principle in these spaces is studied by Isufati [6], Aage and Salunke [1], Kohli et al. [8], Zoto et al. [14] and many more authors.

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## 2 Preliminaries

The objective of this section is to introduce some basic definitions with examples and results which will further be needed to understand next section.

**Definition 2.1 ([13]).** Let $X$ be a nonempty set and let $\tau : X \times X \to [0, \infty)$ be a function satisfying the following conditions:

($\tau_1$) $\tau(x, x) = 0$;
($\tau_2$) $\tau(x, y) = \tau(y, x) = 0$, implies that $x = y$;
($\tau_3$) $\tau(x, y) = \tau(y, x)$ for all $x, y \in X$;
($\tau_4$) $\tau(x, y) \leq \tau(x, z) + \tau(z, y)$ for all $x, y, z \in X$.

If $\tau$ satisfies the conditions from ($\tau_1$) to ($\tau_4$),then $\tau$ is called a metric on $X$. If it satisfies the conditions ($\tau_1$), ($\tau_2$) and ($\tau_4$), it is called a quasi-metric on $X$. If $\tau$ satisfies conditions ($\tau_2$), ($\tau_3$), ($\tau_4$), it is called a dislocated metric (or simply $\tau$ -metric) on $X$ and if $\tau$ satisfies only ($\tau_2$) and ($\tau_4$) then it is called a dislocated quasi-metric (or simply $\tau q$-metric) on $X$.

Nonempty set $X$ together with $\tau q$-metric $\tau$,i.e. $(X, \tau)$ is called a dislocated quasi-metric space. By definition every metric on $X$ is a dislocated metric on $X$, however the converse is not necessarily true as illustrated in following example.

**Example 2.2 ([11]).** Let $X = [0, \infty)$ and define the distance function $\tau : X \times X \to [0, \infty)$ by

$$\tau(x, y) = \max\{x, y\}, \forall x, y \in X$$

.

**Example 2.3 ([11]).** Let $X = [0, 1]$, we define the distance function $\tau : X \times X \to [0, \infty)$ by

$$\tau(x, y) = |x|, \forall x, y \in X$$

.

From example 2.3, we note that a dislocated quasi metric on $X$ need not be dislocated metric on $X$.

**Definition 2.4 ([13]).** Let $\{x_n\}$ be a sequence in dislocated quasi metric space $(X, \tau)$ then

1. $\{x_n\}$ is known as $\tau q$-convergent to $x \in X$ if $\lim_{n\to\infty} \tau(x_n, x) = \lim_{n\to\infty} \tau(x, x_n) = 0$.
   In this case $x$ is called $\tau q$-limit of $\{x_n\}$ and we write $x_n \to x$.
2. $\{x_n\}$ is known as Cauchy sequence in $(X, \tau)$ if for given $\epsilon > 0$, $\exists n_0 \in N$ such that

   $$\forall m, n \geq n_0,\ \tau(x_m, x_n) < \epsilon \text{ or } \tau(x_n, x_m) < \epsilon$$

   that is, $\lim_{m,n\to\infty} \tau(x_m, x_n) = \lim_{m,n\to\infty} \tau(x_n, x_m) = 0$.
   In above definition, if we replace $\tau(x_m, x_n) < \epsilon$ or $\tau(x_n, x_m) < \epsilon$ by $max\{\tau(x_m, x_n), \tau(x_n, x_m)\} < \epsilon$, the sequence $\{x_n\}$ in $\tau q$-metric space $(X, \tau)$ is called 'bi' Cauchy.
3. A $\tau q$-metric space $(X, \tau)$ is called complete if every Cauchy sequence is a $\tau q$- convergent sequence.

**Proposition 2.5 ([13]).** Every convergent sequence in a $\tau q$- metric space is 'bi' Cauchy.

**Lemma 2.6 ([13]).** Every subsequence of $\tau q$- convergent sequence to a point $a_0$ is $\tau q$- convergent to $a_0$.

