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# Boyd-Wong and Meir-Keeler type contractions in a new generalized b-metric space

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**Abstract**

In this paper, we established the Boyd-Wong type and Meir-Keeler type contractions in a new generalized b-metric space. Two types of fixed point theorems are proven, which extend the same results in the metric and b-metric spaces. Some examples and one application are also discussed to show the applicability of the results.

Keywords: generalized b-metric, Boyd-Wong contraction, Meir-Keeler contraction, fixed point
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## 1 Introduction

So far, various generalizations of the metric space concept, which is one of the basic concepts in analysis, have been established. Including $b$-metric space, $E_b$- metric space and etc, see [1, 2, 3, 4, 6, 10, 11].

**Definition 1.1.** Let $X$ be a nonempty set and $d : X \times X \to [0, +\infty)$ for all $x, y \in X$ satisfies:

(i) $d(x, y) = 0$ if and only if $x = y$.
(ii) $d(x, y) = d(y, x)$

then

(iii-1) $(X, d)$ is called metric space if for all $x, y, z \in X$ and a function $s : X \times X \to [1, +\infty)$:

$$d(x, y) \leq d(x, z) + d(z, y).$$

(iii-2) $(X, d)$ is called $b$-metric space if for all $x, y, z \in X$ and a constant $s \geq 1$:

$$d(x, y) \leq s[d(x, z) + d(z, y)].$$

(iii-3) $(X, d)$ is called $E_b$-metric space if for all $x, y, z \in X$ and a function $s : X \times X \to [1, +\infty)$:

$$d(x, y) \leq s(x, y)[d(x, z) + d(z, y)].$$

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Some of these spaces are generalizations of one another, while others may not be related to each other, for example the class of b-metric spaces is larger than the class of metric spaces, since any metric $d$ is a b-metric with each $s \geq 1$. Based on these generalizations, many other related concepts such as the concept of sequence convergence, Cauchy sequence, sequentially compactness and related theorems have been discussed. Among these, we point out to the fixed point theorems such as Banach fixed point theorem and its extensions, where has various types in each of the generalized metric spaces, see [7, 8, 13]. In this paper we consider a generalized metric that introduced by [12] and we prove multiple fixed point theorem that generalized the Boyd-Wong fixed point theorem and Meir-Keeler contraction principle. Some examples and an application is also discussed to show the applicability of the theorems.

## 2 Preliminaries

**Definition 2.1.** [12] Let $X$ as a nonempty set and real numbers $\alpha, \beta \in [1, +\infty)$. A function $d : X \times X \longrightarrow [0, +\infty)$ is called $(\alpha, \beta)$- b-metric if for each $x, y, z \in X$ verifies the following,

(i) $d(x, y) = 0 \Longleftrightarrow x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leq \alpha d(x, z) + \beta d(z, y)$.

The pair $(X, d)$ is called $(\alpha, \beta)$-b-metric space.

**Remark 2.2.** Due to the symmetric property (ii), by (iii) we have,

$$d(x, y) \leq \min \{\alpha d(x, z) + \beta d(z, y), \beta d(x, z) + \alpha d(z, y)\},$$

and so for fixed $\alpha, \beta \in [1, +\infty)$, the spaces $(\beta, \alpha)$-b-metric space and $(\alpha, \beta)$-b-metric space are the same.

**Remark 2.3.** Obviously, $(1, 1)$-b-metric space is exactly the usual metric space and for every $\alpha > 1$, $(\alpha, \alpha)$-b-metric space is just b-metric space. Moreover every b-metric $d_s$ is an $(\alpha, \beta)$-b-metric with $\alpha = s$ and $\beta > s$.

