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New representations of the generalized uniform fuzzy partitions: Generalized normal case

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Abstract

In this research, new representations of basic functions are proposed based on the new types of fuzzy partition and a subnormal generating function. The generalized uniform fuzzy partitions in subnormal case, i.e. in case a generating function K is not normal (generalized normal case), and simpler form of fuzzy transform (FzT) components based on these new representations of the generalized uniform fuzzy partitions are indicated. The main properties of a new uniform fuzzy partition are suggested. New theorems and lemmas are proved.

Keywords: Fuzzy logic, Fuzzy partition, Fuzzy transform, Basic function, The membership functions, Generating function, Ruspini condition
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1 Introduction

FzT is depending basically on fuzzy set theory. The fuzzy set theory was coined by [37] with a suitable tool for modelling the uncertainty phenomenon. The following Ruspini condition

$$\sum_{i=1}^{n} A_i = 1, \text{ for all } x \in D.$$

was proposed by [31]. [21] was proposed the Mamdani's inference rule by a set of linguistic control rules. Thus, the author has a modified inference rule and introduced a method to create a Takagi-Sugeno model with fuzzy partition and linear consequent (rules with consequent being equal to linear expressions) from input-output data [33]. The Takagi-Sugeno fuzzy system computed as:

$$f(x) = \frac{\sum_{i=1}^{n} A_i(x).(c_i + d_i x)}{\sum_{i=1}^{n} A_i(x)}$$

where $c_i,\ d_i$ are real numbers. A satisfactory approximation by a Takagi-Sugeno fuzzy model can be obtained only by refining the partition of close interval [7]. Based on the above background, FzT has been established by [28].

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The difference between FzT and Takagi-Sugeno fuzzy model revealed that FzT provides much more flexible and a satisfactory approximation by leaving partition of close interval unchanged [25].

The core idea of FzT is a fuzzy partition of a universe into fuzzy subsets. The first fuzzy partition of FzT with the Ruspini condition was introduced by [28] and was extensively investigated by [24]. This condition implies normality of the fuzzy partition. Also, the fuzzy partition with the generalized Ruspini condition (fuzzy r-partition) was introduced by [32]. This fuzzy partition was achieved by replacing the partition of unity by fuzzy r-partition. This type of partition was used by [32] and [13] for smoothing or filtering data based on the inverse FzT. Further, a generalized fuzzy partition appeared in connection with the notion of the FzT, where FzT components are polynomials of degree m [27]. By [6], different types of fuzzy partitions are taken into consideration such as B-splines, Shepard kernels, Bernstein basis polynomials and Favard-Szasz-Mirakjan operators. Later, the higher degree FzT based on B-splines was proposed [20] to improve the quality of the function approximation of two variables.

A generalized fuzzy partition was implicitly introduced by [15] with the purpose of meeting the requirements of image compression. Also, a generalized fuzzy partition can also be considered in connection with radial membership functions [22]. Further, necessary and sufficient conditions for modeling the generalized fuzzy partition was provided by [12]. Recently, a new representation formula for basic functions of FzT and a new fuzzy numerical method based on block pulse functions for numerical solution of integral equations were presented by [17]. The approximation method based on the FzT with Shepard-type basic functions for linear Fredholm integral equations was discussed by [39]. New representations of the generalized uniform fuzzy partitions with the normal case to obtain better approximation solutions for solving Cauchy problems were appeared by [4, 2, 3].

FzT is a soft computing method has been developed by Perfilieva [24] that has many applications, for example, in differential and integral equations. FzT for solving ordinary Cauchy problems with one variable was initiated by [28]. Generalization of the Euler method has been discussed by [23] for solving ordinary Cauchy problems. The author has been applied this technique to reef growth and sea level variations models. Further, FzT has been generalized from the case of constant components to the case of polynomial components by [27]. Later, the first and second degree FzT based mid-point rule for solving Cauchy problem and uncertain initial value problem have proposed by [19]. Furthermore, an algorithm to obtain the approximate solutions of second order initial value problems was constructed by [8]. From this idea, FzT for numerical solutions of two point boundary value problems was proposed by [18].