**Lemma 2.7 ([13]).** $\tau q$- limits in $\tau q$- metric spaces are unique.

**Definition 2.8 ([13]).** Let $(X, \tau)$ be a $\tau q$-metric space. A function $f : X \to X$ is called a contraction if there exists $0 \leq \alpha < 1$ such that

$$\tau(f(x), f(y)) \leq \alpha\tau(x, y), \forall x, y \in X$$

.

**Lemma 2.9.** Let $\{a_n\}$ be a sequence in a $\tau q$-metric space $(X,\tau)$ such that

$$\tau(a_n, a_{n+1}) \leq h\tau(a_{n-1}, a_n),$$

where, $0 \leq h < 1$ and $n = 0, 1, 2, 3, \ldots$. Then $\{a_n\}$ is a Cauchy sequence in $(X,\tau)$.

**Proof .**Let $n > m \geq 1$, we have

$$\begin{aligned}
\tau(a_m, a_n) &\leq \tau(a_m, a_{m+1}) + \tau(a_{m+1}, a_{m+2}) + \cdots\cdots + \tau(a_{n-1}, a_n)\\
&\leq (h^m + h^{m+1} + \cdots + h^{n-1})\tau(a_0, a_1)\\
&= h^m(1 + h + h^2 + \cdots + h^{n-m-1})\tau(a_0, a_1)\\
&\leq \frac{h^m}{1-h}\tau(a_0, a_1).
\end{aligned}$$

Since $0 \leq h < 1, h^m \to 0$ as $m \to \infty$. Therefore, $\{a_n\}$ is a Cauchy sequence in $(X,\tau)$.

**Definition 2.10 ([2]).** Let $(X, d)$ be a metric space, a self mapping $T : X \to X$ is called a generalized contraction if and only if for every $x, y \in X$, there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that $sup\{\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 : x, y \in X\} < 1$ and

$$d(Tx, Ty) \leq \alpha_1 d(x, y) + \alpha_2 d(x, Tx) + \alpha_3 d(y, Ty) + \alpha_4[d(x, Ty) + d(y, Tx)]. \tag{2.1}$$

Ćirić [2] proved a unique fixed point theorem for a self mapping which satisfies condition (2.1) in context of metric spaces.

## 3 Main Results

In this section we establish some fixed point theorems for single and pair of continuous self mappings under different contraction conditions using linear and rational expression in dislocated quasi metric spaces. Our results extend and generalize some well-known existing results in the literature in $\tau q$-metric space. Our results generalize that of Lj.B.Ciric [2] .

**Theorem 3.1.** Let $(X,\tau)$ be a complete $\tau q$-metric space and $T : X \to X$ be a continuous function satisfying the following condition:

$$\tau(Tx, Ty) \leq a_1\tau(x, y) + a_2[\tau(x, Tx) + \tau(y, Ty)] + a_3[\tau(x, Ty) + \tau(y, Tx)] + a_4\left[\frac{\tau(x, y)\tau(x, Ty)}{\tau(x, y) + \tau(y, Ty)}\right] + a_5\left[\frac{\tau(x, Ty)\tau(y, Ty)}{\tau(x, y) + \tau(y, Ty)}\right] \tag{3.1}$$

where $a_1, a_2, a_3, a_4, a_5 \geq 0$ with $a_1 + 2a_2 + 4a_3 + a_4 + a_5 < 1$ for all $x, y \in X$, then $T$ has a unique fixed point.

**Proof .** Define a sequence $\{x_n\}$ in $X$ using Picard's iteration as follows:

Let $x_0$ be any arbitrary element in $X, x_1 = T(x_0), x_2 = T(x_1), \cdots, x_{n+1} = T(x_n), \cdots$. Assume that $x_{n+1} \neq x_n$ for any $n$, because if for any n, $x_{n+1} = x_n$, then $x_n$ is a fixed point and there is no need to go further. From (3.1), we have

