Int. J. Nonlinear Anal. Appl. In Press, 15–20
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.28296.3854

# Idempotent multipliers of Figà-Talamanca-Herz algebras

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*(Communicated by Ali Jabbari)*

**Abstract**

For a locally compact group $G$ and $p \in (1, \infty)$, let $B_p(G)$ is the multiplier algebra of the Figà-Talamanca-Herz algebra $A_p(G)$. For $p = 2$ and $G$ amenable, the algebra $B(G) := B_2(G)$ is the usual Fourier-Stieltjes algebra. In this paper, we show that $A_p(G)$ is a Bochner-Schoenberg-Eberlin (BSE) algebra and every clopen subset of $G$ is a synthetic set for $A_p(G)$. Furthermore, we characterize idempotent elements of the Banach algebra $B_p(G)$. This result generalizes the Cohen-Host idempotent theorems for the case of Figà-Talamanca-Herz algebras. Characterization of idempotent elements of $B_p(G)$ is of paramount importance to study homomorphisms in Figà-Talamanca-Herz algebras.

Keywords: Figà-Talamanca-Herz algebra; Multiplier algebra; Idempotent element; Fourier algebra; Fourier-Stieltjes algebra
2020 MSC: 43A22, 43A25, 43A30, 46J10.

## 1 Introduction

Cohen [1, 2] characterized idempotents of the measure algebra $M(G)$ when $G$ is an abelian locally compact group. Host [7] extended Cohen's result for arbitrary (not necessarily abelian) locally compact groups and showed that there is a one to one correspondence between idempotent elements of Fourier-Stieltjes algebra $B(G)$ and elements of the coset ring $\Omega(G)$ of $G$, Ilie [8, 9] applied the Cohen–Host idempotent results to study homomorphisms in Fourier algebras. Looking more precisely at the algebras $M(G)$ and $B(G)$, one can see that these algebras are exactly the multiplier algebras of $L^1(G)$ and $A(G)$, respectively. In other words, Cohen and Host characterized idempotents of the multiplier algebras of $L^1(G)$ and $A(G)$, respectively.

Let $A$ be a commutative algebra, $\Delta(A)$ be the Gelfand space and $M(A)$ be the set of multipliers on $A$, i.e., $T(ab) = aT(b) = T(a)b$ for all $a, b \in A$ and $T$ is a bounded linear map for each $T \in M(A)$ (see [11]). Then for each $T \in M(A)$, there is a unique bounded continuous function $\widehat{T}$ on $\Delta(A)$ such that $\widehat{T(a)}(\varphi) = \widehat{T}(\varphi)\widehat{a}(\varphi)$ for all $a \in A$ and $\varphi \in \Delta(A)$ (see [10]). Let $\widehat{M(A)} = \{\widehat{T} : T \in M(A)\}$. A bounded continuous function $\sigma : \Delta(A) \to \mathbb{C}$ is called a BSE-function if there is a constant $K > 0$ such that for every finite number of $\varphi_1, \varphi_2, \cdots, \varphi_n \in \Delta(A)$ and complex numbers $c_1, c_2, \cdots, c_n$,

$$\left| \sum_{j=1}^{n} c_j \sigma(\varphi_j) \right| \leq K \left\| \sum_{j=1}^{n} c_j \varphi_j \right\|_{A^*}$$

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*Received:* September 2022 *Accepted:* December 2023

holds, where $A^*$ is the dual of $A$. The algebra $A$ is called a Bochner-Schoenberg-Eberlin (BSE) algebra if the set of BSE-functions is $\widehat{M(A)}$ (see [13]).

In this paper, we present Host's results for a larger class of Banach algebras, Figà-Talamanca-Herz algebras $A_p(G)$. We show that there is a one-one correspondence between idempotent elements of $B_p(G)$ and clopen $U$-subsets of $G$ with respect to $A_p(G)$, where $B_p(G)$ is the multiplier algebra of $A_p(G)$.

The rest of this paper is organized as follows. In Section 2, we give a brief introduction of the Figà-Talamanca-Herz algebras $A_p(G)$. In Section 3, we show the correspondence between $U$-sets and idempotent elements of a multiplier algebra of a Banach algebra. Moreover, we present a characterization of idempotent elements of the Banach algebra $B_p(G)$. Finally, conclusions are drawn in Section 4.