**Example 2.4.** Let $X$ as a bounded subset of $\mathbb{R}$ and for some $a > 1$ let $d(x, y) := \begin{cases} a^{|x-y|}; & x \neq y; \\ 0; & x = y \end{cases}$. Then obviously $(X, d)$ is not a metric space in general; for instance, if $X = [0, 5]$, $a = 4$, $x = 0$, $y = 2$ and $z = 1$ then $d(x, y) > d(x, z) + d(y, z)$. Whereas, we will show that for appropriate $\alpha, \beta > 1$, $(X, d)$ is $(\alpha, \beta)$-b-metric space. Choose $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $\max\{p, q\} < a^{diam(X)}$; one can prove that these choices are possible. Using Young's inequality for products, for each $x, y \in X$ we have

$$\begin{aligned} a^{|x-y|} &\leq a^{|x-z|+|z-y|} \\ &= a^{(\frac{1}{p}+\frac{1}{q})|x-z|+|z-y|} \\ &= a^{\frac{1}{p}|x-z|} a^{\frac{1}{q}|z-y|} a^{\frac{1}{q}|x-z|+\frac{1}{p}|z-y|} \\ &\leq (\frac{1}{p} a^{|x-y|} + \frac{1}{q} a^{|y-z|}) a^{\frac{1}{q}|x-z|+\frac{1}{p}|z-y|} \\ &\leq (\frac{1}{p} a^{|x-y|} + \frac{1}{q} a^{|y-z|}) a^{diam(X)} \end{aligned}$$

Hence $(X, d)$ is $(\frac{a^{diam(X)}}{p}, \frac{a^{diam(X)}}{q})$-b-metric space

**Remark 2.5.** Since the parameters $\alpha, \beta$ are involved in the definition of $d$ as an $(\alpha, \beta)$-b-metric ; from now on, we denote an $(\alpha, \beta)$-b-metric $d$ by $d_{\alpha,\beta}$.

## 3 Main results

**Proposition 3.1.** Suppose $(X, d)$ be an arbitrary metric space, $diam(X) = \sup\{d(x, y);\ x, y \in X\} \leq +\infty$ and $g : [0, diam(X)) \to [1, +\infty)$ be a non-decreasing map such that $g(0) = 1$ and for some real constants $\alpha, \beta \geq 1$ we have

$$g(x + y) \leq g(x)^{\alpha} g(y)^{\beta}; \quad \text{for all} \quad x, y > 0. \tag{3.1}$$

Then $(X, d_{\alpha,\beta})$ is an $(\alpha, \beta)$-b-metric space with $d_{\alpha,\beta}(x, y) := \log_{\theta}(g(d(x, y)))$ for each $\theta > 1$.

**Proof .** In this regard, it suffices to study the validity of condition (iii) of the Definition 2.1. For every $x, y, z \in X$, we have

$$\begin{aligned}\log_\theta(g(d(x,y))) &\le \log_\theta(g(d(x,z)+d(y,z))) \\ &\le \log_\theta(g(d(x,z))^\alpha g(d(y,z))^\beta) \\ &= \alpha \log_\theta(g(d(x,z))) + \beta \log_\theta(g(d(y,z))).\end{aligned}$$

This shows condition (iii) of definition 2.1. □

**Remark 3.2.** By letting $g(x) = \theta^x$, since (3.1) is satisfied for $\alpha = \beta = 1$ then $d_{\alpha,\beta}$, that provides in the Proposition 3.1 is exactly equivalent to the metric $d$.

**Example 3.3.** Suppose $(X, d)$ be a bounded metric space then according to the Proposition 3.1, the $(\alpha, \beta)$-b-metric that introduced in the Example 2.4 can be produced by $g : [0, diam(X)) \to [1, +\infty)$ where $g(t) := \theta^{a^t}$.

**Example 3.4.** $g(t) = 1 + t$ with $\alpha = \beta = 1$ and $g(t) = \cosh(t)$ with $\alpha = \beta = 2$ satisfied the conditions of the Proposition 3.1.

Similar to the metric space the concepts of convergence of a sequence, Cauchy sequence and completeness of the $(\alpha, \beta)$-b-metric space is defined as follows:

**Definition 3.5.** Let $\{x_n\}$ be a sequence in an $(\alpha, \beta)$-b-metric space $(X, d_{\alpha,\beta})$ . Then

the sequence $\{x_n\}$ converges in $(X, d_{\alpha,\beta})$ if there exists $x^* \in X$ such that $\lim_{n\to\infty} d_{\alpha,\beta}(x_n, x^*) = 0$.

the sequence $\{x_n\}$ is called a Cauchy in $(X, d_{\alpha,\beta})$ if $\lim_{m,n\to\infty} d_{\alpha,\beta}(x_n, x_m) = 0$.