$$\begin{aligned}
\tau(x_{n+1}, x_{n+2}) =&\tau(Tx_n, Tx_{n+1})\\
\leq& a_1\tau(x_n, x_{n+1}) + a_2[\tau(x_n, Tx_n) + \tau(x_{n+1}, Tx_{n+1})] + a_3[\tau(x_n, Tx_{n+1}) + \tau(x_{n+1}, Tx_n)]\\
&+ a_4\left[\frac{\tau(x_n, x_{n+1})\tau(x_n, Tx_{n+1})}{\tau(x_n, x_{n+1}) + \tau(x_{n+1}, Tx_{n+1})}\right] + a_5\left[\frac{\tau(x_n, Tx_{n+1})\tau(x_{n+1}, Tx_{n+1})}{\tau(x_n, x_{n+1}) + \tau(x_{n+1}, Tx_{n+1})}\right]\\
\leq& a_1\tau(x_n, x_{n+1}) + a_2\tau(x_n, x_{n+1}) + a_2\tau(x_{n+1}, x_{n+2}) + a_3[\tau(x_n, x_{n+1}) + \tau(x_{n+1}, x_{n+2})\\
&+ \tau(x_{n+1}, x_{n+2}) + \tau(x_n, x_{n+1})] + a_4\tau(x_n, x_{n+1}) + a_5\tau(x_{n+1}, x_{n+2})\\
\leq& \frac{[a_1 + a_2 + 2a_3 + a_4]}{[1 - (a_2 + 2a_3 + a_5)]}\tau(x_n, x_{n+1}.
\end{aligned}$$

Let $k = \frac{[a_1+a_2+2a_3+a_4]}{[1-(a_2+2a_3+a_5)]}$. Observe that $0 \leq k < 1$ since $a_1 + 2a_2 + 4a_3 + a_4 + a_5 < 1$. Therefore, $\tau(x_{n+1}, x_{n+2}) \leq k\tau(x_n, x_{n+1})$. Similarly, $\tau(x_n, x_{n+1}) \leq k\tau(x_{n-1}, x_n)$, so we get $\tau(x_{n+1}, x_{n+2}) \leq k^2\tau(x_{n-1}, x_n)$. Proceeding like this, we get $\tau(x_{n+1}, x_{n+2}) \leq k^{n+1}\tau(x_0, x_1)$. Since $0 \leq k < 1 \Rightarrow k^{n+1} \to 0$ as $n \to \infty$. By Lemma 2.9, $\{x_n\}$ is a Cauchy sequence in complete $\tau q$-metric space $X$. So there is a point $p \in X$ such that $p$ is the $\tau q$-limit of $\{x_n\}$, that is $x_n \to p$. Since $T$ is continuous, $\lim_{n\to\infty} Tx_n = Tp$ implies that $\lim_{n\to\infty} x_{n+1} = Tp$. Thus, $Tp = p$. Hence $p$ is a fixed point of $T$.

**Uniqueness:** If $p \in X$ is a fixed point of $T$ then by condition (3.1), we have

$$\begin{aligned}\tau(p,p) =&\tau(Tp,Tp)\\ \leq& a_1\tau(p,p) + a_2[\tau(p,p)+\tau(p,p)] + a_3[\tau(p,p)+\tau(p,p)] + a_4\left[\frac{\tau(p,p)\tau(p,p)}{\tau(p,p)+\tau(p,p)}\right] + a_5\left[\frac{\tau(p,p)\tau(p,p)}{\tau(p,p)+\tau(p,p)}\right]\\ \leq&(a_1+2a_2+2a_3+\frac{a_4}{2}+\frac{a_5}{2})\tau(p,p).\end{aligned}$$

Since $0 \leq (a_1+2a_2+2a_3+\frac{a_4}{2}+\frac{a_5}{2}) < 1$ and $\tau(p,p) \geq 0$, we have, $\tau(p,p) = 0$. Thus, $\tau(p,p) = 0$, if $p$ is a fixed point of $T$. Suppose that $p$ and $q$ are two fixed points of $T(p \neq q)$, that is, $p = Tp$ and $q = Tq$. We have,

$$\begin{aligned}\tau(p,q) =&\tau(Tp,Tq)\\ \leq& a_1\tau(p,q) + a_2[\tau(p,Tp)+\tau(q,Tq)] + a_3[\tau(p,Tq)+\tau(q,Tp)] + a_4\left[\frac{\tau(p,q)\tau(p,Tq)}{\tau(p,q)+\tau(q,Tq)}\right] + a_5\left[\frac{\tau(p,Tq)\tau(q,Tq)}{\tau(p,q)+\tau(q,Tq)}\right]\\ \leq&[a_1+a_3+a_4]\tau(p,q) + a_3\tau(q,p). \qquad (3.2)\end{aligned}$$

