Int. J. Nonlinear Anal. Appl. 16 (2025) 2, 219–234
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.32273.4791

Analysis of student's understanding of the concept of derivative with a discrete approach

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(Communicated by Majid Eshaghi Gordji)

Abstract

This paper studies the engineering students' understanding of the concept of derivative with a discrete approach using sequences introduced by Weigand [21]. In this approach, a step-by-step method of difference sequence for functions defined on Z and Q is proposed. This concept was taught as part of a mathematics course in an engineering college in an Iranian university. HomeWorks with questions based on derivative with a discrete approach were constructed and performed for the participants. Their written answers, which were used to explore the students' mental structures of these mentioned concepts, were analyzed using APOS (Action-Process-Object-Schema) theory and we performed interviews so that the students could explain about their written answers. The results show that students tend to adopt an algorithmic approach when solving derivative problems and students' understanding of the meaning of derivative with a discrete approach was mainly procedural. Also according to the observed mental structures, a suggested genetic decomposition for the derivative with a discrete approach is presented.

Keywords: derivative, difference sequences, APOS theory, genetic decomposition
2020 MSC: 97D70, 97E10

Introduction

One of the most important basic notions in calculus is derivative that depends on the concept of function, limit and rate of change. Giraldo et al. [15] state that there is a difference between explaining a concept that defines some of its features and the formal definition of the concept. They referred to the common description of a function derivative at point a in the domain of the function, which is the slope of the tangent line to the function graph at point $(a, f(a))$. Another common description derivative is the local rate of change, while the derivative physical meaning refers to the speed and acceleration of a moving object in a moment. In contrast to the formal definition of the derivative of the function $f(x)$ at the point $x = a$ is equal to $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$. For most students, it is difficult to understand the formal definitions of the derivative and limit as they are not able to apply the exact definitions in different situations and are only able to solve problems at intuitive level and do not have a deeper understanding of the concepts. Research has shown that the concept of derivative is one of the most difficult concepts in calculus due to the

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Received: September 2023 *Accepted:* December 2023

complexity of its definition [24]. Some students are not able to realize the concept of derivative even after completing the calculus courses [13], their understanding of the derivative is limited to algebraic representations, most of them cannot interpret the graphical derivative and have difficulty relating between algebraic representations and derivative graphs [17].

Previous research on the concept of derivative has shown that most of students have little conceptual or intuitive knowledge of derivative even when their procedural knowledge of differentiation is sufficient [17, 18, 19, 3, 10, 20, 12]. According to the above-mentioned research on the difficulties of understanding the conceptual of derivative, Weigand [21] introduces a new approach to understanding the concept of derivative.

Wiegand [21] offers a step-by-step discrete approach to realize the derivative concept better by interacting with the sequence and difference sequence functions defined on discrete domains $\mathbb{Z}$ and $\mathbb{Q}$ and then by generalizing to domain $\mathbb{R}$. In this method, without explaining the limit concept, the concept of rate of change is represented using discrete examples and the learner's mind is gradually directed towards a better realizing of rate of change and therefore the local rate of change. The sequence of slopes of secants in the discrete-Z_n-functions is presented for the explanation of the function f 's derivative in a specific point in the graph f and therefore obtaining the local change rate.

Acceptable usage of mathematical issues is based on the way a student has mentally built these issues [5]. The present study intends to use the Action-Process-Object-Schema (APOS) theory [3, 11, 16] as an approach to analyze students' mental structures to assess the conception of the derivative notion with a discrete approach. In this research, the aim is to answer the following two question:

1. What are the students' mental constructions while they confront with a derivative discrete approach?
2. How can these mental constructions help design genetic decomposition for building notions while students learn a derivative discrete approach?

Literature review

According to Arnon et al. [2], in most studies that use APOS theory, it can be used as a strictly developmental theory, as well as a strictly analytical evaluative tool, or as both.

Borji and Martínez-Planell [6] used APOS-ACE theory to examine students' conception of implicit function and its derivative. To understand implicit function and its derivative, first a genetic decomposition (GD) was proposed and then using the proposed GD cycle, they created and applied ACE (Activities, Classroom Discussion and Exercises) to assist students perform the mental constructs they lacked. The results showed that students' understanding of implicit function improved. García and Dolores [14] performed a study included students in pre-university stage. They investigated their connection to mathematical issues while sketching the derivative and antiderivative functions graphs. In their opinion, mathematical communication is a cognitive process that by using it one can make a good connection among two or more mathematical opinions, issues, descriptions, theorems or notions. They used two graphical homeworks that included derivative function and anti-derivative function to gather data. The results of this study showed that students use very little visualization to solve graphic tasks. They also stated that students' level of understanding of the FTC is low.

Borji et al. [6] used the APOS-ACE theory to make graphical conception of the function derivative better for students. For this purpose, the experimental group was taught the ACE cycle made using Maple software, and the control group received the concept of derivative in a traditional, speech-based manner. The results of these teachings were assessed through comparison between the experimental group performance with the control group performance. The results of this study showed that the experimental group students could better understand the derivative concept in comparison to the students in the control group. Ariza et al. [1] in a study inspected students' conception of the function-derivative relationship during the time they were passing economic courses. For the evaluation of students' conception, researchers use a fuzzy metric and schema development levels called the Triad that is made up of three stages (Intra, Inter and Trans). This study revealed that conception of economic concepts is in relationship with the ability of the students to do conversion and treatments between algebraic and graphic registers of the function-derivative connection when Summarying concave / convex economic meaning in the graphs of functions using the second derivative. Cetin [8] used APOS theory in programming education to examine students' understanding of the concepts of loops and nested loops. 63 mechanical engineering students participated in this study. An initial genetic decomposition was proposed that included the researchers' understanding and the results of the literature. The findings of this study suggested that the initial genetic decomposition should be modified to include the pre-action concept of loops as a starting step. The results of this study showed that APOS theory is a useful framework for examining the understanding of engineering students related to these concepts. Thomas and Stewart [19] reviewed research on some of the conceptual processes and difficulties students face in learning eigenvalues and eigenvectors. They used the theoretical framework of Tall's three worlds of mathematical thinking, together with the views of Dubinsky's APOS (Action, Process, Object, Schema) theory and Thomas's representational versatility. The results showed that students have sufficient confidence in algebraic and matrix procedures, but the vast majority have no geometric or embodied world view of eigenvectors or eigenvalues, and could not argue the relationship between graphs and eigenvectors.