$(X, d_{\alpha,\beta})$ is called complete $(\alpha, \beta)$-b-metric space if every Cauchy sequence in $(X, d_{\alpha,\beta})$ is convergent.

**Definition 3.6.** Suppose $\alpha, \beta \ge 1$, a function $\psi : [0, +\infty) \to [0, +\infty)$ is called an admissible-contraction function if $\psi^{-1}(0) = \{0\}$ and satisfies at least one of the following conditions:

(S1) $\psi$ is upper-semi continuous function and for each $t > 0$, $\psi(t) < \max\{\frac{1}{\alpha}, \frac{1}{\beta}\}t$.

(S2) $\psi$ is nondecreasing function and for each $t > 0$, $\lim_{n\to\infty} \phi^n(t) = 0$, where $\phi(t) = \min\{\alpha, \beta\}\psi(t)$, and $\phi^n(t) = \phi(\phi^{n-1}(t))$.

Let us denote the family of all admissible-contraction functions with $\Upsilon$.

**Remark 3.7.** One can prove that if $\psi$ satisfied $(S2)$ then we have $\psi(t) < \max\{\frac{1}{\alpha}, \frac{1}{\beta}\}t$ for each $t > 0$. Moreover if $\psi$ be nondecreasing and satisfied $(S1)$ then $\lim_{n\to\infty} \phi^n(t) = 0$ for each $t > 0$, as $\phi$ introduced in $(S_2)$.

**Example 3.8.** $\psi(t) = (\min\{\frac{1}{\alpha}, \frac{1}{\beta}\})t$ and $\psi(t) = \ln(1+(\min\{\frac{1}{\alpha}, \frac{1}{\beta}\})t)$ are examples of admissible-contraction functions.

**Theorem 3.9.** Let $(X, d_{\alpha,\beta})$ be a complete $(\alpha, \beta)$-b-metric space and $T : X \to X$ satisfies

$$d_{\alpha,\beta}(Tx, Ty) \le \psi(d_{\alpha,\beta}(x, y)); \quad \forall x, y \in X;$$

where $\psi \in \Upsilon$. Then $T$ has a unique fixed point in $X$. Moreover, for every $x_0 \in X$, the recursive Picard sequence $x_n = Tx_{n-1}$ is converges to the fixed point of $T$.

**Proof .** For arbitrary $x_0 \in X$ consider the Picard iteration $x_n = Tx_{n-1}$. Then

$$d_{\alpha,\beta}(x_n, x_{n+1}) = d_{\alpha,\beta}(Tx_{n-1}, Tx_n) \le \psi(d_{\alpha,\beta}(x_{n-1}, x_n)). \tag{3.2}$$

If for some $n_0 \in \mathbb{N}$, $d_{\alpha,\beta}(x_{n_0-1}, x_{n_0}) = 0$ then obviously $x_{n_0}$ be a fixed point of $T$. Hence we suppose that for each $n$, $d_{\alpha,\beta}(x_{n_0-1}, x_{n_0}) \ne 0$, thus from (3.2) and due to the properties of $\psi$ we obtain $d_{\alpha,\beta}(x_n, x_{n+1}) < d_{\alpha,\beta}(x_{n-1}, x_n)$, which means that the positive sequence $c_n := d_{\alpha,\beta}(x_n, x_{n+1})$ is decreasing and so converges to some $c \ge 0$. If $c > 0$, in the case where the function $\psi$ has condition $(S_1)$, from (3.2) and due to the uppersemi continuity of $\psi$, we have $c \le \psi(c)$, where contradicts with conditions $(S_1)$; hence $c = 0$.