Similarly,

$$\tau(q,p) \leq [a_1+a_3+a_4]\tau(q,p) + a_3\tau(p,q). \qquad (3.3)$$

Subtract (3.3) from (3.2)

$$\begin{aligned}|\tau(p,q)-\tau(q,p)| &\leq |(a_1+a_3+a_4)-a_3||\tau(p,q)-\tau(q,p)|\\ &\leq |a_1+a_4||\tau(p,q)-\tau(q,p)|\end{aligned}$$

Since $|a_1+a_4| < 1$, we have $|\tau(p,q)-\tau(q,p)| = 0$. This implies that $\tau(p,q) = \tau(q,p)$. Putting $\tau(p,q) = \tau(q,p)$ in (3.2), we get $\tau(p,q) \leq [a_1+2a_3+a_4]\tau(p,q)$. Since $0 \leq a_1+2a_3+a_4 < 1$, we have $\tau(p,q) = 0$. Similarly, $\tau(q,p) = 0$. Hence $p = q$. ☐

**Example 3.2.** Let $X = [0,1]$ and complete $\tau q$-metric is defined by $\tau(x,y) = |x|$ for all $x, y \in X$ and define the continuous self mapping $T : X \to X$ by $Tx = \frac{x}{2}$.
Suppose $a_1 = \frac{1}{3}, a_2 = \frac{1}{10}, a_3 = \frac{1}{12}, a_4 = \frac{1}{15}, a_5 = \frac{1}{20}$. Then $T$ satisfies all the conditions of Theorem 3.1, and $x = 0$ is the unique fixed point of $T$ in $X$.

**Theorem 3.3.** Let $(X,\tau)$ be a complete $\tau q$-metric space and $T : X \to X$ be a continuous function satisfying the following condition:

$$\begin{aligned}\tau(Tx,Ty) \quad\leq\quad & b_1\tau(x,y) + b_2[\tau(x,Tx)+\tau(y,Ty)]\left[\frac{\tau(x,Ty)}{\tau(x,y)+\tau(y,Ty)}\right]\\ &+b_3[\tau(x,Ty)+\tau(y,Tx)]\left[\frac{\tau(x,Ty)}{\tau(x,y)+\tau(y,Ty)+\tau(x,Ty)}\right], \qquad (3.4)\end{aligned}$$

where $b_1, b_2, b_3 \geq 0$ with $b_1 + 2b_2 + 4b_3 < 1$ for all $x, y \in X$. Then $T$ has a unique fixed point.

**Proof .** Define a sequence $\{x_n\}$ in $X$ using Picard's iteration as follows:

Let $x_0$ be any arbitrary element in $X, x_1 = T(x_0), x_2 = T(x_1), \cdots, x_{n+1} = T(x_n), \cdots$. Assume that $x_{n+1} \neq x_n$ for any $n$, because if for any n, $x_{n+1} = x_n$, then $x_n$ is a fixed point and there is no need to go further. From (3.4), we have

$$\begin{aligned}\tau(x_{n+1},x_{n+2}) =&\tau(Tx_n,Tx_{n+1})\\ \leq& b_1\tau(x_n,x_{n+1})+b_2[\tau(x_n,Tx_n)+\tau(x_{n+1},Tx_{n+1})]\left[\frac{\tau(x_n,Tx_{n+1})}{\tau(x_n,x_{n+1})+\tau(x_{n+1},Tx_{n+1})}\right]\\ &+b_3[\tau(x_n,Tx_{n+1})+\tau(x_{n+1},Tx_n)]\left[\frac{\tau(x_n,Tx_{n+1})}{\tau(x_n,x_{n+1})+\tau(x_{n+1},Tx_{n+1})+\tau(x_n,Tx_{n+1})}\right]\\ \leq& b_1\tau(x_n,x_{n+1})+b_2[\tau(x_n,x_{n+1})+\tau(x_{n+1},x_{n+2})]+b_3[\tau(x_n,x_{n+2})+\tau(x_{n+1},x_{n+1})]\\ \leq& b_1\tau(x_n,x_{n+1})+b_2\tau(x_n,x_{n+1})+b_2\tau(x_{n+1},x_{n+2})+b_3\tau(x_n,x_{n+1})\\ &+b_3\tau(x_{n+1},x_{n+2})+b_3\tau(x_{n+1},x_{n+2})+b_3\tau(x_n,x_{n+1})\\ \leq&\frac{[b_1+b_2+2b_3]}{[1-(b_2+2b_3)]}\tau(x_n,x_{n+1}).\end{aligned}$$