FzT of two variables based on finite differences method was used by [14] for solving a type of partial differential equations with Dirichlet boundary conditions and initial conditions. Also, the first degree FzT of two variables was introduced by [11]. By [29], the partial derivatives using the first FzT were approximated and modification of the Canny edge detector was proposed. Furthermore, the uniform stability result for the vibrations of a telegraph equation using FzT of two variables was proposed by [10]. The composition of inverse and direct discrete FzT method was extended to numerical solution of Fredholm integral equations and Volterra Fredholm integral equations [9]. The general form of the higher order FzT was constructed by [38] for solving differential and integral equations using any arbitrary basis functions. The FzT has investigated for solving Volterra population growth model using the approximation for the Caputo derivative [5]. A new numerical method based FzT was demonstrated to solve a class of delay differential equations by mean of the Picard-like numerical scheme [34]. FzT was considered to approximate solution of boundary value problems by minimizing the integral squared error in 2-norm [1]. In [35], the dynamical properties of a two neuron system with respect to FzT and a single delay have been investigated. The conditions under which quasi-consensus in a multi-agent system with sampled data based on FzT were proposed by [36].

The motivation of the proposed study comes from [4, 2, 30, 3]. In [4], new fuzzy numerical methods to solve Cauchy problem was considered and the authors showed that the error can be reduced by FzT and NIM with respect to new generalized uniform fuzzy partitions, namely power of the triangular and raised cosine generalized uniform fuzzy partitions, where generating functions are normal (see also [30] for another approach). Also, two basic approximation methods, modified Euler method and Trapezoidal rule, with help of FzT for solving SODEs are analyzed in detail by [2, 3]. For this purpose, more generally, new generalized uniform fuzzy partitions are proposed in this study, where a generating function is not normal.

The membership functions in underlying fuzzy partitions are often called basic functions. There has been a growing interest in investigating the properties of fuzzy partitions. However, the problem arises on how one can effectively construct the basic function of fuzzy partitions. In this paper, new representations of basic functions are proposed. This is achieved by introducing new generalized uniform fuzzy partitions, where a generating function is not normal.

The paper is organized as follows. The main part of the paper is Section 3, new representations of basic functions, including new representations of the generalized uniform fuzzy partitions in generalized normal case, i.e. in case a generating function K is not normal, and simpler form of FzT components based on these new representations of the generalized uniform fuzzy partitions Later, fuzzy partitions models with the Ruspini condition. Concluding remarks are summarized in Section 4.

2 Basic Concepts

In this section, we give some definitions and introduce the necessary notation following [26], which will be used throughout the paper. Throughout this section, we deal with an interval $[a, b] \subset \mathbb{R}$ of real numbers.

Definition 2.1. (generalized uniform fuzzy partition) Let $t_i \in [a, b]\,,\ i = 1, \ldots, n$, be fixed nodes such that $a = t_1 < \ldots < t_n = b,\ t_0 = t_1,\ t_n = t_{n+1},\ n \geq 2$ and $[t_i - h, t_i + h] \subseteq [a, b]$. We say that the fuzzy sets $A_i : [a, b] \to [0, 1]$ constitute a generalized fuzzy partition of $[a, b]$ if the following conditions are fulfilled:

1. (positivity and locality) – $A_i\,(t) > 0$ if $t \in (t_{i-1}, t_{i+1})$ and $A_i\,(t) = 0$ if $t \in [a, b] \setminus (t_{i-1}, t_{i+1})$;
2. (continuity) – A_i is continuous on $[t_{i-1}, t_{i+1}]$;
3. (covering) – for $t \in [a, b]\,,\ \sum_{i=1}^{n} A_i(t)\ > 0$.