Further, in the case where the function $\psi$ has condition $(S_2)$, without loose of generality suppose $\min\{\alpha, \beta\} = \alpha$. From (3.2) and due to the monotonicity of $\psi$ we have

$$\alpha d_{\alpha,\beta}(x_n, x_{n+1}) \le \phi(d_{\alpha,\beta}(x_{n-1}, x_n)) \le \phi(\alpha d_{\alpha,\beta}(x_{n-1}, x_n))... \le \phi^n(d_{\alpha,\beta}(x_0, x_1))$$

and so,

$$0 \le \alpha c < \limsup_{n\to\infty} \phi^n(d_{\alpha,\beta}(x_0, x_1) = 0,$$

thus $c = 0$. Hence for each $\varepsilon > 0$, there exists a positive integer $m = m(\varepsilon)$ such that for all $n \ge m$, $c_n \le \max\{\frac{\varepsilon-\alpha\psi(\varepsilon)}{\beta}, \frac{\varepsilon-\beta\psi(\varepsilon)}{\alpha}\}$. Without loose of generality, suppose $c_n \le \frac{\varepsilon-\alpha\psi(\varepsilon)}{\beta}$, for all $n \ge m$. Let

$$M_\varepsilon := \{x \in X; d_{\alpha,\beta}(x, x_{m(\varepsilon)}) \le \varepsilon\}.$$

If $y \in M_\varepsilon$ we prove that $Ty \in M_\varepsilon$. Indeed,

$$\begin{aligned} d_{\alpha,\beta}(Ty, x_{m(\varepsilon)}) &\le \alpha d_{\alpha,\beta}(Ty, Tx_{m(\varepsilon)}) + \beta d_{\alpha,\beta}(x_{m(\varepsilon)+1}, x_{m(\varepsilon)}) \\ &\le \alpha\psi(d_{\alpha,\beta}(y, x_{m(\varepsilon)})) + \beta c_{m(\varepsilon)} \\ &\le \alpha\psi(\varepsilon) + \beta\frac{\varepsilon - \alpha\psi(\varepsilon)}{\beta} = \varepsilon. \end{aligned}$$

Obviously $x_{m(\varepsilon)} \in M_\varepsilon$, therefore $x_{m(\varepsilon)+1} = Tx_{m(\varepsilon)} \in M(\varepsilon)$. Hence we deduce that for each $n \ge m(\varepsilon)$; $x_n \in M(\varepsilon)$, i.e., $d_{\alpha,\beta}(x_n, x_{m(\varepsilon)}) \le \varepsilon$. Thus $\{x_n\}$ is a Cauchy sequence in the complete $(\alpha,\beta)$-b-metric space $X$ and so it converges to some $x^* \in X$. We will show that $x^*$ is the unique fixed point of $T$.

$$\begin{aligned} d_{\alpha,\beta}(x^*, Tx^*) &\le \alpha d_{\alpha,\beta}(x^*, Tx_n) + \beta d_{\alpha,\beta}(Tx_n, Tx^*) \\ &\le \alpha d_{\alpha,\beta}(x^*, x_{n+1}) + \beta\psi(d_{\alpha,\beta}(x_n, x^*)). \end{aligned}$$

Since the right hand side of the above inequality converge to zero as $n$ tends to infinity, we derive that $d_{\alpha,\beta}(x^*, Tx^*) = 0$. Foe uniqueness, suppose for tow distinct $x^*, y^* \in X$ we have $T(x^*) = x^*$ and $T(y^*) = y^*$ then

$$d_{\alpha,\beta}(x^*, y^*) = d_{\alpha,\beta}(Tx^*, Ty^*) \le \psi(d_{\alpha,\beta}(x^*, y^*)) \le \min\{\frac{1}{\alpha}, \frac{1}{\beta}\}d_{\alpha,\beta}(x^*, y^*).$$

Thus $\min\{\frac{1}{\alpha}, \frac{1}{\beta}\} \ge 1$ where is a contradiction by the assumptions. Hence the fixed point problem has a unique solution in $X$. □

**Remark 3.10.** In the metric space, since $\alpha = \beta = 1$, Theorem 3.9 coincide with the Boyd-Wong fixed point Theorem; see Theorem 1.7 and Theorems 1.8 in [5].

