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Qualifications and stationarity for nonsmooth multiobjective problems with switching constraints

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Abstract

This paper aims to study a broad class of multiobjective mathematical problems with switching constraints in which all emerging functions are assumed to be locally Lipschitz. First, we are interested in some Abadie, Guignard, and Cottle types qualification conditions for the problem. Then, these constraint qualifications are applied to obtain several stationarity conditions. The results are based on Clarke's subdifferential.

Keywords: Multiobjective optimization, Stationarity conditions, Switching constraints, Constraint qualification, Clarke subdifferential.
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1 Introduction

In this paper, for the first time, we consider the multiobjective mathematical programming with switching constraints (MMPSC, in brief) which is defined as

$$(MP): \qquad \begin{array}{lll} \min & \big(f_1(x), \cdots, f_m(x)\big) & \\ s.t. & g_j(x) \leq 0, & j \in J, \\ & G_k(x)H_k(x) = 0, & k \in K, \end{array}$$

where J and K are finite index sets with $J \cup K \neq \emptyset$, and the functions f_i, g_j, H_k and G_k, for $i \in I := \{1, \cdots, m\}$, $j \in J$, and $k \in K$, are locally Lipschitz from $\mathbb{R}^n$ to $\mathbb{R}$. The problem (MP) is said to be smooth if f_i, g_j, G_k, and H_k as $(i, j, k) \in I \times J \times K$ are continuously differentiable functions. We will suppose that the feasible set of (MP) is nonempty, i.e.,

$$\digamma := \big\{x \in \mathbb{R}^n \mid g_j(x) \leq 0,\ G_k(x)H_k(x) = 0, \quad j \in J,\ k \in K\big\}.$$

If $m = 1$, the problem (MP) reduces to mathematical programming with switching constraints (MPSC, in brief) which is introduced by Mehlitz [11]. It should be noted that the general form of an MPSC which has been considered in [11] includes equality constraints $h_t(x) = 0,\ t \in T$ for some finite index set T. Since adding these constraints to problem (MP) does not increase the technical problems of the issue and just prolongs the formulas, we ignore them

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and just deal with problem (MP). Also, compared to the MPSC considered in [3], the problem (MP) has additional inequality constraints as $g_j(x) \leq 0$ for $j \in J$, and this makes it possible to define some constraint qualifications (CQ, in brief) that were not possible to define in [3], such as the some Cottle type CQs; the definitions will be presented in the next sections.

Various CQs, optimality conditions, and exact penaltization for MPSC were studied in [10, 11]. Shikhman [17] studied MPSC from the topological perspective and established some interesting theorems from Morse theory. Li and Guo [9] investigated Mordukhovich stationary conditions for MPSC under some weak CQs. Some relaxation schemes for MPSC was explored [6]. Very recently, the Fréchet normal cone of the feasible set of MPSC is estimated by Jafariani *et al.* [5]. In the previous works referenced earlier, all functions that define MPSC are continuously differentiable. Gorgini and Kanzi [3] (resp. [4]) investigated Abadie (resp. Guignard) type CQs and stationary conditions for MPSC with non-differentiable locally Lipschitz functions, for the first time. The results of [3] are presented in terms of Mordukhovich subdifferential.

The first work about MMPSC with continuously differential data is [14]. There stated some CQs and stationarity conditions at weakly efficient solutions of the problem. It should be noted that there are no articles that study CQs and stationarity conditions for nonsmooth MMPSCs. In this paper, we are trying to fill this gap, i.e., we introduce, categorize, and compare various CQs and prove some stationarity conditions at weakly efficient solutions and properly efficient solutions of (MP).

Since the feasible set of an MPSC is not necessarily convex, even under the criteria of convexity of the functions that construct it, applying standard methods of convex analysis is not applicable here. Therefore, we take the nonsmooth analysis approach, and we assume that all emerging functions of (MP) are locally Lipschitz. To choose a suitable subdifferential, we select the Clarke subdifferential because its calculation rules are known.