## 2 Figà-Talamanca-Herz algebras

We assume the reader is familiar with the preliminary notions of Banach algebras and multipliers, but for the sake of accessibility, we list the main definitions and theorems without proofs which will be referred to later on. Our main goal in this section is to give a brief introduction of the Figà-Talamanca-Herz algebra and corresponding multiplier algebra.

**Definition 2.1.** For a locally compact group $G$ and $p \in (1,\infty)$, let $p\prime \in (1,\infty)$ be dual to $p$, i.e., $\frac{1}{p}+\frac{1}{p\prime}=1$. The *Figà-Talamanca-Herz algebra* $A_p(G)$ consists of those functions $f : G \to \mathbb{C}$, for which there are sequences $(\xi_n)_{n=1}^{\infty}$ in $L^p(G)$ and $(\eta_n)_{n=1}^{\infty}$ in $L^{p\prime}(G)$ such that

$$\sum_{n=1}^{\infty} \|\xi_n\|_{L^p(G)}\, \|\eta_n\|_{L^{p\prime}(G)} < \infty \tag{2.1}$$

and

$$f = \sum_{n=1}^{\infty} \xi_n * \check{\eta}_n, \tag{2.2}$$

where $\check{\eta}_n : G \to \mathbb{C}$ is defined by

$$\check{\eta}_n := \eta_n(x^{-1}) \quad \forall x \in G.$$

**Remark 2.2.** The norm on $A_p(G)$ is defined as the infimum over all sums (2.1) such that (2.2) holds. Indeed, if $f \in A_p(G)$, then

$$\|f\|_{A_p(G)} = \inf \left\{ \sum_{n=1}^{\infty} \|\xi_n\|_{L^p(G)} \|\eta_n\|_{L^{p\prime}(G)} \;\Big|\; f = \sum_{n=1}^{\infty} \xi_n * \check{\eta}_n \right\}.$$

Herz [6] showed that $A_p(G)$ is closed under pointwise multiplication and hence is a Banach algebra. Eymard [3] previously studied the case $p = p\prime = 2$; in this case, $A(G) := A_2(G)$ is called a *Fourier algebra* of $G$.

Let $\lambda_{p\prime} : G \to \mathcal{B}(L^{p\prime}(G))$ be the regular left representation of $G$ on $L^{p\prime}(G)$. Using integration, $\lambda_{p\prime}$ can be extended to a representation of $L^1(G)$ on $L^{p\prime}(G)$. The algebra of *p′-pseudomeasures* $\mathbf{PM}_{p\prime}(G)$ is defined as the $w^*$-closure of $\lambda_{p\prime}(L^1(G))$ in $\mathcal{B}(L^{p\prime}(G))$. There is a canonical duality $\mathbf{PM}_{p\prime}(G) \cong A_p(G)^*$ via

$$\langle \xi * \check{\eta}, T\rangle := \langle T\eta, \xi\rangle \qquad \Big(\xi \in L^{p\prime}(G),\ \eta \in L^p(G),\ T \in \mathbf{PM}_{p\prime}(G)\Big).$$

For more details, see [3, 5, 6, 12]. If $p = 2$, then $VN(G) := \mathbf{PM}_2(G)$ is known as the *group von Neumann algebra* of $G$.

Assume that $G$ is a locally compact group and $p \in (1,\infty)$. The surjective mapping $\Gamma_1$ is defined by

$$\Gamma_1 : L^p(G) \hat{\otimes} L^{p\prime}(G) \to A_p(G)$$

$$\Gamma_1(\varphi)(x) = \textstyle\int_G \varphi(xy, y)dy,$$

where $\varphi \in L^p(G) \hat{\otimes} L^{p\prime}(G)$ and $x \in G$. Also, we have $\|\Gamma_1\| = 1$ and $A_p(G) \cong (L^p(G) \hat{\otimes} L^{p\prime}(G))/\mathrm{Ker}\Gamma_1$. Let $\xi \in L^p(G)$ and $\eta \in L^{p\prime}(G)$ be two complex functions on $G$. Setting

$$\xi \otimes \eta(x, y) := \xi(x)\eta(y) \qquad \forall x, y \in G,$$

we have

$$\Gamma_1(\xi \otimes \eta) = \xi * \check{\eta}.$$

**Definition 2.3.** Let $\mathcal{A}$ be a Banach algebra of complex functions on the set $X$. The complex function $u$ is called a `multiplier function` of $\mathcal{A}$ if

1. $u\varphi \in \mathcal{A}$, for all $\varphi \in \mathcal{A}$,
2. $\psi \in \mathcal{A}$, where $\psi : \mathcal{A} \to \mathcal{A},\ \varphi \mapsto u\psi$.