**Example 3.11.** Let $X := [0, \frac{1}{2}] \cup \{1\}$ and $T : X \to X$ with $Tx = \begin{cases} 1; & x \in \mathbb{Q} \cap X \setminus \{1, 0\}; \\ 0; & x \in \{1, 0\} \cup \mathbb{Q}' \cap X \end{cases}$. Then obviously $T$ doesn't satisfies the Banach contraction in the metric space $(X, d)$; where $d(x, y) = |x - y|$. Indeed, if $1 \ne x \in X \cap \mathbb{Q}, y \in X \cap \mathbb{Q}'$ we have

$$d(Tx, Ty) = 1 \nleq k|x - y| = kd(x, y) < \frac{k}{2},$$

for each $k \in [0, 1]$. Whereas, if we equipped $X$ with a $(\alpha, \beta)$-b-metric such as

$$d_{\alpha,\beta}(x, y) = \begin{cases} e^{-|x-y|}; & x, y \in (0, \frac{1}{2}]; \\ 1; & y \in \{0, 1\}, x \in (0, \frac{1}{2}]; \\ e^{-\frac{3}{2}}, & (x, y) = (0, 1); \\ 0; & x = y. \end{cases}$$

then $(X, d_{\alpha,\beta})$ is a complete $(\alpha, \beta)$-b-metric space with $\alpha = e^{\frac{1}{2}}$ and $\beta = 1$; Moreover, Then $T$ satisfies

$$d_{\alpha,\beta}(Tx, Ty) \le e^{-\frac{1}{2}} d_{\alpha,\beta}(x, y) \tag{3.3}$$

that Theorem 3.9 can be applied to show the existence of a unique fixed point, which is evidently zero. Indeed, if $x, y, z \in (0, \frac{1}{2}]$ then

$$e^{-|x-y|} \leq e^{|y-z|}e^{-|x-z|} < e^{\frac{1}{2}}e^{-|x-z|} + e^{-|y-z|},$$

which means $d_{\alpha,\beta}(x, y) \leq e^{\frac{1}{2}}d_{\alpha,\beta}(x, z) + d_{\alpha,\beta}(y, z)$. Moreover, if $x, y \in (0, \frac{1}{2}]$ and $z\{0, 1\}$, then $e^{-|x-y|} \leq e^{\frac{1}{2}} + 1 \leq$, which means $d_{\alpha,\beta}(x, y) \leq e^{\frac{1}{2}}d_{\alpha,\beta}(x, z) + d_{\alpha,\beta}(y, z)$. The other conditions of Definition 2.1 is obvioulу satisfied. Now we examine the validity of (3.3). For every $x, y \in X$ we have

$$d_{\alpha,\beta}(Tx, Ty) = \begin{cases} d_{\alpha,\beta}(1, 1) = 0; & x, y \in \mathbb{Q} - \{1, 0\}; \\ d_{\alpha,\beta}(0, 0) = 0; & x, y \in \mathbb{Q}'; \\ d_{\alpha,\beta}(1, 0) = e^{-\frac{3}{2}}; & x \in \mathbb{Q} - \{1, 0\}, y \in \mathbb{Q}'; \\ d_{\alpha,\beta}(0, 1) = e^{-\frac{3}{2}}; & x \in \{1, 0\}, y \in \mathbb{Q}; \\ d_{\alpha,\beta}(0, 0) = 0; & x \in \{1, 0\}, y \in \mathbb{Q}'. \end{cases}$$

Thus, evidently in all above cases the contraction (3.3) is satisfied.

**Definition 3.12.** Let $(X, d_{\alpha,\beta})$ be a complete $(\alpha, \beta)$-b-metric space. Then the mapping $T : X \to X$ is said to be $(\alpha, \beta)$-Meir-Keeler contraction if for any $\varepsilon > 0$, there exists $\delta > ((\max\{\alpha, \beta\})^3 - 1)\varepsilon$ such that for all $x, y \in X$,

$$\text{if } \ \epsilon \leq d_{\alpha,\beta}(x, y) < \varepsilon + \delta \ \text{ then } \ d_{\alpha,\beta}(Tx, Ty) < \varepsilon.$$

**Remark 3.13.** Obviously $(\alpha, \beta)$-Meir-Keeler contraction coincides with the Meir-Keeler contraction in a metric space $(X, d)$ with $\alpha = \beta = 1$; see [9].