The structure of subsequent sections of this paper is as follows: In Sect. 2, we define the required definitions and preliminary results, which are requested in the sequel. Section 3 is focused on the definition of several CQs and their interrelations. The stationarity conditions conditions, that are presented in Section 4, are divided in to three categories: weak stationarity conditions, Mordukhovich stationarity conditions, and strong stationarity conditions. Also, we compare our stated results with earlier results in Section 4.

2 Preliminaries

In this section, we briefly address some notations, basic definitions, and standard preliminaries which are used in the sequel, from[1, 15].

For a non-empty subset D of $\mathbb{R}^n$, its negative polar cone and its orthogonal cone are defined respectively as

$$D^{\preceq} := \{x \in \mathbb{R}^n \mid \langle x, d\rangle \leq 0, \quad \forall d \in D\},$$

$$D^{\perp} := D^{\preceq} \cap (-D)^{\preceq} = \{x \in \mathbb{R}^n \mid \langle x, d\rangle = 0, \quad \forall d \in D\},$$

where, $\langle \cdot, \cdot \rangle$ denotes the standard inner-product in $\mathbb{R}^n$. With convention $\emptyset^{\preceq} = \emptyset^{\perp} = \mathbb{R}^n$, it is easy to see ([15, Section 14]) that $D^{\preceq}$ and $D^{\perp}$ are closed convex cones for each $D \subseteq \mathbb{R}^n$. The convex hull, the convex cone, the closure, and the closed convex cone of $D \subseteq \mathbb{R}^n$ are respectively denoted by $conv(D)$, $cone(D)$, $\overline{D}$, and $\overline{cone}(D)$. Also, put

$$D^{\prec} := \big\{x \in \mathbb{R}^n \mid \langle x, d\rangle < 0, \quad \forall d \in D \setminus \{0_n\}\big\},$$

where the zero vector in $\mathbb{R}^n$ is denoted by 0_n. It is simple to check (see , e.g., [15]) that if D_1 and D_2 are two subsets of $\mathbb{R}^n$, we have

$$(D_1 \cup D_2)^{\preceq} = D_1^{\preceq} \cap D_2^{\preceq} \quad \& \quad (D_1 \cup D_2)^{\prec} = D_1^{\prec} \cap D_2^{\prec}, \tag{2.1}$$

$$D_1 \subseteq D_2 \implies D_2^{\prec} \subseteq D_1^{\prec} \;\; \& \;\; D_2^{\preceq} \subseteq D_1^{\preceq}, \tag{2.2}$$

$$D_1^{\prec} \neq \emptyset \implies \overline{D_1^{\prec}} = D_1^{\preceq}, \tag{2.3}$$

$$D_1^{\prec} = \big(conv(D_1)\big)^{\prec} = \big(cone(D_1)\big)^{\prec} \quad \& \quad D_1^{\preceq} = \big(conv(D_1)\big)^{\preceq} = \big(cone(D_1)\big)^{\preceq}. \tag{2.4}$$

It should be mentioned [15, Theorem 6.9] that if $\Pi := \{D_\ell \mid \ell \in \mathcal{L}\}$ is a collection of convex sets in $\mathbb{R}^n$, then:

$$cone\big(\bigcup_{\ell\in\mathcal{L}} D_\ell\big) = \Big\{\sum_{r=1}^{n} \lambda_r D_{\ell_r} \mid D_{\ell_r} \in \Pi \ \ \& \ \lambda_r \geq 0,\ \forall r \in \{1,\cdots,n\}\Big\}, \tag{2.5}$$

$$conv\big(\bigcup_{\ell\in\mathcal{L}} D_\ell\big) = \Big\{\sum_{r=1}^{n+1} \lambda_r D_{\ell_r} \mid D_{\ell_r} \in \Pi \ \ \& \ \lambda_r \geq 0 \ \ \& \ \sum_{r=1}^{n+1} \lambda_r = 1,\ \forall r \in \{1,\cdots,n+1\}\Big\}. \tag{2.6}$$

Theorem 2.1. For a given $D \subseteq \mathbb{R}^n$,

- if D is a finite set, $cone(D)$ is closed.
- if D is compact and $0_n \notin conv(D)$, $cone(D)$ is closed.