Fuzzy sets $A_1, \ldots,\ A_n$ are called basic functions. It is important to remark that by conditions of locality and continuity, $\int_a^b A_i(t)dt > 0$. A generalized uniform fuzzy partition of $[a, b]$ is defined for equidistant nodes, i.e., for all $i = 1, \ldots, n-1,\ t_i = t_{i+1} + h$, where $h = (b - a)\,/\,(n - 1)$ and two additional properties are satisfied,

4. $A_i\,(t_i - t) = A_i\,(t_i + t)$ for all $t \in [0, h]\,,\ i = 2, \ldots, n-1$;
5. $A_i\,(t) = A_{i-1}\,(t - h)$ and $A_{i+1}\,(t) = A_i\,(t - h)$ for all $t \in [t_i,\, t_{i+1}]\,,\ i = 2, \ldots, n-1$;

then the fuzzy partition is called h-uniform generalized fuzzy partition.

Definition 2.2. (generating function) A function $K : [-1, 1] \to [0, 1]$ is called a generating function if it is assumed to be even, continuous and $K\,(t) > 0$ if $t \in (-1, 1)$. The function $K : [-1, 1] \to \mathbb{R}$ is even if for all $t \in [0, 1]\,,\ K\,(-t) = K\,(t)$.

The following definition recall the concept of the generalized fuzzy partition which can be easily extended to the interval $[a, b]$. We assume that $[a, b]$ is partitioned by $A_1, \ldots,\ A_n$, according to Definition 2.1.

Definition 2.3. A h-uniform generalized fuzzy partition of interval $[a, b]$, determined by the triplet $(K,\, h,\, a)$, can be defined using generating function K (Definition 2.2). Then, basic functions of a h-uniform generalized fuzzy partition are shifted copies of K defined by

$$A_i\,(t) = K\left(\frac{t - t_i}{h}\right), \quad t \in [t_i - h,\, t_i + h]\,,$$

for all $i = 1, \ldots, n$. The parameter h is called the bandwidth or the shift of the fuzzy partition and the nodes $t_i = a + ih$ are called the central point of the fuzzy sets $A_1, \ldots,\ A_n$.

Remark 2.4. A h-uniform fuzzy partition is called Ruspini if the following condition

$$A_i\,(t) + A_{i+1}\,(t) = 1,\ i = 1, \ldots, n-1, \tag{2.1}$$

holds for any $t \in [t_i, t_{i+1}]$. This condition is often called Ruspini condition.

3 New Representations for Basic Functions of FzT

Let us recall the basic facts of an FzT of a continuous real function f as presented by [23, 24]. The first step in the definition of the FzT of f involves the selection of a fuzzy partition of the domain $[a, b]$ by a finite number $n \geq 2$ of fuzzy sets $B_k(t),\ k = 1, \ldots, n$. In those papers, five axioms specified $B_k(t),\ k = 1, \ldots, n$, in the fuzzy partition: normality, locality, continuity, unimodality (monotonicity) and orthogonality (Ruspini condition). A fuzzy partition is called uniform if the fuzzy sets $B_k(t),\ k = 2, \ldots, n-1$, are shifted copies of symmetrized B_1 (More details can be found in [23]). The membership functions $B_k(t),\ k = 1, \ldots, n$, in a fuzzy partition are called basic functions.

Later, a generalized fuzzy partition appeared in connection with the notion of a higher-degree FzT [27]. Furthermore, summarize both these notions in [26]. Three axioms specify $B_k(t)$, $k = 1, \ldots, n$, in the fuzzy partition: positivity and locality, continuity and covering. Recently, the different conditions for generalized uniform fuzzy partitions was proposed [12, 26] while another approach was demonstrated by [30] where a function can be reconstructed from its F-transform components. In the following, we modify the definition h-uniform generalized fuzzy partition.