Weller et al. [22] use the APOS theory pedagogical strategy, which is the ACE (Activities, Classroom Discussion, and Exercises) teaching cycle, designed a decimal repetition unit for teachers to understand the relationship between rational numbers (fractions or integers) and its decimal expansion. The results of this study showed that the group that received the experimental instruction (ACE teaching cycle), compared to the control group that received the traditional teaching, had a significant improvement in the relationship between rational numbers and its decimal expansions. Weller et al. [23] continued the previous study [22] to determine the strength and stability of students' beliefs over time about the relationship between rational numbers (fraction or integer) and its decimal expansion, 47 interviews were conducted 4 months after the training. The results showed that students who received APOS-based guidelines developed stronger and more stable beliefs (over time). Cetin [9] in a study examined students' understanding of the concept of limit using APOS theory. The main purpose of this study was to investigate first-year calculus students' understanding of the formal limit concept and the change in their understanding after following an instruction designed by the researcher and based on APOS Theory. He used the case study method to explore the research questions. The students had five weeks of instruction in the fall semester of 2007-2008, and they met for 2 hours each week in a computer lab to study in groups and then attend classes for 4 hours. Students worked on programming activities in the computer lab to reflect on the concept of limit before receiving the formal definition of limit in class. To determine changes in students' perceptions of this concept, a questionnaire on limits including open-ended questions was administered as a pretest and posttest. At the end of the instruction, a semi- structured interview protocol developed by the researcher was administered to all of the students to explore their understanding in depth. Students' answers in this questionnaire were analyzed quantitatively and qualitatively. The results of the interview were analyzed using the APOS framework. The results of the study showed that students thinking reflected what was predicted by the preliminary genetic decomposition. The instruction was found to play a positive role in facilitating students' understanding of the limit concept. Asiala, Cottrill, et al. [4] used the APOS theory to analyze students' graphical conception of the derivative concept. They interviewed about derivatives with 41 engineering, science, and mathematics students who had at least passed two terms of calculus. In this research, the pedagogical tactics of APOS theory called ACE teaching cycle was used. The results of this study showed that the use of ACE teaching cycle with computer tasks which are created intently has been sensibly effectual in helping students to make a reasonably powerful process understanding of the function and a graphical conception of the derivative. The students that used the teaching cycle ACE showed a powerful process perception in understanding the symbol $f(x)$ and in interpreting the relation among the derivative, its graph, and its function graph. This research uses APOS theory both as an analytical evaluation and as a development tool. Breidenbach et al. [7] conducted a study entitled "Development of the process conception of function" using APOS theory, and how it is applied to this concept. They refer to an instructional treatment, using computers, that leads to substantial improvements in students' understanding of the concept of function. The results of this study show that students seem to develop a concept of a process of a function that they use to solve certain mathematical problems. The above study is an example of using APOS theory as a developmental tool.

Theoretical framework

In this section, we explain the two theoretical frameworks used in the research (Discrete Derivative Approach, APOS Theory).

Discrete Derivative Approach

Weigand [21] explained a theory of a discrete approach to calculus, which creates the concept of the mean change rate. This theory is based on a discussion of different sequences by considering discrete functions. Wiegand [21] explaines that discrete sequences and their difference sequences support the creation of basic opinions of the concepts of derivative, derivative function and change rate. Wiegand [21] states about the discrete approach that "In developing the first levels of this concept, the limit concept is only used in an intuitive sense. All calculations can be done on a discrete algebraic level. This concept of a discrete approach to the concept of derivative is a preparation for the understanding of derivatives of real functions. It creates a better understanding of the meaning of the rate of change or the difference quotient, it puts the limit or approximation process into practice by explicitly working with sequences (or discrete functions), and—in this sense—it goes beyond an intuitive level of understanding". Wiegand [21] introduces five levels for applying a discrete method to understand the derivative concept.

Level 1: Difference sequences

In this level, the goal is introducing the difference sequences $\Delta a_n = a_{n+1}+a_n \quad (n \in N)$ concept. As assumed $\Delta n = 1$,Δa_n can be considered as the change rate, which is used in daily issues, such as the mean temperature of air per year, which can be shown as a table and a graph.

Level 2: The concept of (quadratic) Z-functions

The sequence with the domain N was previously defined. The concept of sequence is now extended to the functions described on Z. The function $f : Z \to R$ is called the Z-function. The functions $y = f(z)$ are extended sequences described on numbers $z \in Z$.

Example 0.1. For the function: $f(z) = z^2 - 2z + 3$. The difference of Z-function is as follows (Figs. 1, 2): $D_f(z) = f(z + 1) - f(z) = 2z + 1 - 2$.

Example 0.2. For $f(z) = az^2 + bz + c$, we can reach $D_f(z) = 2az + a + b$ and find that $D_f(z)$ does not depend on the parameter c.

Level 3: Polynomial Z-functions

This notion can also be extended to polynomial functions of a superior degree.

Example 0.3. For $f(z) = az^3 + bz^2 + cz + d$ we have: $D_f(z) = 3az^2 + (3a + 2b)z + a + b + c$, where D_f is a quadratic function that does not depend on the parameter d.

Level 4: Exponential functions

For exponential functions $E(z) = a^z (a \in R^+, z \in Z)$. Its difference function is:

$$D_E(z) = E(z+1) - E(z) = a^{z+1} - a^z = a^z(a-1) = E(z)(a-1).$$

To achieve $D_E(z)$, it is enough to multiply $E(z)$ by the factor $(a - 1)$.

Level 5: Transfer the idea from Z to Q and R

In this section, domains in the form of

$$Z_k = \left\{-\frac{2}{k}, -\frac{1}{k}, 0, \frac{1}{k}, \frac{2}{k}\right\}$$

are selected which are subsets of $[-2, 2]$.

$$Z = \left\{-\frac{2}{10}, -\frac{1}{10}, 0, \frac{1}{10}, \frac{2}{10}, \dots\right\} \qquad \left(z_{10} = \frac{z}{10}\ ,\ z \in Z\right)$$

$z_{10} \in Z_{10}$ and the function f is defined as $f_{10} : Z_{10} \to R$. To obtain the change rate of consecutive values, this time we limit the inertvals to $\frac{1}{10}$ instead of 1 unit , and the difference-quotient-Z_{10}-function is obtained:

$$D_f(z_{10}) = \frac{f\left(z_{10} + \frac{1}{10}\right) - f(z_{10})}{\frac{1}{10}} \qquad (z_{10} \in Z_{10})$$

We can generalize this problem to an interval of the length $\frac{1}{n}$, $n \epsilon N$, and the difference-quotient-Z_n-function D_{fn} is obtained as follows:

$$D_{fn}(z_n) = \frac{f\left(z_n + \frac{1}{n}\right) - f(z_n)}{\frac{1}{n}} \ (z_n \in \mathrm{Z}_n) = \left\{\dots, -\frac{1}{n}, -\frac{2}{n}, 0, \frac{1}{n}, \frac{2}{n}, \dots\right\}.$$