The Bouligand tangent cone (or contingent cone) of $D \neq \emptyset$ at $x_0 \in \overline{D}$ is denoted by $\Gamma(D, x_0)$,

$$\Gamma(D, x_0) := \Big\{u \in \mathbb{R}^n \mid \exists t_r \downarrow 0,\ \exists u_r \to u \text{ such that } x_0 + t_r u_r \in D \ \ \forall r \in \mathbb{N}\Big\}.$$

It is worth mentioning that $\Gamma(D, x_0)$ is a closed cone, while it is not necessarily convex. The Clarke directional derivative of locally Lipschitz function $\phi : \mathbb{R}^n \to \mathbb{R}$ at $x_0 \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$ is defined by

$$\phi^\circ(x_0; d) := \limsup_{u\to x_0,\ t\downarrow 0} \frac{\phi(u + td) - \phi(u)}{t},$$

and the Clarke subdifferential of ϕ at x_0 is defined by

$$\partial_c\phi(x_0) := \{\xi \in \mathbb{R}^n \mid \phi^\circ(x_0; d) \geq \langle \xi, d\rangle, \quad \forall d \in \mathbb{R}^n\}.$$

The Clarke subdifferential of a locally Lipschitz function ϕ is always a non-empty convex compact set. It is worth mentioning that if $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function, we have $\partial_c\varphi(x_0) = \big\{\nabla\varphi(x_0)\big\}$ for all $x_0 \in \mathbb{R}^n$, where $\nabla\varphi(x_0)$ denotes the gradient of φ at x_0 (see [1]). Let us recall some important properties of Clarke subdifferential from [1], which are widely used in what follows.

Let ϕ_1 and ϕ_s be locally Lipschitz functions from $\mathbb{R}^n$ to $\mathbb{R}$ and $x_0 \in \mathbb{R}^n$. Then

$$\partial_c\big(\max\{\phi_1, \phi_2\}\big)(x_0) \subseteq conv\Big(\partial_c\phi_1(x_0) \cup \partial_c\phi_2(x_0)\Big), \tag{2.7}$$

$$\partial_c\big(\lambda_1\phi_1 + \lambda_2\phi_2\big)(x_0) \subseteq \lambda_1\partial_c\phi_1(x_0) + \lambda_2\partial_c\phi_2(x_0), \qquad \forall \lambda_1, \lambda_2 \in \mathbb{R}, \tag{2.8}$$

$$\phi_1^\circ(x_0; \nu) = \max\big\{\langle \nu, \xi\rangle \mid \xi \in \partial_c\phi_1(x_0)\big\}. \tag{2.9}$$

A locally Lipshitz function $\phi : \mathbb{R}^n \to \mathbb{R}$ is said to be ∂_c-pseudolinear at $x_0 \in \mathbb{R}^n$ if ϕ is both ∂_c-pseudoconvex and ∂_c-pseudoconcave at x_0, i.e., for all $x \in \mathbb{R}^n$ one has

$$\begin{cases} \phi(x) < \phi(x_0) \implies \langle \xi, x - x_0\rangle < 0, \quad \forall \xi \in \partial_c\phi(x_0) \\ \phi(x) > \phi(x_0) \implies \langle \xi, x - x_0\rangle > 0, \quad \forall \xi \in \partial_c\phi(x_0) \end{cases}.$$

In other word, ϕ is ∂_c-pseudolinear at x_0 if

$$\exists \xi \in \partial_c\phi(x_0), \ \ \langle \xi, x - x_0\rangle = 0 \qquad \implies \phi(x) = \phi(x_0).$$

It should be noted that the concept of ∂_c-pseudolinearity is introduced and is characterized in [12].