**Remark 3.14.** From the definition 3.12, evidently we deduce that if $T$ be an $(\alpha, \beta)$-Meir-Keeler contraction then $d_{\alpha,\beta}(Tx, Ty) < d_{\alpha,\beta}(x, y)$, for every distinct $x, y \in X$ that are non fixed points.

**Theorem 3.15.** Let $(X, d_{\alpha,\beta})$ be a complete $(\alpha, \beta)$-b-metric space and let $T$ be a $(\alpha, \beta)$-Meir-Keeler contraction mapping. Then $T$ has a unique fixed point on $X$.

**Proof .** For arbitrary $x_0 \in X$ consider the Picard iteration $x_n = Tx_{n-1}$. If for some $n_0 \in \mathbb{N}$, $d_{\alpha,\beta}(x_{n_0-1}, x_{n_0}) = 0$ then obviously $x_{n_0}$ be a fixed point of $T$. Hence we suppose that for each $n$, $d_{\alpha,\beta}(x_{n_0-1}, x_{n_0}) \neq 0$. Regarding the Remark 3.14 we have

$$d_{\alpha,\beta}(x_n, x_{n+1}) = d_{\alpha,\beta}(Tx_{n-1}, Tx_n) < d_{\alpha,\beta}(x_{n-1}, x_n). \tag{3.4}$$

Thus $c_n := d_{\alpha,\beta}(x_n, x_{n+1})$ is a positive decreasing sequence where converges to some $c \geq 0$. If $c > 0$ then due to the assumption there exists $\delta > (\alpha^3 - 1)\varepsilon$ such that for all $x, y \in X$,

$$\text{if } \ c \leq d_{\alpha,\beta}(x, y) < c + \delta \ \text{ then } \ d_{\alpha,\beta}(Tx, Ty) < c. \tag{3.5}$$

For given $\delta$, since the decreasing sequence $c_n \to c$, there exists $m \in \mathbb{N}$, where

$$c \leq c_n \leq c + \delta; \quad \text{for all } \ n \geq m \tag{3.6}$$

Hence by choosing $x = x_n$, $y := x_{n+1}$ with $n \geq m$ in (3.5), we obtain $c_{n+1} < c$ which contradicts (3.6). Hence we conclude that $c_n \to 0$ as $n \to \infty$. We have to prove that $\{x_n\}$ is a Cauchy sequence. Otherwise, there exists $\varepsilon > 0$ such that for any $M > 0$, there exists $n > m > M$ such that $d_{\alpha,\beta}(x_n, x_m) > 2\alpha^3\varepsilon$. For given $\varepsilon$ due to the assumption there exists $\delta > (\alpha^3 - 1)\varepsilon$ such that

$$\text{if } \ \varepsilon \leq d_{\alpha,\beta}(x, y) < \varepsilon + \delta \ \text{ then } \ d_{\alpha,\beta}(Tx, Ty) < \varepsilon. \tag{3.7}$$

Since $c_n \to 0$, we can choose $M > 0$ such that for all $k \geq M$, $d_{\alpha,\beta}(x_k, x_{k+1}) < \eta$, where $0 < \eta \leq \dfrac{\min\{(1 - \alpha^3)\varepsilon + \delta, \alpha^3\varepsilon\}}{\alpha^2\beta + \alpha\beta - \beta}$.