We denote by $B_p(G)$ the Banach algebra including all $u \in \mathcal{C}(G)$ which are multiplier functions of the Figà-Talamanca-Herz algebra $A_p(G)$. The Banach algebra $B_p(G)$ is called a *multiplier algebra* of $A_p(G)$. We remind that, in the case $p = 2$ and $G$ amenable, the algebra $B(G) := B_2(G)$ is the Fourier-Stieltjes algebra of $G$, which was previously studied by Eymard [3].

Here, we recall Theorems 2.4 and 2.5 from [12] and Theorems 2.7 and 2.8 from [6] without proofs, which state essential properties of the Figà-Talamanca-Herz algebra $A_p(G)$.

**Theorem 2.4.** [12] Locally compact group $G$ is amenable if and only if one of the following statements holds:

1. for every $p \in (1, \infty)$, the algebra $A_p(G)$ has a bounded approximate identity (with the bound 1) with elements belonging to $\mathcal{C}_c(G)$;
2. for every $p \in (1, \infty)$, the algebra $A_p(G)$ has a bounded approximate identity (with the bound 1);
3. for every $p \in (1, \infty)$, the algebra $A_p(G)$ has a multiplicative bounded approximate identity (with the bound 1);
4. for at least one $p \in (1, \infty)$, one of the above three statements holds.

**Theorem 2.5.** [12] Let $G$ be an amenable locally compact group and $p \in (1, \infty)$. Then the Banach algebra $A_p(G)$ is a closed ideal of $B_p(G)$.

**Definition 2.6.** A Banach algebra $\mathcal{A}$ is called a `regular Tauberian` algebra of functions on the set $G$ if the following statements hold:

1. for given compact subset $K \subseteq G$ and for disjoint subset $F \subseteq G$ ($K \cap F = \varnothing$), there is a function $f \in \mathcal{A}$ such that $f \equiv 1$ on $K$ and $f \equiv 0$ on $F$;
2. the subset of compact support elements in $\mathcal{A}$ is dense in $\mathcal{A}$;
3. if $T$ is a continuous multiplicative functional on $\mathcal{A}$ with support $\{x\} \subset G$, then $T = \delta_x$. Indeed, for every $f \in \mathcal{A}$, $\langle f, T\rangle = f(x)$.

**Theorem 2.7.** [6] Let $G$ be a locally compact group and $p \in (1, \infty)$. Then $A_p(G)$ is a semi-simple regular Tauberian algebra of functions on $G$.

**Theorem 2.8.** [6] Let $\mathcal{A}$ be a regular Tauberian algebra of functions on a locally compact group $G$. Then the regular maximal ideals space of $\mathcal{A}$ is isomorphic to $G$.

As a result of the above theorems, the *spectrum* of $A_p(G)$ is equal to $G$.

## 3 Characterization of idempotents of $B_p(G)$

In this section, we aim to present the correspondence between the $U$-sets and the idempotent elements of $\mathcal{M}(\mathcal{A})$, where $\mathcal{A}$ is a commutative semi-simple Banach algebra and $\mathcal{M}(\mathcal{A})$ is the multiplier algebra of $\mathcal{A}$ [14]. Furthermore, we show that $A_p(G)$ is a `BSE`-algebra and every clopen subset of $G$ is a synthetic set for $A_p(G)$. We give a characterization of idempotent elements of $B_p(G)$ and show the correspondence between idempotent elements of $B_p(G)$ and clopen $U$-subsets of $G$ with respect to $A_p(G)$. This result generalizes the Host's idempotent theorem which characterizes idempotents of Fourier-Stieltjes algebras. Let us start this section with some necessary definitions.