Example 0.4. For the quadratic Z-functions

$$D_{fn}(z_n) = a{z_n}^2 + bz_n + c$$

we have:

$$D_f(z_n) = 2az_n + b + \frac{a}{n}.$$

Now if $n \to +\infty$, we have:

$$D_f(z_n) = 2az + b$$

which will be the derivative of the function. This idea is similar for the exponential-Z-function $E(z_{10}) = a^{z_{10}}$ and we obtain its difference-quotient-Z_{10}-function as:

$$D_E(z_{10}) = \frac{E\left(z_{10} + \frac{1}{10}\right) - E(z_{10})}{\frac{1}{10}} = \frac{a^{z_{10}+\frac{1}{10}} - a^{z_{10}}}{\frac{1}{10}} = a^{z_{10}} \frac{a^{\frac{1}{10}} - 1}{\frac{1}{10}}.$$

If $\frac{a^{\frac{1}{10}}-1}{\frac{1}{10}} = 1$ or $a = 2/5937$ is assumed, the function $E(z_{10})$ will be equal to $D_E(z_{10})$. Considering the exponential function $E(z_n) = a^{z_n}$ we have:

$$D_E(z_n) = \frac{E\left(z_n + \frac{1}{n}\right) - E(z_n)}{\frac{1}{n}} = \frac{a^{z_n+\frac{1}{n}} - a^{z_n}}{\frac{1}{n}} = a^{z_n} \frac{a^{\frac{1}{n}} - 1}{\frac{1}{n}}.$$

But when $n \to \infty$, we have:

$$\lim_{n\to\infty} \frac{a^{\frac{1}{n}} - 1}{\frac{1}{n}} = 1$$

and this means that $D_E(z_n)$ is the derivative of the exponential function. Then the local change rate for $x_0 \in D \subset R$ is obtained. The sequence of the difference quotient is formed for a real-valued difference.

$$D_n(x_0) = \frac{f\left(x_0 + \frac{1}{n}\right) - f(x_0)}{\frac{1}{n}} \quad (n \in N)$$

Example 0.5. For the function $(x) = ax^2 + bx + c$, at the point $(x_0, f(x_0))$ we have:

$$D_n(x_0) = \frac{f\left(x_0 + \frac{1}{n}\right) - f(x_0)}{\frac{1}{n}} = \frac{a(x_0+\frac{1}{n})^2 + b\left(x_0+\frac{1}{n}\right) + c - ax_0{}^2 - bx_0 - c}{\frac{1}{n}} = 2ax_0 + b + \frac{a}{n}.$$

Therefore, according to the graph of f, the sequence $D_n(x_0)$ is explained as the sequence of the slope of the secants at a point.

APOS theory

It is confirmed that APOS theory is beneficial to provide a precise explanation of how to build several mathematical concepts, so it is useful in realizing how to make a concept of a discrete derivative approach. APOS theory is developed upon Piaget's principle that a person uses special mental pathways to build mental principles to learn various concepts, including mathematical concepts, and then uses these structures to address problematic situations in mathematics. According to this principle, one can create a mental structure for any mathematical concept which is suitable for that concept and can be used to understand, learn and apply that concept [2]. According to this theory, mental structures for learning mathematics include actions, processes, objects, and schemas. Also, the mental mechanisms used to construct these mental structures include interiorization, coordination and encapsulation. The main steps of the following notion are used in APOS theory:

1 The concept of Action:

An Action is the transformation of a mathematical object that has already been constructed and that one perceives as an external component. An Action is an external in the sense that each step of the transformation must be done explicitly and be guided by external instructions. In addition, each step creates the next step, meaning that the action steps cannot be imagined yet and none can be ruled out [2]. Therefore, at this stage, the learner will be able to calculate the difference function with the domain Z and the difference quotient function with the domain Q and R step by step.

2 The concept of process:

When an action is repeated and the learner reflects it, the action is internalized as a process. At this time the learner has an internal structure to perform the same action. This means that , an internal structure is constructed which performs the same action, but this time it is not necessarily guided by an external stimulus, and the control is in the learner's own hand. These connections will have meaning with other mathematical knowledge that will allow the individual to imagine the process and predict the results without the need for explicit execution. Meaningful communication in a concept of process allows the individual to connect different representations and justify the process. Different processes may be coordinated to form new processes. A process may be created by internalizing several different actions and coordinating the resulting processes. Therefore, at this stage, the learner will be able to imagine some steps of calculating the differential function with the domain Z and the difference quotient function with the domain Q and R in his mind and calculate the difference quotient function $D_n(x_0)$ by considering the graph of the function f to Interpret the sequence of the slope of the secants at a point $(x_0, f(x_0))$.

3 The concept of the object:

If the learner can reflect on the actions that have been applied to a particular process and realize that each process can be considered as a totality on which changes (actions or processes) can be applied and if he can really make these changes, it is said that the learner understands the process in the form of an object. At this stage we say that the process is encapsulated as an object. In addition, to perform an action or process on an object, it is often necessary to de-encapsulate that object into the process from which it was derived. Therefore, at this stage, the learner will be able to understand the formal definition of the derivative with a discrete approach, which is the limit of the difference quotient sequence at infinity.

4 The concept of the Schema:

Actions, processes, objects, and other previously constructed mental structures dealing with a particular mathematical notion may be organized into a coherent structure called a schema. A schema is brought into play in response to a problem situation that the individual perceives involves the specific mathematical notion. The Schema is coherent in the sense that the different components are interrelated in a way that allows the individual to determine which problem situations are pertinent to the schema. Asiala et al. [4] claim that a person's schema is all knowledge that he or she consciously or unconsciously relates to a particular mathematical subject; for example, a person may have a functional schema or a derivative schema. Therefore, at this stage, the learner will be able to calculate the difference quotient function and then obtain the limit of the function at infinity and the derivative of the function. Also, at this level, the learner can calculate the extremum points of a quadratic function.

The notion which is related to APOS theory for analyzing students' conception of a mathematical notion is called Genetic Decomposition (GD). In Arnon et al.'s (2014) opinion, a genetic decomposition is defined as a hypothetical theory that explains the internal constructions and pathways that students may require to create to realize a special mathematical notion. The genetic decomposition is defined according to the internal constructions (Action, Process, Object, Schema) and mechanisms (interiorization, encapsulation, coordination, etc.) of the theory. A genetic decomposition for a concept is not exclusive. Various genetic decompositions might be recommended for the same notion.

Methodology

The goal of this study is to explore the meaning of derivative in a discrete learning environment from students' point of view. By the use of the APOS theoretical analyses we hoped to be able to help the development of empirical learning of conceptual development of derivative in a discrete learning environment, for the group of students. The type of this research was a single qualitative case study research to investigate the development of the concept of derivative in a discrete learning environment. The students' population in this study was 30 engineering volunteers of one of Iran's universities. In the first stage, students answered to homeworks containing questions that were designed to Summary the students' mental constructions during the learning of the derivative in a discrete learning environment. However 9 students refused to complete the study and so only answers from 21 students were collected and analysed. The researcher then carefully analyzed the students' an swers to the questions. Some of the students were selected for interviews based on their answers, in order to explain their responses to clarify their level of conception of derivative in a discrete learning environment. The participants selected for the interviews were chosen from the students that we couldn't determine their understandings level according to APOS. These interviews helped us so that we could verify our suggested genetic decomposition for the understanding of derivative in a discrete learning environment and to identify the students' mental structures.