Since $d_{\alpha,\beta}(x_n, x_m) > 2\alpha^3\varepsilon > (\alpha^2\beta + \alpha\beta + \beta)\eta + \alpha^3\varepsilon$ and moreover for each $k \in \{m, m + 1, ..., n\}$, we have

$$d_{\alpha,\beta}(x_m, x_{j+1}) \leq \alpha d_{\alpha,\beta}(x_m, x_j) + \beta d_{\alpha,\beta}(x_j, x_{j+1}) \leq \alpha d_{\alpha,\beta}(x_m, x_j) + \eta. \tag{3.8}$$

Hence we conclude that there exists $j \in \{m, m+1, ..., n\}$ such that

$$(\beta + \alpha\beta)\eta + \alpha^2\varepsilon < d_{\alpha,\beta}(x_m, x_j) < (\alpha^2\beta + \alpha\beta + \beta)\eta + \alpha^3\varepsilon < \varepsilon + \delta. \tag{3.9}$$

Hence by $(\alpha, \beta)$-Meir-Keeler condition we insert that $d_{\alpha,\beta}(x_{m+1}, x_{j+1}) < \varepsilon$. On the other hand, we have

$$\begin{aligned} d_{\alpha,\beta}(x_m, x_j) &\le \beta d_{\alpha,\beta}(x_m, x_{m+1}) + \alpha d_{\alpha,\beta}(x_{m+1}, x_j) \\ &\le \beta d_{\alpha,\beta}(x_m, x_{m+1}) + \alpha^2 d_{\alpha,\beta}(x_{m+1}, x_{j+1}) + \alpha\beta d_{\alpha,\beta}(x_{j+1}, x_j) \\ &\le (\beta + \alpha\beta)\eta + \alpha^2\varepsilon; \end{aligned}$$

which contradicts (3.9). This contradiction proves that $x_n$, must be a Cauchy sequence and so for some $x \in X$, $x_n \to x$ as $n \to \infty$. Obviously $x$ is a fixed point of $T$. Indeed, by Remark 3.14, T is a continuous operator, thus we have

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T(\lim_{n\to\infty} x_n) = Tx.$$

Further, by Remark 3.14, evidently we conclude that $T$ has a unique fixed point. □

**Remark 3.16.** In the metric space, since $\alpha = \beta = 1$, Theorem 3.15 coincide with the Meir-Keeler fixed point Theorem; see [9].

## 4 An application in the integral equations

We shall establish the existence of a solution to the following type of integral equation:

$$y(t) = \varsigma(t) + \int_0^1 \omega(t, s)\varrho(s, y(s))ds, \tag{4.1}$$

where $\varsigma : [0, 1] \to \mathbb{R}$, $\omega : [0, 1]^2 \to [-1, 1]$ and $\varrho : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions and for all $s \in [0, 1]$ and $y_1, y_2 \in \mathbb{R}$satisfies

$$ln(\frac{|\rho(s, y_1) - \rho(s, y_2)|}{1 + |\rho(s, y_1) - \rho(s, y_2)|}) + \frac{|\rho(s, y_1) - \rho(s, y_2)|}{1 + |\rho(s, y_1) - \rho(s, y_2)|} + 1 \le ln(\frac{|y_1 - y_2|}{1 + |y_1 - y_2|}) + \frac{|y_1 - y_2|}{1 + |y_1 - y_2|}. \tag{4.2}$$

**Proof .** Firstly we show that $C([a, b])$ with $(\alpha, \beta)$-b- metric $d_{\alpha,\beta}(y_1, y_2) = \frac{d(y_1,y_2)}{1+d(y_1,y_2)} e^{\frac{d(y_1,y_2)}{1+d(y_1,y_2)}}$, where $d(y_1, y_2) := \max\{|y_1(t) - y_2(t)|; t \in [0, 1]\}$, is an $(\alpha, \beta)$-b- metric space. Indeed,