3.1 Generalized Uniform Fuzzy Partitions with the Generalized Normal Case

Let us recall the h-uniform generalized fuzzy partition of real line can be defined using generating function K. Then, basic functions of the h-uniform generalized fuzzy partition are shifted copies of K. On the basis of Definition 2.1 can be also defined using a generating function $\lambda\beta K(t)$ where $\beta = 1/K(0)$, $K(0) \neq 0$, $\beta > 0$ and $\lambda > 0$ (in general, not necessarily satisfy normal and Ruspini condition) which is that $K(t)$ assumed to be even, continuous and $K(t) > 0$ if $t \in (-1, 1)$. Therefore, we will modify the basic functions of the h-uniform generalized fuzzy partition are shifted copies of $\lambda\beta K$ defined by

$$A_k(t, t_0) = \lambda\beta K\left(\frac{t - t_0}{h} - k\right), \ t \in [t_k - h, t_k + h], \ k \in \mathbb{Z}. \tag{3.1}$$

The parameter h is bandwidth of the fuzzy partition and $t_0 + kh = t_k$. The concept of the h-uniform generalized fuzzy partition can be easily extended to the interval $[a, b]$ as follows.

Definition 3.1. Let $t_1 < \ldots < t_n$ be fixed nodes within $[a, b] \subset \mathbb{R}$, such that $t_1 = a$, $t_n = b$ and $n \geq 2$. We consider nodes $t_1, \ldots, t_n$ are equidistant, with distance (shift) $h = (b - a)/(n - 1)$. A system of fuzzy sets $B_1, \ldots, B_n$: $[a, b] \to [0, 1]$ be a generalized uniform fuzzy partitions of $[a, b]$ if it is defined by

$$B_k(t) = \begin{cases} A_k(t, a), & t \in [a, b], \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} \lambda\beta K\left(\frac{t - t_k}{h}\right), & t \in [a, b], \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

where $t_k = a + (k - 1)h$, $\beta = 1/K(0)$, $K(0) \neq 0$, $\beta > 0$ and $\lambda > 0$. In the sequel, a generating function denote by K and basic functions of FzT denote by B_k, $k = 1, \ldots, n$.

Lemma 3.2. If basic functions B_k, $k = 1, \ldots, n$, of a h-uniform generalized fuzzy partition are shifted copies of $\lambda\beta K$ defined by (3.2). Then, for each $k = 1, \ldots, n$, $B_k(t_k) = \lambda$, $t_k \in [t_k - h, t_k + h]$.

Proof . By (3.2), we get $B_k(t_k) = \lambda\beta K\left(\frac{t_k - t_k}{h}\right) = \lambda$. □

3.2 Simpler form of F-transform Components Based on Generalized Uniform Fuzzy Partitions with the Generalized Normal Case

In this subsection, we present the main principles of FzT with respect to new representations of h-uniform generalized fuzzy partition. Further, we will show that FzT components with respect to new representations of h-uniform generalized fuzzy partition can be simplified and approximated of an original function, say f.

Definition 3.3. Let f be a continuous function on $[a, b]$ and $B_k(t)$, $k = 1, \ldots, n$, be h-uniform generalized fuzzy partition of $[a, b]$, $n \geq 2$. A vector of real numbers $F[f] = (F_1, F_2, \ldots, F_n)$ given by

$$F_k = \frac{\int_a^b f(t) B_k(t)\, dt}{\int_a^b B_k(t)\, dt}, \tag{3.3}$$

for $k = 1, \ldots, n$ is called the direct FzT of f with respect to B_k.

In the following, we will simplify the representation (3.3).