Results and discussion

The first two questions of the homeworks were designed to collect information about students' understanding of the definitions of difference function and the difference quotient sequence.

1. Think about the difference function $\boldsymbol{D_f(z)}$ with domain Z and the difference quotient sequence $\boldsymbol{D_f(z_{10})}$ and answer the following questions:
   1.1 The difference function $\boldsymbol{D_f(z)}$ with domain Z means ………………
   1.2 The difference quotient sequence $\boldsymbol{D_f(z_{10})}$ means …………………

Question 1.1: The meaning of $\boldsymbol{D_f(z)}$

Almost all participants responded to question 1.1. The students' answers to question 1.1 is categorized in table 1. Category I included all the answers that represent $D_f(z)$ as a result. In this category, $D_f(z)$ was considered as a solution resulting from the process of the function $f(z)$ with domain Z. A typical answer in this category is "$D_f(z)$ is the difference function of $f(z)$ with domain Z". Answers categorized in category II show $D_f(z)$ as a command or instruction to perform an operation. Most students' answers indicate that "$D_f(z)$ means the difference function of the function $f(z)$ with domain z". In the answer of one of the participants, which was also classified in category II_b, a symbolic example as a means to define $D_f(z)$ by putting $f(z) = z^2 + 2z + 1$ was used, and then by this means he found its difference function. In both categories, the description of the difference function is the rate of change between the points $(z, f(z))$, $(z + 1, f(z + 1))$. For example, one participant explicitly demonstrated the concept of difference function when he wrote: "$D_f(z)$ means difference function of the function f (z) with domain Z. In other words, the difference function $D_f(z)$ can be considered as the slope of the function $f(z)$ between the two points $(z, f(z))$ and $(z + 1, f(z + 1))$".

Table 1: Categorization of students' answers to Question 1.1

Students' answers	Frequency
Category I: $D_f(z)$ means difference function of f(z) with domain Z	18
Category II: $D_f(z)$ is a special function which is the rate of change between points $(z, f(z))$ and $(z+1, f(z+1))$	1
Example: $f(z) = z^2 + 2z + 1$, $D_f(z) = f(z+1) - f(z) = 2z + 3$, $f(3) = 16, f(4) = 25$, $D_f(3) = 9$	
Category III: No response	2

Reza was one of the students whose answer falls into the category II (see Fig. 1). When he was asked to explain his written answer, the following conversation took place:

A1: Interviewer: *What does* $D_f(z)$ *mean?*
A2: Reza: $D_f(z)$ *is the difference function of the function* $f(z)$ *with domain* z.
A3: Interviewer: *How do you obtain* $D_f(z)$*?*
A4: Reza: *According to the instructions, to obtain* $D_f(z)$ *we have:* $D_f(z) = f(z + 1) - f(z)$.
A5: Interviewer: *What do you know about the difference function?*
A6: Reza: *What I do know is that the difference function of the* $f(Z)$ *function is an instruction to obtain the rate of change between two consecutive points. For the quadratic z-function, the difference function that we obtain according to the instruction, is a linear function with domain* z.
A7: Interviewer: *Well, according to the above answer, what can you say about the polynomial z-functions with degree* n*?*
A8: Reza: *We can conclude that the difference function of a polynomial* Z*-function with degree* n*, is a function with degree* $n - 1$.

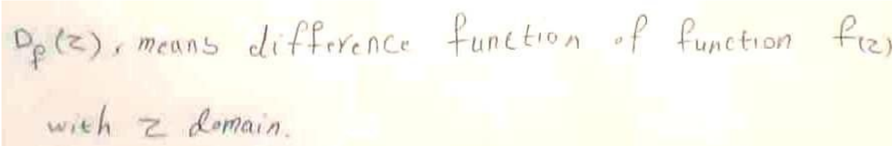

Figure 1: Reza's answer to question 1.1

Summary 1: Interview with Reza

From the interview, it was observed that Reza mentioned the symbol $D_f\,(z)$ as a difference function (A2) and an instruction to find the rate of change (A6). Reza seemed to make sense of his mathematics via external cues. His conception of difference function seemed to be confined to performing precise procedures. He had an action conception of $D_f\,(z)$, as a symbol which served as an external stimulus that caused him to reach the concept of difference function. Regarding the meaning of the difference function, in A6 Reza says: "What I do know is that the difference function of the f(z) function is an instruction to obtain the rate of change between two consecutive points.".This proposition states that Reza has considered the difference function as an instruction for finding the rate of change between the two points $(z, f(z))$, $(z+1, f(z+1))$ of the function $f(z)$. The following is an Summary of the interview with student Zahra, whose answer was in category I (see Fig. 2). She emphasizes on the concept of the difference function as the rate of change between two points.
A9: Interviewer: *What do you understand of the difference function?*
A10: Zahra: *The difference function gives us the slope between two points $(z, f(z))$ and $(z+1, f(z+1))$ in the function $f(z)$. It sounds like you are using the relation $m = \frac{y_2-y_1}{x_2-x_1}$ to get the slope between two points and similarly we can use the difference function to obtain the slope between the mentioned two points.*

Summary 2: Interview with Zahra

From Zahra's point of view, the concept of difference function was the same as the rate of change between the points $(z, f(z))$ and (z+1, f(z+1)), We classified her written answer to the meaning of $D_f\,(z)$ (Fig. 2) as a result, but the interview showed that Zahra mentioned the symbol $D_f\,(z)$ as a way to find the rate of change. This answer was very similar to many other students' answers. Most students have discovered $D_f\,(z)$ as an instruction to perform an action. The symbol $D_f\,(z)$ is used as an external symbol to perform an action that determines the slope of the line between the points $(z, f(z))$ and $(z+1, f(z+1))$.