**Definition 3.1.** Let $\mathcal{A}$ be a semi-simple Banach algebra and let $\mathcal{M}(\mathcal{A})$ be the multiplier algebra of $\mathcal{A}$. For a multiplier $T$ in $\mathcal{M}(\mathcal{A})$, let $\hat{T}$ be the `Gelfand transform` of $T$. Then we define

$$\Delta(\hat{T}) := \{f \in \sigma(\mathcal{A}) \mid \hat{T}(f) \neq 0\}.$$

Let $\mathcal{A}$ be a function algebra on a topological space $X$ and $\mathcal{A}_c$ the set of all $f \in \mathcal{A}$ whose support of $f$ is compact in $X$. For a closed subset $S \subseteq X$, we define

$$\begin{aligned} J(S): &= \{f \in \mathcal{A}_c \mid (\text{supp } f) \cap S = \varnothing\}, \\ I(S): &= \{f \in \mathcal{A} \mid f(S) \subseteq \{0\}\}. \end{aligned}$$

A closed subset $S \subseteq X$ is called `synthetic` for $\mathcal{A}$ if $\overline{J(S)} = I(S)$.

The semi-simple commutative Banach algebra $\mathcal{A}$ with an approximate identity is called a `BSE`-algebra if every $u \in \mathcal{A}^{**}$ for which $\hat{u}$ is continuous on $\sigma(\mathcal{A})$, belongs to $\Lambda(\mathcal{A})$, where

$$\Lambda(\mathcal{A}) = \{m \in \mathcal{A}^{**} \mid m\mathcal{A} \subseteq \mathcal{A}\}.$$

Moreover, $\hat{u}$ is defined by

$$\hat{u} \colon \sigma(\mathcal{A}) \to \mathbb{C}$$

$$\hat{u}(f) = \langle u, f \rangle.$$

**Definition 3.2.** Let $\mathcal{A}$ be a commutative Banach algebra and $\sigma(\mathcal{A})$ the spectrum of $\mathcal{A}$. An open subset $O \subseteq \sigma(\mathcal{A})$ is called a `U-set` with respect to $\mathcal{A}$ if there exists a constant $c > 0$ such that for every compact subset $K \subseteq O$, there is $a \in \mathcal{A}$ that $\|a\| \leq c$ and $\hat{a} = 1$ on $K$.

For a regular `BSE`-algebra $\mathcal{A}$ with an approximate identity for which all clopen subsets of $\sigma(\mathcal{A})$ are synthetic, Ulger [14] showed that the family of all clopen $U$-subsets of $\sigma(\mathcal{A})$, denoted by $\mathcal{R}(\sigma(\mathcal{A}))$, forms a ring.

**Definition 3.3.** We say that a subset $O \subseteq \sigma(\mathcal{A})$ supports an idempotent multiplier if for some idempotent element, namely, $\theta \in \mathcal{M}(\mathcal{A})$, $O = \sigma(\hat{\theta})$ holds.

Let $\mathcal{A}$ be a regular `BSE`-algebra with an approximate identity for which all clopen subsets of $\sigma(\mathcal{A})$ are synthetic. The clopen subset $O \subseteq \sigma(\mathcal{A})$ supports an idempotent multiplier if and only if $O$ is a $U$-set with respect to $\mathcal{A}$ [14].

As we mentioned before, Host [7] identified idempotents of $B(G)$. This was an extension of Cohen's work in which $G$ is abelian.

**Theorem 3.4.** (Host Idempotent Theorem) For locally compact group $G$, the element $u \in B(G)$ is an idempotent if and only if $\hat{u}$ is a characteristic function of some elements of coset ring of $G$. Indeed,

$$\delta_X \in B(G) \Longleftrightarrow X \in \Omega(G).$$

The following theorem is an immediate result of the Host's idempotent theorem [14].

**Theorem 3.5.** Let $G$ be a locally compact amenable group and $O \subseteq G$ a clopen set. Then $O$ belongs to the coset ring $\Omega(G)$ of $G$ if and only if $O$ is a $U$-set with respect to the Fourier algebra $A(G)$.

For a locally compact amenable group $G$, we recall that $B_p(G)$ is the Banach algebra including all $u \in \mathcal{C}(G)$ which are multiplier functions of the Figà-Talamanca-Herz algebra $A_p(G)$, i.e., $B_p(G) = \mathcal{M}(A_p(G))$. It was previously shown that $A_p(G)$ is a regular semi-simple Banach algebra with an approximate identity and that the Gelfand spectrum of this algebra is isomorphic to $G$.