Lemma 3.4. Let $f \in C([a, b])$ and according to Definition 3.1, fuzzy sets B_k, $k = 1, \ldots, n$, $n \geq 2$, be a h-uniform generalized fuzzy partition of $[a, b]$ with a generating function K, then representation (3.3) of direct FzT can be simplified for $k = 1, \ldots, n$ as follows

$$F_k = \frac{\int_{-1}^{1} f(th + t_k) K(t)\, dt}{\int_{-1}^{1} K(t)\, dt} = \frac{\int_{-h}^{h} f(t + t_k) K(\frac{t}{h})\, dt}{\int_{-h}^{h} K(\frac{t}{h})\, dt}. \tag{3.4}$$

Proof . By Definition 3.1, we get

$$B_k(t) = \lambda\beta K\left(\frac{t-t_k}{h}\right), \quad t \in [t_k - h, t_k + h],$$

for $k = 1, \ldots, n$,$t_0 = t_1$, $t_{n+1} = t_n$, and substituting $u = \frac{t-t_k}{h}$ and then substituting $t = s/h$. Thus, we get

$$\int_{t_{k-1}}^{t_{k+1}} f(t)\, B_k(t)\, dt = \lambda\beta h \int_{-1}^{1} f(th + t_k)\, K(t)\, dt = \lambda\beta \int_{-h}^{h} f(t + t_k)\, K(\frac{t}{h})\, dt,$$

$$\int_{t_{k-1}}^{t_{k+1}} B_k(t)\, dt = \lambda\beta h \int_{-1}^{1} K(t)\, dt = \lambda\beta \int_{-h}^{h} K(\frac{t}{h})\, dt,$$

and its corresponding results with representation (3.3). $\square$

If $\lambda > 0$, the lemma 3.2 still hold by choosing suitable constant λ, satisfying $\lambda = 1/\left(\int_{-1}^{1} \beta K(t)dt\right)$, where $\int_{-1}^{1} \beta K(t)dt > 0$. So, we will restrict ourselves to h-uniform generalized fuzzy partition with $0 < \lambda = 1/\left(\int_{-1}^{1} \beta K(t)dt\right)$, where $\int_{-1}^{1} \beta K(t)dt \neq 0$. In the following, we will simplify the above given expressions for the coefficients $F[f] = (F_1,\ F_2, \ldots,\ F_n)$ in the representation (3.3). This fact is very important for applications which are more flexible and consequently easier to use.

Corollary 3.5. Let the assumptions of Lemma 3.4 be fulfilled and $0 < \lambda = 1/\left(\int_{-1}^{1} \beta K(t)dt\right)$, where $\int_{-1}^{1} \beta K(t)dt \neq 0$. Then, the coefficients $F[f] = (F_1,\ F_2, \ldots,\ F_n)$ in the expression (3.3) of the FzT component F_k of f as follows:

$$F_k = \frac{1}{h}\int_a^b f(t)\, B_k(t)\, dt = \frac{\lambda\beta}{h}\int_a^b f(t)\, K\left(\frac{t-t_k}{h}\right) dt, \tag{3.5}$$

for $k = 1, \ldots, n$, where interval $[a, b]$ is partitioned by the h-uniform generalized fuzzy partition $B_1, \ldots, B_n$.

Proof . Let $k \in \{1, \ldots, n\}$ and consider set of fuzzy sets $B_k(t)$ be the h-uniform generalized fuzzy partition of $[a, b]$ defined by (3.2). Using the proof of Lemma 3.4, we get

$$\int_{t_{k-1}}^{t_{k+1}} B_k(t)\, dt = \int_{t_{k-1}}^{t_{k+1}} A_k(t, a), dt = \int_{t_k-h}^{t_k+h} \lambda\beta K\left(\frac{t-t_k}{h}\right) dt = h\lambda \int_{-1}^{1} \beta K(t)\ dt = h, \tag{3.6}$$

where $0 < \lambda = 1/\left(\int_{-1}^{1} \beta K(t)dt\right)$, $\int_{-1}^{1} \beta K(t)dt \neq 0$, h is bandwidth of the fuzzy partition and $t_k = a + (k-1)h$ and then its corresponding in the expression (3.3). $\square$