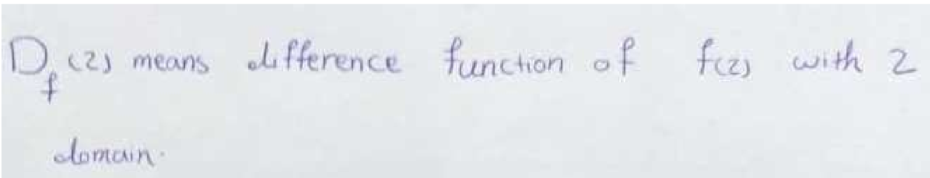

Figure 2: Zahra's written answer to question 1.1

Question 1.2: The meaning of $D_f\,(z_{10})$

Regarding the meaning of the difference quotient sequence, only three students interpreted it as the sequence of the slope of the secants in a point. The majority mentioned slope of the line in their definition. The responses to the meaning of $D_f\,(z_{10})$ in Table 2, categorized in category II_a, were those that mentioned the sequence of the slope of secants at a point, while the responses in category II_b, were those that didn't mention the sequence of the slope of secants at a point. In fact, the distinction between categories II_a and II_b was blurred at the time of the interview, and even students who did not mention the sequence of the slope of secants at a point showed their reflection in this regard. According to the example of Category I In Table 2, the student decided to use $f\,(z_{10}) = 2{z_{10}}^2 - z_{10} + 1$ to explain the meaning of $D_f\,(z_{10})$. Unfortunately, he neglected to obtain the difference quotient function, for which he had to restrict distances to $\frac{1}{10}$ instead of 1. This means that he had no clear understanding of the concept of the difference quotient function $D_f\,(z_{10})$; So he had not yet conceptualized $D_f\,(z_{10})$ as an process. He also went on to show that he does not even have the concept of action of difference quotient function, otherwise he would have applied what he had learned in the classroom. Javad was another student, when Javad was asked to explain his answer, in his written reply, he did not explicitly mention $D_f\,(z_{10})$ as the difference quotient sequence, but he wrote "In the function $f(z_{10})\left(\boldsymbol{Z_{10}} = \frac{\boldsymbol{z}}{\boldsymbol{10}}, \boldsymbol{z} \in \boldsymbol{Z}\right)$ to get $D_f(z_{10})$, we have to restrict the distances to $\frac{1}{10}$". During the interview, Javad explained his written answer in detail and said:

A11: Interviewer: *What does the difference quotient sequence $D_f(z_{10})$ mean?*

A12: Javad: *According to the concept of difference function, we can say that the difference quotient function is the same as the slope of the function with domain z_{10}.*

A13: Interviewer: *What is the answer that we get if we obtain the limit of the difference quotient function in infinity?*

A14: Javad: *If I am not making a mistake, what we obtain is the rate of change of the graph of the function with domain z_{10}.*

Summary 3: Interview with Javad

This interview shows that the concept of the difference quotient function, which is "the difference quotient function is the same as the slope of the function with domain z_{10}" has been evoked in Javad. Javad made no connections between the difference quotient function and the concept of derivative. This is confirmed by his claim that the final value obtained when the difference quotient function tends to infinity is the rate of change of the function's graph. The hesitation and the use of the words, "if I am not making a mistake", might be interpreted as an indication of the lack of at least a process level of understanding when evaluating a difference quotient function. Javad relied on an algorithm to evaluate the difference quotient function, so it can be said that he is still operating in an action level. Therefore, according to the table 2 students' level of understanding based on the APOS theory, it is at the process level of the derivative concept with a discrete approach. At this level, the learner is able to, in addition to calculating the difference quotient function, interpret it as a sequence of the slope of secants at a point. **Question 2.1**

Table 2: Categorization of students' answers to Question 1.2

Students' answers	Frequency
Category I: $D_f(z_{10})$ means the difference quotient function $f(Z_{10})$	3
Example: $f(z_{10}) = 2{z_{10}}^2 - z_{10} + 1$, $D_f(z_{10}) = f(z_{10}+1) - f(z_{10}) = 4z_{10} + 1$	
Category II: $D_f(z_{10})$ means the difference quotient function of the function $f(Z_{10})$ $\left(Z_{10} = \frac{z}{10}, z \in Z\right)$	
II_a: *mention* the sequence of the slope of secants *at a point*	10
II_b: *did not mention the sequence of the slope of secants at a point*	5
Category III: No response	3

In question 2.1 students required to link visual representation of graphs (the Z-function and the difference quotient sequence) with meaning of finding the rate of change. In question 2.1.1 they were asked to draw the graph of the function $f(z_{10}) = 0/1 \quad z_{10}^3 - z_{10} + 1$, Which is a third degree function with domain Z_{10}. They were then asked in question 2.1.2 to obtain the difference quotient sequence $D_{f10}(z_{10})$ and then draw a graph of it. Students were asked to evaluate and then explain the relationship between the derivative concept and the Z-function and the difference quotient sequence. Table 3 gives their responses to question 2.1. Referring to the function $f(z_n)$ and the difference quotient sequence $D_f(z_n)$, five students mentioned the concept of the rate of change of successive values.

Table 3: Categorization of students' answers to question 2.1

Students' answers	Frequency
Category I: **Refers to the rate of change**	
I_a: Drawing a correct graph of the functions $f(z_n)$ and $D_f(z_n)$	11
I_b: Drawing an incorrect graph of the functions $f(z_n)$ and $D_f(z_n)$	2
Category II: **No refers to the rate of change**	
II_a: Drawing a correct graph of the functions $f(z_n)$ and $D_f(z_n)$	4
II_b: Drawing an incorrect graph of the functions $f(z_n)$ and $D_f(z_n)$	2
Category III: **No response**	2

Mohammad's response belongs to category I_a and he had a complete understanding of the function $f(z_n)$ and the difference quotient sequence $D_f(z_n)$ as the rate of change of successive values. Mohammad, firstly, referred to the difference quotient sequence denoted by the symbol $D_f(z_n)$. He also interpreted the sequence $D_f(z_n)$ according to the graph of f as the sequence of the slope of the secants (Fig. 3). He also pointed out that if $n \to \infty$, the derivative of the function would be obtained, which meant that he had internalized the action, from the external stimulus $\lim_{n\to\infty} D_f(z_n)$ to a process of being able to perform a procedure mentally. He had further encapsulated this process as an object when he referred to the limit of the difference quotient sequence as a function's derivative. Mohammad displayed a completely mathematically correct answer and we observed that having at least one object conception of the difference quotient sequence is a prerequisite for understanding the concept of derivative in a discrete learning environment.