Let $k=\frac{[b_1+b_2+2b_3]}{[1-(b_2+2b_3)]}$. Observe that $0\leq k<1$ since $b_1+2b_2+4b_3<1$. Therefore, $\tau(x_{n+1},x_{n+2})\leq k\tau(x_n,x_{n+1})$. Similarly, $\tau(x_n,x_{n+1})\leq k\tau(x_{n-1},x_n)$, so we get $\tau(x_{n+1},x_{n+2})\leq k^2\tau(x_{n-1},x_n)$.

Proceeding like this, we get $\tau(x_{n+1},x_{n+2})\leq k^{n+1}\tau(x_0,x_1)$. Since $0\leq k<1\Rightarrow k^{n+1}\to 0$ as $n\to\infty$. By Lemma 2.9, $\{x_n\}$ is a Cauchy sequence in complete $\tau q$-metric space $X$. So there is a point $p\in X$ such that $p$ is the $\tau q$-limit of $\{x_n\}$, that is $x_n\to p$. Since $T$ is continuous, we have $\lim_{n\to\infty}Tx_n=Tp$. This implies $\lim_{n\to\infty}x_{n+1}=Tp$. Thus, $Tp=p$. Hence $p$ is a fixed point of $T$.

**Uniqueness:** If $p\in X$ is a fixed point of $T$ then by condition (3.4), we have

$$\begin{aligned}\tau(p,p)=&\tau(Tp,Tp)\\ \leq& b_1\tau(p,p)+b_2[\tau(p,Tp)+\tau(p,Tp)]\left[\frac{\tau(p,Tp)}{\tau(p,p)+\tau(p,Tp)}\right]+b_3[\tau(p,Tp)+\tau(p,Tp)]\left[\frac{\tau(p,Tp)}{\tau(p,p)+\tau(p,Tp)+\tau(p,Tp)}\right]\\ \leq&\left(b_1+b_2+\frac{2}{3}b_3\right)\tau(p,p).\end{aligned}$$

Since $0\leq(b_1+b_2+\frac{2}{3}b_3)<1$ and $\tau(p,p)\geq 0$, we have, $\tau(p,p)=0$. Thus, $\tau(p,p)=0$, if $p$ is a fixed point of $T$. Suppose that $p$ and $q$ are two fixed points of $T(p\neq q)$, that is, $p=Tp$ and $q=Tq$. We have,

$$\begin{aligned}\tau(p,q)=&\tau(Tp,Tq)\\ \leq& b_1\tau(p,q)+b_2[\tau(p,Tp)+\tau(q,Tq)]\left[\frac{\tau(p,Tq)}{\tau(p,q)+\tau(q,Tq)}\right]+b_3[\tau(p,Tq)+\tau(q,Tp)]\left[\frac{\tau(p,Tq)}{\tau(p,q)+\tau(q,Tq)+\tau(p,Tq)}\right]\\ \leq&[b_1+\frac{b_3}{2}]\tau(p,q)+\frac{b_3}{2}\tau(q,p).\end{aligned}\tag{3.5}$$

Similarly,

$$\tau(q,p)\quad\leq[b_1+\tfrac{b_3}{2}]\tau(q,p)+\tfrac{b_3}{2}\tau(p,q).\tag{3.6}$$