3 Constraint Qualifications

As the first point of this section, we introduce some symbols. Motivated by [3, 8, 16], for each $L \subseteq J \cup K$ and $\varphi \in \{g, G, H\}$, we put

$$\sigma_L^{\varphi} := \bigcup_{\ell \in L} \partial_c \varphi_\ell(\hat{x}).$$

Throughout this article, for a fixed feasible point $\hat{x} \in \digamma$, we define the following index sets:

$$\begin{aligned}
J(\hat{x}) &:= \{j \in J \mid g_j(\hat{x}) = 0\},\\
K_G &:= \{k \in K \mid H_k(\hat{x}) \neq 0,\ G_k(\hat{x}) = 0\},\\
K_H &:= \{k \in K \mid H_k(\hat{x}) = 0,\ G_k(\hat{x}) \neq 0\},\\
K_{GH} &:= \{k \in K \mid H_k(\hat{x}) = 0,\ G_k(\hat{x}) = 0\}.
\end{aligned}$$

As mentioned in the Introduction, the existence of multiplicative constraints $G_k(x)H_k(x) = 0$ creates some difficulties in the analysis of problem (MP). Therefore, we are looking for classical problems that do not have multiplicative constraints. As explained in [3], we consider the following two tightened problems for this purpose (a tightened problem for (MP) is a multiobjective optimization problem that its feasible set is contained in $\digamma$):

$$\begin{array}{llll}
(MP_1): & \min & \big(f_1(x), \cdots, f_m(x)\big) & \\
& s.t. & g_j(x) \leq 0, & j \in J,\\
& & G_k(x) = 0, & k \in K_G \cup K_{GH},\\
& & H_k(x) = 0, & k \in K_H \cup K_{GH},
\end{array}$$

$$\begin{array}{llll}
(MP_{K_*}): & \min & \big(f_1(x), \cdots, f_m(x)\big) & \\
& s.t. & g_j(x) \leq 0, & j \in J,\\
& & G_k(x) = 0, & k \in K_G \cup K_*,\\
& & H_k(x) = 0, & k \in K_H \cup (K_{GH} \setminus K_*),
\end{array}$$

where $K_* \subseteq K_{GH}$ is chosen arbitrarily. Considering the above problems, we introduce the following Abadie and Guignard types CQs.

Definition 3.1. We say that (MP) satisfies the

(i): weak Abadie (resp. weak Guignard) CQ, denoted by ACQ$_W$ (resp. GCQ$_W$) at $\hat{x}$, if

$$\big(\sigma^g_{J(\hat{x})}\big)^{\preceq} \cap \big(\sigma^G_{K_G \cup K_{GH}}\big)^{\perp} \cap \big(\sigma^H_{K_H \cup K_{GH}}\big)^{\perp} \subseteq \Gamma(\digamma, \hat{x}),$$

$$\Big(\ resp.\ \ \big(\sigma^g_{J(\hat{x})}\big)^{\preceq} \cap \big(\sigma^G_{K_G \cup K_{GH}}\big)^{\perp} \cap \big(\sigma^H_{K_H \cup K_{GH}}\big)^{\perp} \subseteq \overline{cone}\big(\Gamma(\digamma, \hat{x})\big)\ \ \Big).$$

(ii): ACQ$_{K_*}$ (resp. GCQ$_{K_*}$) at $\hat{x}$, for some $K_* \subseteq K_{GH}$, if

$$\big(\sigma^g_{J(\hat{x})}\big)^{\preceq} \cap \big(\sigma^G_{K_G \cup K_*}\big)^{\perp} \cap \big(\sigma^H_{K_H \cup (K_{GH}\setminus K_*)}\big)^{\perp} \subseteq \Gamma(\digamma, \hat{x}),$$

$$\Big(\ resp.\ \ \big(\sigma^g_{J(\hat{x})}\big)^{\preceq} \cap \big(\sigma^G_{K_G \cup K_*}\big)^{\perp} \cap \big(\sigma^H_{K_H \cup (K_{GH}\setminus K_*)}\big)^{\perp} \subseteq \overline{cone}\big(\Gamma(\digamma, \hat{x})\big)\ \ \Big).$$