The following, on the other hand, is an excerpt of interview with Reza, whose response fell into category I_b:

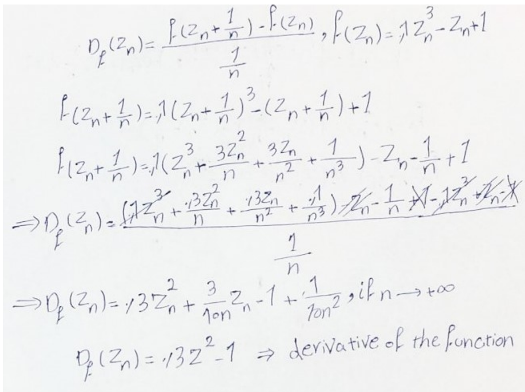

Figure 3: Mohammad's answer to get $D_f\left(z_n\right)$

A15: **Interviewer:** *What is the relationship between depicted graphs and derivatives?*
A16: Reza:*I did not realize what I wrote here. (Reza had written this "The difference quotient sequence is the slope of the line between two points").*
A17: **Interviewer:** *What does this slope give you? (It means that if $n \to \infty$, in the difference quotient sequence, the derivative of the function is obtained).*
A18: Reza: *A difference quotient sequence graph can be drawn. Then, in the difference quotient sequence, the value of Z can be substituted and the derivative of the function can be obtained at any point of Z.*
A19: **Interviewer:** *In the graph, what does the difference quotients function define?*
A20: Reza: *I don't know.*

Reza had drawn an incorrect graph but the explanation he had written, as reflected in A16, meant that his answer was classified as belonging to Category I_b. His verbal responses though, displayed a different conception. Firstly, he stated that he did not realize what he had written (A16). Secondly, in A18, without referring to the limit of the difference quotient sequence to obtain the derivative function, he referred to substitution. Therefore, it is inconceivable that he means that he substitutes the given values of z in the difference quotient sequence, the same concept that students usually use after obtaining the limit of the difference quotient sequence. Lastly, he stated explicitly that he did not understand what was defined in relation to the concept of derivative with the graph of the drawn difference quotient sequence. Reza has not internalized the concept of the difference quotient sequence, as well as taking the limit from difference quotient sequence, which is the derivative of the function. He was aware of the action of substituting of the value Z for the difference quotient sequence to obtain the function's derivative, but seemed to have neglected the process of finding the limit of the difference quotient sequence through which the derivative function is obtained. Reza was operating at an action conception level since his reasoning was based mainly on an algorithm (A18) with an intention to evaluate the concept of derivative. Therefore, according to the table 3, the level of understanding of these students based on the APOS theory is at the level of the object of the derivative concept with a discrete approach, because at this level, the learner is able to understand the formal definition of the derivative function with a discrete approach.

Question 3
3. The equation of the motion is $f\left(t\right) = \frac{1}{2}t^2 - 3t + 1$. T is in seconds and f is in meters.
Obtain the moving local speed at $t = 7$.
Question 3 includes the definition of a derivative function as a moving instantaneous velocity and the basic techniques to obtain a derivative function using a discrete approach. Students' responses were categorized in (Table 4). 18 out of 22 participants, responded by evaluating the derivative function. The variations within these 18 students' responses were in obtaining the moving velocity equation and explicit referring to the notion of derivative function. Mehdi was one of the students from Category I_{cc}^{i}.

He had shown the notion of instantaneous velocity and had a good understanding of the derivative function but was wrong in obtaining the derivative function. The following is an Summary from the interview with him:
A21: **Interviewer:** *In question 3, to obtain the instantaneous velocity, how do you obtain the derivative function*

Table 4: Categorization of students' answers to question 3

Students' answers	Frequency
Category I: Evaluation of the derivative function at time $t = 7$	
Category I_a: Correct solution with notion of instantaneous velocity reflected	11
Category I_{bb}^i: Correct differentiation but with no notion of instantaneous velocity	5
Category I_{cc}^i: Errors in differentiation but with notion of instantaneous velocity	2
Category I_{dd}^i: Errors in differentiation and no notion of instantaneous velocity	2
Category III: Incorrect approach/Left blank	1

using the discrete approach?
A22: Mehdi: *To obtain the derivative function, we first obtain the difference quotient sequence and then substitute the value of t into difference quotient sequence.*
A23: **Interviewer:** *In the discrete approach, how do you interpret the difference quotients sequence?*
A24: Mehdi: *The difference quotient sequence is the same as the rate of change.*

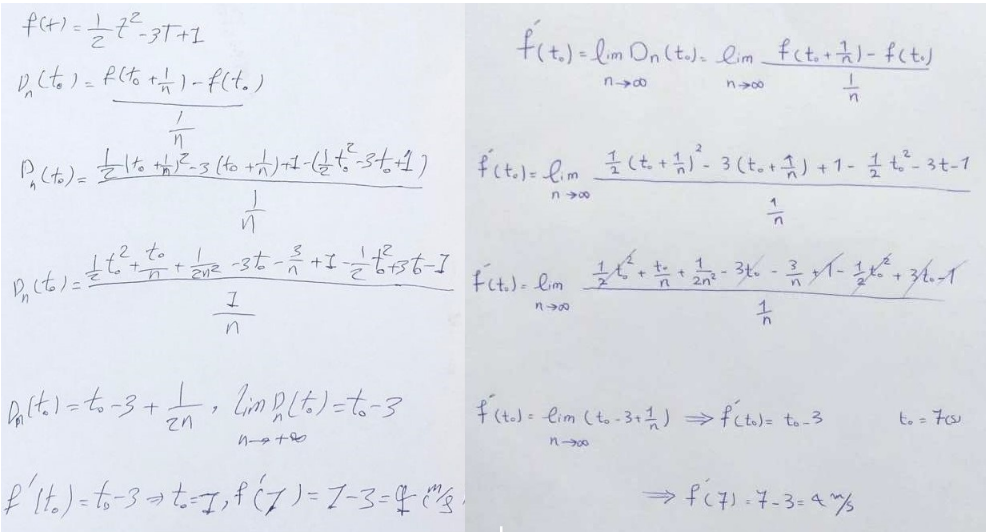

Figure 4: Correct answer to question 3

Summary 4: Interview with Mehdi

In A21, the researcher meant to prompt Mehdi to realize the mistake he had made obtaining the derivative function. Mehdi had not created a complete scheme of the derivative function as the moving instantaneous velocity.His answer in A22 showed that, although he knew he had to obtain the derivative function, he had difficulty in obtaining it which was obtaining the limit of the difference quotient sequence in infinity. So this knowledge had not been interiorised. It was ultimately probing in A22 that evoked this appropriate conception in Mehdi. Students who explicitly state that they are calculating the function's derivative or provide a final answer, consider instantaneous velocity as the function's derivative. For example, the responses of 11 students are in category I_a, who had calculated the derivative of the function correctly and then were able to obtain the moving instantaneous velocity. The two students' solutions are similar. In the first case, the student obtained the difference quotient sequence at first and then calculated its limit at infinity, and in the second case, directly obtained the limit of the difference quotient sequence at infinity. Therefore, they were able to form both the difference quotient function scheme and the limit scheme and obtain the velocity equation correctly, and according to the APOS theory, they have the concept of the derivative function scheme. First, they knew that to find the instantaneous velocity they needed to find the derivative function. Second, they correctly identified the moving instantaneous velocity and correctly obtained the derivative function using the discrete approach. Therefore, according to the table 4, the level of understanding of these students based on APOS theory is at the schema level of the derivative concept with a discrete approach. Because at this level, the learner was able to form both the schema of the difference quotient function and the schema of the limit of this function. Fig. 4 shows the responses of two students who, according to APOS, have the concept of schema for achieving the derivative function. Both went step by step in receiving their answers. First, they knew they needed to find a derivative function to find instantaneous velocity. Second, they correctly identified the moving instantaneous velocity and obtained the derivative function correctly using the discrete approach.