Here, we are ready to state and prove our main results. By Lemma 3.6, we show that for a locally compact group $G$, $A_p(G)$ is a BSE-algebra and every clopen subset of $G$ is synthetic for $A_p(G)$. Furthermore, via Theorem 3.7, we characterize idempotent elements of the multiplier algebra $B_p(G)$ using $U$-sets.

**Lemma 3.6.** Let $G$ be a discrete amenable locally compact group and $p \in (1, \infty)$. Then $A_p(G)$ is a `BSE`-algebra and every clopen subset of $G$ is a synthetic set for $A_p(G)$.

**Proof .** Assume $C_{\mathrm{BSE}}(\sigma(A_p(G)))$ is the set of all $\varphi \in C(\sigma(A_p(G)))$ for which there exists a constant $\beta > 0$ such that for every $c_1, \cdots, c_n \in \mathbb{C}$ and $f_1, \cdots, f_n \in \sigma(A_p(G))$, the inequality

$$\left|\sum_{i=1}^{n} c_i \varphi(f_i)\right| \leq \beta \left\|\sum_{i=1}^{n} c_i f_i\right\|$$

holds true. Using the definition of BSE-algebras in [13], $A_p(G)$ is a BSE-algebra if

$$\hat{M}(A_p(G)) = \left\{\hat{T} : \sigma(A_p(G)) \to \mathbb{C} \quad | \quad T \in \mathcal{M}(A_p(G))\right\} = C_{\mathrm{BSE}}(\sigma(A_p(G))).$$

Applying [3, Lemma 2.13] for the algebra $B_p(G)$, one can show that there exists a positive constant $\beta$ such that the inequality

$$\left|\sum_{i=1}^{n} c_i u(x_i)\right| \leq \beta \left\|\sum_{i=1}^{n} c_i \delta_{x_i}\right\| \tag{3.1}$$

holds for every $c_1, \cdots, c_n \in \mathbb{C}$ and $x_1, \cdots, x_n \in G$, where $u \in B_p(G)$ and $\|u\| \leq 1$. Therefore, the inequality (3.1) shows that $A_p(G)$ is a BSE-algebra.

On the other hand, since $G$ is amenable, using [4, Corollary 5.4], every clopen subset of $G$ is a synthetic set for $A_p(G)$, which completes the proof. □

Summing all results up, our characterization of idempotents of the multiplier algebra $B_p(G)$ is presented as follows:

**Theorem 3.7.** Let $G$ be a locally compact amenable group and $p \in (1, \infty)$. Then $\delta_X \in B_p(G)$ if and only if $X$ is a $U$-set with respect to $A_p(G)$. Indeed,

$$\delta_X \in B_p(G) \Longleftrightarrow X \in \mathcal{R}(G).$$

**Proof .** For the locally compact amenable group $G$, the Figà-Talamanca-Herz algebra $A_p(G)$ is a regular semi-simple BSE-algebra with an approximate identity. Also, Lemma 3.6 states that every clopen subset of $G$ is synthetic for $A_p(G)$. Thus each clopen subset $X \subseteq G$ supports an idempotent multiplier if and only if $X$ is a $U$-set with respect to $A_p(G)$.

On the other hand, since $B_p(G)$ is a Banach algebra with the pointwise addition and multiplication, it is clear that the idempotent elements of $B_p(G)$ are the characterization functions of some clopen subsets, i.e., each idempotent element of $B_p(G)$ is of the form of $\delta_X$ for some clopen subset $X \subseteq G$. Furthermore, since the elements of $B_p(G)$ are continuous, for every $\delta_X \in B_p(G)$, the set $X$ is a clopen set. Therefore, $\delta_X \in B_p(G)$ if and only if $X$ is a $U$-set with respect to $A_p(G)$. □

## 4 Conclusions

In this work, for a locally compact amenable group $G$ and $p \in (1, \infty)$, we considered Figà-Talamanca-Herz algebra $A_p(G)$ and its multiplier algebra $B_p(G)$, as natural generalizations of Fourier and Fourier-Stieltjes algebras, respectively. We showed that $A_p(G)$ is a "BSE"-algebra and every clopen subset of $G$ is a synthetic set for $A_p(G)$. Moreover, we gave a characterization of idempotent elements of $B_p(G)$ and showed the correspondence between idempotent elements of $B_p(G)$ and clopen $U$-subsets of $G$ with respect to $A_p(G)$. This result generalizes the Host's idempotent theorem which characterizes the idempotents of Fourier-Stieltjes algebras.

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