Subtract (3.6) from (3.5), we get

$$|\tau(p,q)-\tau(q,p)|\leq|b_1||\tau(p,q)-\tau(q,p)|$$

Since $|b_1|<1$, we have $|\tau(p,q)-\tau(q,p)|=0$. This implies that $\tau(p,q)=\tau(q,p)$. Putting $\tau(p,q)=\tau(q,p)$ in (3.5), we get $\tau(p,q)\leq[b_1+b_3]\tau(p,q)$. Since $0\leq b_1+b_3<1$, we have $\tau(p,q)=0$. Similarly, $\tau(q,p)=0$. Hence $p=q$. □

**Example 3.4.** Let $X=[0,1]$ with a complete $\tau q$-metric defined by $\tau(x,y)=|x|$ for all $x,y\in X$ and let $T:X\to X$ such that $Tx=\frac{x}{2}$ be a continuous self mapping. Suppose $b_1=\frac{21}{50},b_2=\frac{1}{10},b_3=\frac{1}{12}$. Then $T$ satisfies all the conditions of Theorem 3.2, and $x=0$ is the unique fixed point of $T$ in $X$.

**Theorem 3.5.** Let $(X,\tau)$ be a complete $\tau q$-metric space and let $A$ and $B$ be two continuous self mapping $A, B : X \to X$ satisfying the following condition:

$$\begin{aligned}\tau(Ax,By) \leq & c_1\tau(x,y) + c_2[\tau(x,Ax)+\tau(y,By)]\left[\frac{\tau(x,By)}{\tau(x,y)+\tau(y,By)}\right]\\ & + c_3[\tau(x,By)+\tau(y,Ax)]\left[\frac{\tau(x,By)}{\tau(x,y)+\tau(y,By)+\tau(x,Ax)}\right], \end{aligned}\tag{3.7}$$

where $c_1, c_2, c_3 \geq 0$ with $c_1 + 2c_2 + 4c_3 < 1$ for all $x, y \in X$. Then $A, B$ has a unique common fixed point.

**Proof .** Let $x_0 \in X$ be arbitrary, we define a sequence $\{x_n\}$ in $X$ as follows:

$x_1 = A(x_0)$, $x_2 = B(x_1)$, $x_3 = A(x_2), \cdots$ , $x_{2n} = B(x_{2n-1})$, $x_{2n+1} = A(x_{2n}), \cdots$ for all $n \in N$. We will show that $\{x_n\}$ is a Cauchy sequence in $X$. From (3.7) we have,

$$\begin{aligned}\tau(x_{2n+1},x_{2n+2}) =& \tau(Ax_{2n},Bx_{2n+1})\\ \leq & c_1\tau(x_{2n},x_{2n+1}) + c_2[\tau(x_{2n},Ax_{2n})+\tau(x_{2n+1},Bx_{2n+1})]\left[\frac{\tau(x_{2n},Bx_{2n+1})}{\tau(x_{2n},x_{2n+1})+\tau(x_{2n+1},Bx_{2n+1})}\right]\\ & + c_3[\tau(x_{2n},Bx_{2n+1})+\tau(x_{2n+1},Ax_{2n})]\left[\frac{\tau(x_{2n},Bx_{2n+1})}{\tau(x_{2n},x_{2n+1})+\tau(x_{2n+1},Bx_{2n+1})+\tau(x_{2n},Ax_{2n})}\right]\\ \leq & c_1\tau(x_{2n},x_{2n+1}) + c_2[\tau(x_{2n},x_{2n+1})+\tau(x_{2n+1},x_{2n+2})]\\ & + c_3[\tau(x_{2n},x_{2n+1})+\tau(x_{2n+1},x_{2n+2})+\tau(x_{2n+1},x_{2n+2})+\tau(x_{2n},x_{2n+1})]\\ \leq & \frac{[c_1+c_2+2c_3]}{[1-(c_2+2c_3)]}\tau(x_{2n},x_{2n+1}).\end{aligned}$$

Let $k = \frac{[c_1+c_2+2c_3]}{[1-(c_2+2c_3)]}$. Observe that $0 \leq k < 1$ since $c_1+2c_2+4c_3 < 1$. Therefore, $\tau(x_{2n+1},x_{2n+2}) \leq k\tau(x_{2n},x_{2n+1})$. Similarly, $\tau(x_{2n},x_{2n+1}) \leq k\tau(x_{2n-1},x_{2n})$. Proceeding like this, we get $\tau(x_{2n+1},x_{2n+2}) \leq k^{2n+1}\tau(x_0,x_1)$. Since $0 \leq k < 1$, we have, $k^{2n+1} \to 0$ as $n \to \infty$.