Lemma 3.6. Let $f \in C[a, b]$. Then for any $\varepsilon > 0$ there exist $n_\varepsilon \in \mathbb{N}$ and $B_1, \ldots, B_{n_\varepsilon}$ be basic functions form the h-uniform generalized fuzzy partition of $[a, b]$. Let F_k, $k = 1\ldots, n$, be the integral FzT components of f with respect to $B_1, \ldots, B_{n_\varepsilon}$. Then for each $k = 1\ldots, n_\varepsilon - 1$ the following estimations hold: $|f(t) - F_i| \leq \varepsilon$ for each $t \in [a, b] \cap [t_k, t_{k+1}]$ and $i = k,\ k+1$.

Proof . see [24]. $\square$

Corollary 3.7. Let the conditions of Lemma 3.6 be fulfilled. Then for each $k = 1\ldots, n_\varepsilon - 1$ the following estimations hold: $|F_k - F_{k+1}| < \varepsilon$.

Proof . According to [24, 16], let $t \in [a, b] \cap [t_k, t_{k+1}]$. Then by Lemma 3.6, for any $k = 1, \ldots, n-1$ we obtain

$$|f(t) - F_k| < \varepsilon/2 \quad \text{and} \quad |f(t) - F_{k+1}| < \varepsilon/2.$$

Thus,

$$|F_k - F_{k+1}| \leq |f(t) - F_k| + |f(t) - F_{k+1}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\square$

The following theorem estimates the difference between the original function and its direct FzT with respect to the h-uniform generalized fuzzy partition.

Theorem 3.8. Let $f(t) \in C^2[a,b]$ and the conditions of Lemma 3.4 be fulfilled. Then for $k = 1, \ldots, n$

$$F_k = \lambda f(t_k) + \mathcal{O}(h^2), \tag{3.7}$$

where $0 < \lambda = 1/\left(\int_{-1}^{1} \beta K(t)dt\right)$ and $\int_{-1}^{1} \beta K(t)dt \neq 0$.

Proof . By locality condition of definition of h-uniform generalized fuzzy partition, Corollary 3.5, Lemma 3.2, and according to [23], using the trapezoid formula with nodes t_{k-1}, t_k, t_{k+1} to the numerical computation of the integral, we get for $k = 1, \ldots, n$ and $0 < \lambda = 1/\left(\int_{-1}^{1} \beta K(t)dt\right)$

$$\begin{aligned} F_k &= \frac{1}{h}\int_{t_{k-1}}^{t_{k+1}} f(t)\, B_k(t)\, dt, \\ &= \frac{1}{h}\frac{h}{2}\left(f(t_{k-1})\, B_k(t_{k-1}) + 2f(t_k)\, B_k(t_k) + f(t_{k+1})\, B_k(t_{k+1})\right) + \mathcal{O}(h^2), \\ &= f(t_k)\, B_k(t_k) + \mathcal{O}(h^2) = \lambda f(t_k) + \mathcal{O}(h^2). \end{aligned} \tag{3.8}$$

□

Corollary 3.9. Let $f(t) \in C^2[a,b]$ and the conditions of Lemma 3.4 be fulfilled. Let moreover, f be Lipschitz continuous with respect to t, i.e. there exists a constant $L \in \mathbb{R}$, such that for all $t \in [a,\, b]$ and $t, t' \in \mathbb{R}$,

$$|f(t) - f(t')| \leq L|t - t'|. \tag{3.9}$$

Then for $k = 1, \ldots, n$

$$\left| f(t) - \frac{1}{\lambda}F_k \right| \leq Lh + \frac{h^2}{6\lambda}M,$$

where $0 < \lambda = 1/\left(\int_{-1}^{1} \beta K(t)dt\right)$, $\int_{-1}^{1} \beta K(t)dt \neq 0$, $M = \max\limits_{t\in[t_{k-1},\, t_{k+1}]} |f''(t)|$ and $|t - t_k| < h$ whenever $t \in [t_{k-1},\, t_{k+1}]$.