Question 4

4.1 If $f(x) = x^3$ and $g\,(x) = x^2 + 1$, obtain the derivative of $f(g\,(x))$ using the discrete approach?

4.2 Obtain the derivative of $f(x) = (x+2)^2$ using the discrete approach?

In question 4, students were asked to obtain the function's derivative using a discrete approach. In this question, the student should think more analytically about this issue. Therefore, to solve problem 4.1, it is necessary to encapsulate both the functions $f(x)$ and $g(x)$ to form the rule $f(g(x))$ in order to obtain the derivative. Students' responses to question 4.1 are categorized as follows (Table 6).

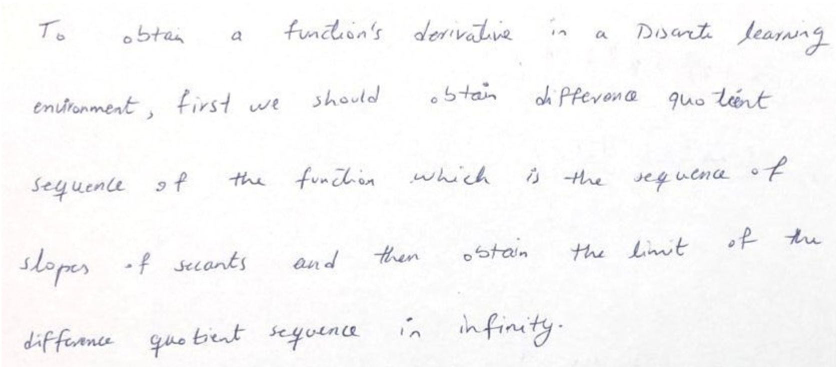
To obtain a function's derivative in a Discrete learning environment, first we should obtain difference quotient sequence of the function which is the sequence of slopes of secants and then obtain the limit of the difference quotient sequence in infinity.

Figure 5: Alireza's strategies justification

Table 5: Categorization of students' answers to question 4.1

Students' answers	Frequency
Category I: Correct solution provided	
Category I^{i}_{aa} :Reference to the formation of the rule of the function f (g (x)) and the correct differentiation	6
Category II: Incorrect solution provided/Blank	
CategoryII^{ii}_{aa}: Correct formation of the function rule $f(g(x))$ but differentiation error	9
CategoryII^{ii}_{bb} : Incorrect formation of the function rule $f(g(x))$	4
CategoryII^{ii}_{cc} : Blank	2

Fifteen of the respondents correctly formulated the rule function $f(g(x))$ and and 6 people were able to do both the rule of the function $f(g(x))$ and the derivation correctly. Three presented algorithmic arguments for the differentiation of the function $f(g(x))$, whose arguments were mainly procedural. The following are examples of reasons given: "If we want to obtain the derivative of the function $f(g(x))$ using the discrete approach, we must obtain the limit of its difference function." Therefore, this student did not consider level 5 of the discrete approach, which requires us to obtain the difference quotient function and then calculate its limit at infinity. The following is an Summary from interview with Sara.

A25: **Interviewer:** *How do you form the rule of the function $f(g(x))$?*

A26: **Sara:** *The function $f(g(x))$ is a combination of two functions $f(x)$ and $g(x)$. Therefore, to form a rule, it is sufficient to put the function $g(x)$ in the function $f(x)$ instead of the variable x and obtain it.*

A27: **Interviewer:** *All right, now how do you get the derivative of it by using the discrete approach?*

A28: **Sara:** *We must first obtain $D_n(x_0)$ and then calculate its limit at infinity.*

A29: **Interviewer:** *Why do we get $D_n(x_0)$?*

A30: **Sara:** *We do not know why $D_n(x_0)$ should be obtained at first and ect. We know that according to the instruction to obtain a function's derivative using the discrete approach, one must first obtain $D_n(x_0)$ and then calculate its limit at infinity.*

Summary 5: Interview with Sara

The admission by Sarah in line A30 that "We do not know why $D_n(x_0)$ should be obtained at first" indicated a purely algorithmic approach without underlying reasons for methods applied. In such cases, learning is highly mechanical and it focuses on the procedure only. At this stage, Sara displayed action conception of differentiation using a discrete approach since her decision to solve this question was based solely on the explanations given in the discrete derivative instruction, with no mathematical basis to support the choice. Therefore, according to the table 6, the level of understanding of these students based on APOS theory is at the process level of the concept of derivative with a discrete approach, because at this level, most of the students were not able to understand the formal definition of derivative with a discrete approach.

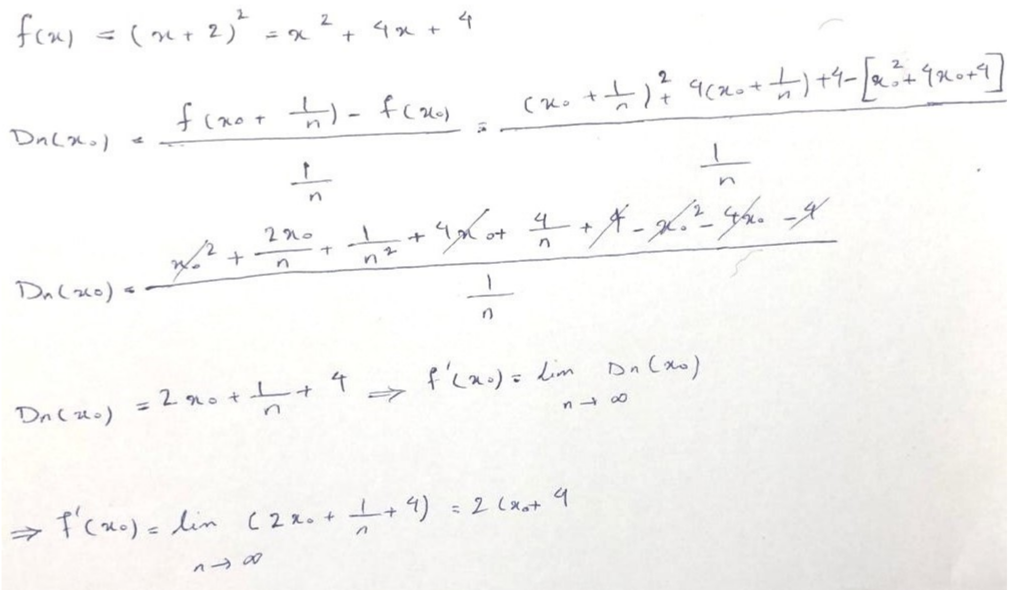

Figure 6: Alireza's repones to Question 4.2

Fig. 5 is an example of a response where Alireza was justifying her choices for the strategies used to obtain the function's derivative.