By Lemma 2.9, $\{x_n\}$ is a Cauchy sequence in complete $\tau q$-metric space $X$. So there is a point $p \in X$ such that $p$ is the $\tau q$-limit of $\{x_n\}$, that is $x_n \to p$. Also, the subsequences $\{A(x_{2n})\} \to p$ and $\{B(x_{2n-1})\} \to p$. Since $A, B : X \to X$ are continuous, we get $A(p) = p$ and $B(p) = p$. Thus, $p$ is a fixed point $A$ and $B$.

**Uniqueness of common fixed point:** Let $p \in X$ be a fixed point of $A$ and $B$ then by condition (3.7), we have

$$\begin{aligned}\tau(p,p) =& \tau(Ap,Bp)\\ \leq & c_1\tau(p,p) + c_2[\tau(p,Ap)+\tau(p,Bp)]\left[\frac{\tau(p,Bp)}{\tau(p,p)+\tau(p,Bp)}\right] + c_3[\tau(p,Bp)+\tau(p,Ap)]\left[\frac{\tau(p,Bp)}{\tau(p,p)+\tau(p,Bp)+\tau(p,Ap)}\right]\\ \leq & \left(c_1+c_2+\frac{2}{3}c_3\right)\tau(p,p).\end{aligned}$$

Since $0 \leq (c_1 + c_2 + \frac{2}{3}c_3) < 1$ and $\tau(p,p) \geq 0$, we have, $\tau(p,p) = 0$. Thus, $\tau(p,p) = 0$, if $p$ is a fixed point of $T$. Now let $p$ and $q$ be two fixed points of $A$ and $B$ $(p \neq q)$, then we have

$$\begin{aligned}\tau(p,q) =& \tau(Ap,Bq)\\ \leq & c_1\tau(p,q) + c_2[\tau(p,Ap)+\tau(q,Bq)]\left[\frac{\tau(p,Bq)}{\tau(p,q)+\tau(q,Bq)}\right] + c_3[\tau(p,Bq)+\tau(q,Ap)]\left[\frac{\tau(p,Bq)}{\tau(p,q)+\tau(q,Bq)+\tau(p,Ap)}\right]\\ \leq & [c_1+c_3]\tau(p,q) + c_3\tau(q,p).\end{aligned}\tag{3.8}$$

Similarly,

$$\tau(q,p) \leq [c_1+c_3]\tau(q,p) + c_3\tau(p,q).\tag{3.9}$$

Subtract (3.9) from (3.8), we get

$$\begin{aligned}|\tau(p,q)-\tau(q,p)| \leq & |(c_1+c_3)-c_3||\tau(p,q)-\tau(q,p)|\\ \leq & |c_1||\tau(p,q)-\tau(q,p)|.\end{aligned}$$

Since $|c_1| < 1$, we have $|\tau(p,q) - \tau(q,p)| = 0$. Hence, $\tau(p,q) = \tau(q,p)$. Putting $\tau(p,q) = \tau(q,p)$ in (3.8), we get $\tau(p,q) \leq [c_1 + 2c_3]\tau(p,q)$. Since $0 \leq c_1 + 2c_3 < 1$, we have $\tau(p,q) = 0$. Similarly, $\tau(q,p) = 0$. Hence $p = q$. $\square$

**Example 3.6.** Let $X = [0,1]$ with a complete $\tau q$-metric defined by $\tau(x,y) = |x|$, for all $x, y \in X$ and let $A, B : X \to X$ be continuous self mappings defined by $Ax = \frac{x}{5}$ and $Tx = 0$ for all $x \in X$. Suppose that $c_1 = \frac{21}{50}, c_2 = \frac{1}{10}, c_3 = \frac{1}{12}$. Then $A$ and $B$ satisfy all the conditions of Theorem 3.3, so $x = 0$ is the unique common fixed point of $A$ and $B$ in $X$.

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