Proof . By the assumption f has continuous second order derivatives on $[a,b]$ and is Lipschitz continuous with respect to t. Therefore, using the trapezoid rule and let us choose a value of k in the range $1 \leq k \leq n$ and $t \in [t_{k-1},\, t_{k+1}]$, we get for $0 < \lambda = 1/\left(\int_{-1}^{1} \beta K(t)dt\right)$

$$\begin{aligned} \left| f(t) - \frac{1}{\lambda}F_k \right| &= \left| f(t) - \frac{1}{h\lambda}\int_{t_{k-1}}^{t_{k+1}} f(t)\, B_k(t)\, dt \right| \\ &= \left| f(t) - \frac{1}{h\lambda}\left[h\lambda f(t_k) - \frac{h^3}{12}\left(f''(\xi_{k-1}) + f''(\xi_{k+1})\right)\right]\right| \\ &\leq |f(t) - f(t_k)| + \frac{h^2}{12\lambda}2M \\ &\leq L\,|t - t_k| + \frac{h^2}{6\lambda}M \ \leq Lh + \frac{h^2}{6\lambda}M, \end{aligned} \tag{3.10}$$

where $\xi_{k-1} \in (t_{k-1}, t_k)$, $\xi_{k+1} \in (t_k, t_{k+1})$ and $M = \max\limits_{t\in[t_{k-1},\, t_{k+1}]} |f''(t)|$. □

Remark 3.10. In view of (3.10), if $0 < \lambda \leq 1$. Then, $\left| f(t) - \frac{1}{\lambda}F_k \right| \leq Lh + \frac{h^2}{6}M$.

Definition 3.11. Let $F[f] = (F_1,\ F_2, \ldots,\ F_n)$ be direct FzT of a function $f \in C[a,b]$ with respect to the fuzzy partition $B_k(t)$, $k = 1, \ldots, n$ of $[a,b]$. Then, the function $\hat{f}$ defined on $[a,b]$

$$\hat{f}(t) = \frac{\sum_{k=1}^{n} F_k B_k(t)}{\sum_{k=1}^{n} B_k(t)}, \tag{3.11}$$

is called the inverse FzT of f.

The following lemma estimates the difference between the original function and its inverse FzT.

$K_{C_2^m}(t)$	β	λ	$B_k = \lambda\beta K\left(\frac{t-t_k}{h}\right)$
$(1+\cos(\pi t))^m$	$\frac{1}{2^m}$	$\frac{\sqrt{\pi}\Gamma(m+1)}{2\Gamma\left(m+\frac{1}{2}\right)}$	$\left(\frac{\sqrt{\pi}\Gamma(m+1)}{2\Gamma\left(m+\frac{1}{2}\right)}\right)\frac{1}{2^m}\left(1+\cos\left(\pi\frac{t-t_k}{h}\right)\right)^m$

Table 1: Example 3.1

Lemma 3.12. Let the assumptions of Theorem 3.8 and let $\hat{f}(t)$ be the inverse FzT of f with respect to the fuzzy partition of $[a,b]$ is given by Definition 3.1 . Then, the following estimation holds for $t \in [a,b]$ and $k = 1,\ldots,n$

$$\hat{f}(t) = \lambda f(t_k) + \mathcal{O}\left(h^2\right), \tag{3.12}$$

where $0 < \lambda = 1/\left(\int_{-1}^{1}\beta K(t)dt\right)$ and $\int_{-1}^{1}\beta K(t)dt \neq 0$.