Table 6: Categorization of students' answers to question 4.2

Students' answers	Frequency
Category I: Referring to the limit of difference quotient function and correct differentiation	6
Category II: Incorrect solution provided/Blank	
Category II_{aa}^{ii}: Referring to the limit of the difference quotient function, but the error in differentiation of the function	10
Category II_{bb}^{ii}: Blank	5

In question 4.2, Alireza stated that to obtain the derivative, $D_n(x_0)$ of $f(x) = x + 2$ must be obtained and then calculate its limit at infinity. Alireza wrote her solution for the derivative of the function $f(x) = (x+2)^2$ as mentioned (see Fig. 6). This answer showed that Alireza's solution to differentiation of the function was to first open the function with respect to the quadratic union, then obtain the difference quotient sequence, and finally calculate its limit at infinity. Sara displayed a coherent collection of processes required to solve this type of problem. Therefore, according to the table 6, the level of students based on APOS theory is at the object level of the concept of derivative with a discrete approach, because at this level, many students were able to formally understand the definition of derivative with a discrete approach, but they could not get the derivative of the function correctly.

Conclusions

In this study, we used APOS theory as a lens to provide more insight into our findings of how to understand derivative by considering the concept of derivative discrete approach. To answer the research question, What are the students' mental constructions while they confront with a derivative discrete approach?, we observed that most students tend to use an algorithmic method when solving derivative problems. Also students' understanding of the meaning of derivative with a discrete approach was mainly procedural and based on the fact that the derivative function

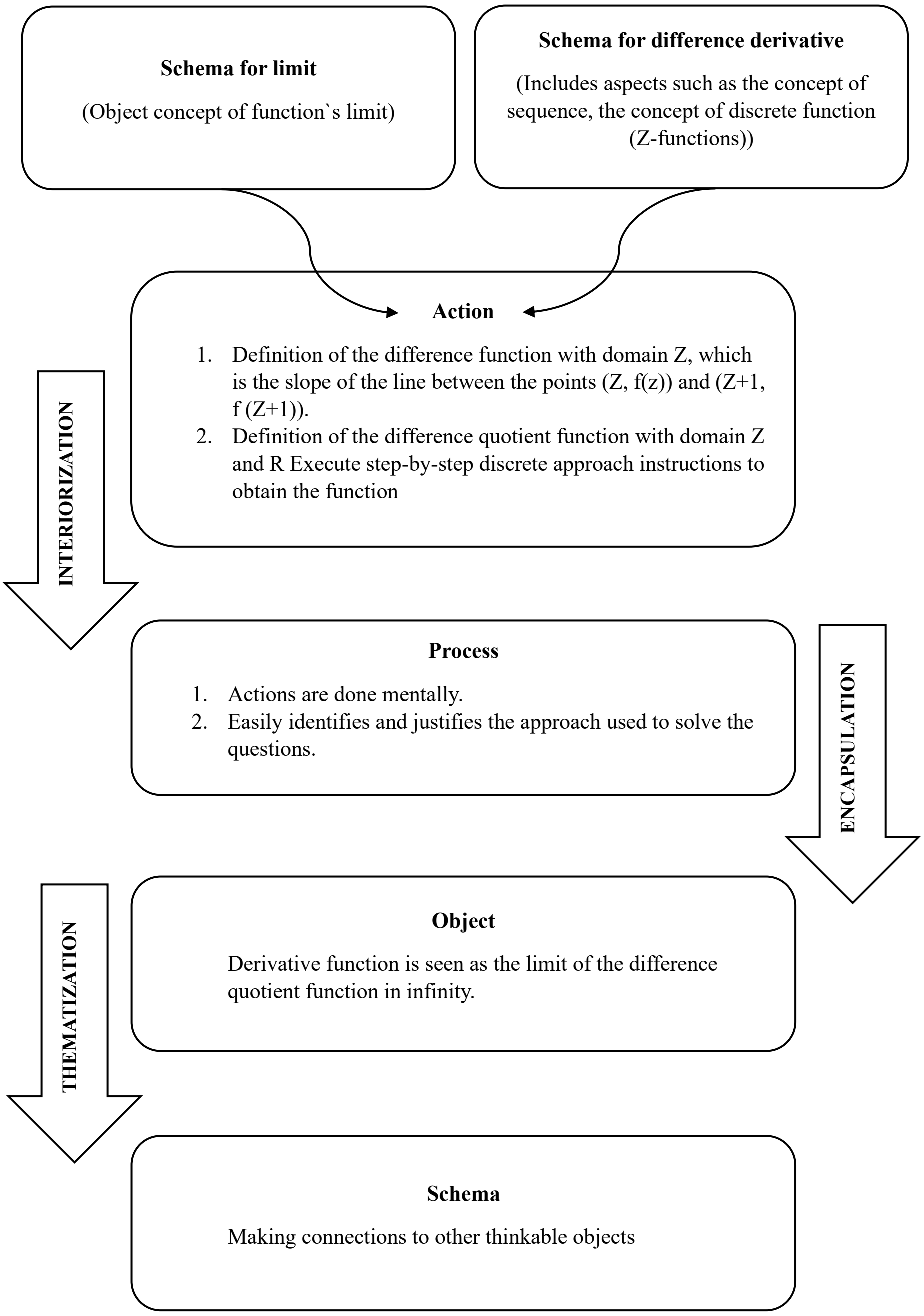

Figure 7: Model for the Genetic Decomposition for Derivative with a Discrete Approach

is the limit of the difference quotient function in infinity. According to the results obtained from the qualitative analysis using interviews, the students' level of understanding based on APOS theory was that most of the students are at the process and object level of the derivative concept with a discrete approach, and some have reached a higher level and have formed a derivative schema in their minds. Explanations about the second question, how can these mental constructions help design genetic decomposition for building notions while students learn a derivative discrete approach? are demonstrated in Fig. 7. The results showed that the schemas of difference quotient functions and limit are prerequisites for learning derivative in a discrete learning environment. To better understand the derivative in a discrete learning environment, as confirmed by Weigand [21], students must work on the Z-function and its difference quotient function, which leads to a better understanding of the relationship between function and derivative function. Students should also extender an object or process conception of difference quotient function which allows them to decipher and reflect on fundamental features such as the rate of change and the sequence of the slope of the secants and graphical representations. For the limit schema, students should have an object level of conception of the limit of function. Such conception should assist students to consider the behavior of functions when obtaining the limit of functions such as, limit in infinity. In this case, the limit of the difference quotient function in infinity represents the process that is encapsulated in an object. These observations led to designing the proposed genetic decomposition shown in Fig. 7.

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