$$\begin{aligned} d_{\alpha,\beta}(y_1, y_2) = \frac{d(y_1, y_2)}{1 + d(y_1, y_2)} e^{\frac{d(y_1,y_2)}{1+d(y_1,y_2)}} &\le \left(\frac{d(y_1, z)}{1 + d(y_1, z)} + \frac{d(z, y_2)}{1 + d(z, y_2)}\right) e^{\frac{d(y_1,y_2)}{1+d(y_1,y_2)}} \\ &\le \frac{d(y_1, z)}{1 + d(y_1, z)} \frac{1}{p} e^{\frac{d(y_1,z)}{1+d(y_1,z)}} e^{\frac{1}{q} \cdot \frac{d(z,y_2)}{1+d(z,y_2)}} + \frac{d(z, y_2)}{1 + d(z, y_2)} \frac{1}{q} e^{\frac{d(z,y_2)}{1+d(z,y_2)}} e^{\frac{1}{p} \cdot \frac{d(y_1,z)}{1+d(y_1,z)}} \\ &\le \frac{1}{p} e^{\frac{1}{q}} d_{\alpha,\beta}(y_1, z) + \frac{1}{q} e^{\frac{1}{p}} d_{\alpha,\beta}(z, y_2); \end{aligned}$$

where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Evidently, $C([a, b])$ with $d_{\alpha,\beta}$ is a complete $(\alpha, \beta)$-b- metric. Now, let the well defined operator $\Theta : C([a, b]) \to C([a, b])$ with $\Theta(y) = \varsigma(t) + \int_0^1 \omega(t, s)\varrho(s, y(s))ds$, we have to show that $\Theta$ satisfies the condition of the Theorem 3.9. From (4.2),

$$d_{\alpha,\beta}(\Theta(y_1), \Theta(y_2)) \le \frac{\int_0^1 |\omega(t, s)||\varrho(s, y_1(s)) - \varrho(s, y_2(s))|ds}{1 + \int_0^1 |\omega(t, s)||\varrho(s, y_1(s)) - \varrho(s, y_2(s))|ds} e^{\frac{\int_0^1 |\omega(t,s)||\varrho(s,y_1(s)) - \varrho(s,y_2(s))|ds}{1 + \int_0^1 |\omega(t,s)||\varrho(s,y_1(s)) - \varrho(s,y_2(s))|ds}}.$$

Since $\max_{t,s} |\omega(t, s)| \le 1$ and the function $f(t) = \frac{t}{1+t}$ is decreasing, we deduce that

$$\begin{aligned} d_{\alpha,\beta}(\Theta(y_1), \Theta(y_2)) &\le \frac{\sup_s(|\varrho(s, y_1(s)) - \varrho(s, y_2(s))|)}{1 + \sup_s(|\varrho(s, y_1(s)) - \varrho(s, y_2(s))|)} e^{\frac{\sup_s(|\varrho(s,y_1(s)) - \varrho(s,y_2(s))|)}{1 + \sup_s(|\varrho(s,y_1(s)) - \varrho(s,y_2(s))|)}} \\ &\le \sup_s \left( \frac{|\varrho(s, y_1(s)) - \varrho(s, y_2(s))|}{1 + |\varrho(s, y_1(s)) - \varrho(s, y_2(s)|)} e^{\frac{|\varrho(s,y_1(s)) - \varrho(s,y_2(s))|}{1 + |\varrho(s,y_1(s)) - \varrho(s,y_2(s)|)}} \right) \end{aligned} \tag{4.3}$$

On the other hand, according to (4.2), we deduce that

$$\frac{|\varrho(s,y_1(s))-\varrho(s,y_2(s))|}{1+|\varrho(s,y_1(s))-\varrho(s,y_2(s)|)}e^{\dfrac{|\varrho(s,y_1(s))-\varrho(s,y_2(s))|}{1+|\varrho(s,y_1(s))-\varrho(s,y_2(s)|)}} \leq e^{-1}\frac{|y_1-y_2|}{1+|y_1-y_2|}e^{\frac{|y_1-y_2|}{1+|y_1-y_2|}}. \tag{4.4}$$

Therefore from (4.3), (4.4) we derive

$$d_{\alpha,\beta}(\Theta(y_1),\Theta(y_2)) \leq e^{-1}d_{\alpha,\beta}(y_1,y_2).$$

Since for each $t>0$, $e^{-\frac{1}{t}}-t<0$, we have $e^{-1}<\max\{\frac{p}{e^{\frac{1}{q}}},\frac{q}{e^{\frac{1}{p}}}\}$. So the condition of the Theorem 3.9 is fulfilled with $\psi(t)=e^{-1}t$. $\square$

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