Proof . Let $t \in [a,b]$ so that $t \in [t_k, t_{k+1}]$ for some $k = 1,\ldots,n$. By Theorem 3.8,

$$\begin{aligned}\hat{f}(t) - \lambda f(t_k) &= \frac{\sum_{k=1}^{n} F_k B_k(t)}{\sum_{k=1}^{n} B_k(t)} - \lambda f(t_k) = \frac{\sum_{k=1}^{n} F_k B_k(t)}{\sum_{k=1}^{n} B_k(t)} - \frac{\sum_{k=1}^{n} \lambda f(t_k) B_k(t)}{\sum_{k=1}^{n} B_k(t)},\\ &= \frac{\sum_{k=1}^{n}(F_k - \lambda f(t_k)) B_k(t)}{\sum_{k=1}^{n} B_k(t)} = \mathcal{O}\left(h^2\right).\end{aligned}$$

□

Theorem 3.13. Let $f \in C[a,b]$. Thus for any $\varepsilon > 0$ there exist $n_\varepsilon \in \mathbb{N}$ and $B_1,\ldots,B_{n_\varepsilon}$ be the h-uniform generalized fuzzy partition of $[a,b]$ defined by (3.2). Then, the following estimations hold $\left|\hat{f}(t) - f(t)\right| < \varepsilon$ for each $t \in [a,b] \cap [t_k, t_{k+1}]$.

Proof . From the proof of Lemma 3.12 and then using Lemma (3.6) in the sense that for all $k = 1,\ldots,n$,

$$\left|\hat{f}(t) - f(t)\right| = \frac{\sum_{k=1}^{n}|F_k - f(t)| B_k(t)}{\sum_{k=1}^{n} B_k(t)} < \varepsilon.$$

□

Remark 3.14. According to Definition (3.1), it is easy to see that the inverse FzT $\hat{f}(t_k) = F_k$ for all $k = 1,\ldots,n$.

On the basis of Definition 3.1, necessary steps of a new method to construct generalized uniform fuzzy partitions of $[-1,1]$ for solve case K is not normal in the following.

1. Select the generating function K which is assumed to be even, continuous and $K(t) > 0$ if $t \in (-1,1)$.
2. Specify the value $\beta = 1/K(0)$, where $K(0) \neq 0$ to get the normal generating function K and then compute the value $\lambda = 1/\left(\int_{-1}^{1}\beta K(t)dt\right)$, where $\int_{-1}^{1}\beta K(t)dt \neq 0$.
3. If conditions $\beta > 0$ and $\lambda > 0$ holds, then construct generalized uniform fuzzy partitions of $[-1,1]$ by $\lambda\beta K(t)$.

Example 3.1. Let $K : \mathbb{R} \to [0,1]$ be defined by

$$K(t) = (1+\cos(\pi t))^m.$$

One can see in Tabel 1 the h-uniform generalized fuzzy partition of $[a,b]$ determined by Definition 3.1.

The following remark need for modified Trapezoidal rule based on FzT and NIM to solve SODEs.

Remark 3.15. In view of Eq. (3.6), $\int_{t_{k-1}}^{t_{k+1}} B_k(t)\,dt = h$. This means that $\int_{t_k}^{t_{k+1}} B_k(t)\,dt = \frac{h}{2}$.

Important property of the direct FzT as well as inverse FzT is their linearity, namely, given $f,\ g \in C[a,b]$ and $\alpha,\ \beta \in R$, if $h = \alpha f + \beta g$, then $F[h] = \alpha F[f] + \beta F[g]$ and $\hat{h} = \alpha\hat{f} + \beta\hat{g}$.

4 Conclusion

More generally, a generating function K can be also defined by K is not normal and in general, not necessarily satisfying normal and Ruspini condition. Therefore, the basic functions of the uniform generalized fuzzy partitions were modified in this paper. In this case, the main principles of FzT with respect to new representations of generalized uniform fuzzy partition were modified. Also, FzT components were simplified and approximated of an original function with respect to new representations of generalized uniform fuzzy partition. Necessary steps of a new method to construct generalized uniform fuzzy partitions were presented for solving case K is not normal. Finally, the procedures of uniform fuzzy partitions models with the Ruspini condition were proposed.

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