(iii): strong Abadie (resp. strong Guignard) CQ, denoted by ACQ$_S$ (resp. GCQ$_S$) at $\hat{x}$, if

$$\big(\sigma^g_{J(\hat{x})}\big)^{\preceq} \cap \big(\sigma^G_{K_G}\big)^{\perp} \cap \big(\sigma^H_{K_H}\big)^{\perp} \subseteq \Gamma(\digamma, \hat{x}),$$

$$\Big(\ resp.\ \ \big(\sigma^g_{J(\hat{x})}\big)^{\preceq} \cap \big(\sigma^G_{K_G}\big)^{\perp} \cap \big(\sigma^H_{K_H}\big)^{\perp} \subseteq \overline{cone}\big(\Gamma(\digamma, \hat{x})\big)\ \ \Big).$$

We observe that the ACQ$_W$, ACQ$_{K_*}$, and ACQ$_S$ are generalizations of Abadie-type CQs, introduced in [3], to the multiobjective cases. Also, the GCQ$_W$, GCQ$_{K_*}$, and GCQ$_S$ are the counterpart of CQs introduced in [14, 11] for the nonsmooth case. It should be noted that, unlike ACQ$_W$, GCQ$_W$, ACQ$_S$, and GCQ$_S$, the ACQ$_{K_*}$ and GCQ$_{K_*}$ are dependent on the partial $K_{GH} = K_* \cup (K_{GH} \setminus K_*)$ of K_{GH}. Motivated by [3], we define the following CQs that are not dependent on K_*.

Definition 3.2. We say that (MP) satisfies the ACQ$_\sharp$ (resp. GCQ$_\sharp$) at $\hat{x}$, if

$$(\sigma^g_{J(\hat{x})})^{\preceq} \cap \bigcup_{K_* \subseteq K_{GH}} \Big[\big(\sigma^G_{K_G \cup K_*}\big)^{\perp} \cap \big(\sigma^H_{K_H \cup (K_{GH} \setminus K_*)}\big)^{\perp} \Big] \subseteq \Gamma(F, \hat{x}),$$

$$\Big(\textit{resp.} \;\; (\sigma^g_{J(\hat{x})})^{\preceq} \cap \bigcup_{K_* \subseteq K_{GH}} \Big[\big(\sigma^G_{K_G \cup K_*}\big)^{\perp} \cap \big(\sigma^H_{K_H \cup (K_{GH} \setminus K_*)}\big)^{\perp} \Big] \subseteq \overline{cone}\big(\Gamma(F, \hat{x})\big) \;\; \Big).$$

The following theorem can characterize the ACQ$_\sharp$ and GCQ$_\sharp$ as a simpler form, without using K^*.

Theorem 3.3. The following equality holds:

$$\bigcup_{K_* \subseteq K_{GH}} \Big[\big(\sigma^G_{K_G \cup K_*}\big)^{\perp} \cap \big(\sigma^H_{K_H \cup (K_{GH} \setminus K_*)}\big)^{\perp} \Big] = \big(\sigma^G_{K_G}\big)^{\perp} \cap \big(\sigma^H_{K_H}\big)^{\perp} \cap \Bigg(\bigcap_{k \in K_{GH}} \Big[\big(\partial_c G_k(\hat{x})\big)^{\perp} \cup \big(\partial_c H_k(\hat{x})\big)^{\perp} \Big] \Bigg).$$

Proof . " $\subseteq$ ": Let

$$\nu \in \bigcup_{K_* \subseteq K_{GH}} \Big[\big(\sigma^G_{K_G \cup K_*}\big)^{\perp} \cap \big(\sigma^H_{K_H \cup (K_{GH} \setminus K_*)}\big)^{\perp} \Big].$$

Thus, there exists a subset K^* of K_{GH} such that

$$\begin{aligned} \nu \;\; \in \;\; & \big(\sigma^G_{K_G \cup K_*}\big)^{\perp} \cap \big(\sigma^H_{K_H \cup (K_{GH} \setminus K_*)}\big)^{\perp} = \big(\sigma^G_{K_G}\big)^{\perp} \cap \big(\sigma^G_{K_*}\big)^{\perp} \cap \big(\sigma^H_{K_H}\big)^{\perp} \cap \big(\sigma^H_{K_{GH} \setminus K_*}\big)^{\perp} \\ = \;\; & \big(\sigma^G_{K_G}\big)^{\perp} \cap \big(\sigma^G_{K_*}\big)^{\perp} \cap \Big(\bigcup_{k \in K^*} \partial_c G_k(\hat{x}) \Big)^{\perp} \cap \Big(\bigcup_{k \in K_{GH} \setminus K_*} \partial_c H_k(\hat{x}) \Big)^{\perp} \\ = \;\; & \big(\sigma^G_{K_G}\big)^{\perp} \cap \big(\sigma^G_{K_*}\big)^{\perp} \cap \Big(\bigcap_{k \in K^*} \big(\partial_c G_k(\hat{x})\big)^{\perp} \Big) \cap \Big(\bigcap_{k \in K_{GH} \setminus K_*} \big(\partial_c H_k(\hat{x})\big)^{\perp} \Big), \end{aligned}$$

where the last equality holds by (2.1). Now, if $k_0 \in K_{GH}$ is given, then $k_0 \in K^*$ or $k_0 \in K_{GH} \setminus K_*$. The above inclusion shows that

$$\begin{cases} \nu \in \big(\sigma^G_{K_G}\big)^{\perp} \cap \big(\sigma^G_{K_*}\big)^{\perp} \cap \big(\partial_c G_{k_0}(\hat{x})\big)^{\perp}, & \text{if } k_0 \in K^* \\ \nu \in \big(\sigma^G_{K_G}\big)^{\perp} \cap \big(\sigma^G_{K_*}\big)^{\perp} \cap \big(\partial_c H_{k_0}(\hat{x})\big)^{\perp}, & \text{if } k_0 \in K_{GH} \setminus K_*. \end{cases}$$

Consequently,

$$\begin{aligned} \nu \;\; \in \;\; & \Big(\big(\sigma^G_{K_G}\big)^{\perp} \cap \big(\sigma^G_{K_*}\big)^{\perp} \cap \big(\partial_c G_{k_0}(\hat{x})\big)^{\perp} \Big) \cup \Big(\big(\sigma^G_{K_G}\big)^{\perp} \cap \big(\sigma^G_{K_*}\big)^{\perp} \cap \big(\partial_c H_{k_0}(\hat{x})\big)^{\perp} \Big) \\ = \;\; & \big(\sigma^G_{K_G}\big)^{\perp} \cap \big(\sigma^H_{K_H}\big)^{\perp} \cap \Big[\big(\partial_c G_{k_0}(\hat{x})\big)^{\perp} \cup \big(\partial_c H_{k_0}(\hat{x})\big)^{\perp} \Big]. \end{aligned}$$

Since $k_0 \in K_{GH}$ was arbitrary chosen, the last inclusion implies that

$$\nu \in \big(\sigma^G_{K_G}\big)^{\perp} \cap \big(\sigma^H_{K_H}\big)^{\perp} \cap \Bigg(\bigcap_{k \in K_{GH}} \Big[\big(\partial_c G_k(\hat{x})\big)^{\perp} \cup \big(\partial_c H_k(\hat{x})\big)^{\perp} \Big] \Bigg),$$

and hence

$$\bigcup_{K_* \subseteq K_{GH}} \Big[\big(\sigma^G_{K_G \cup K_*}\big)^{\perp} \cap \big(\sigma^H_{K_H \cup (K_{GH} \setminus K_*)}\big)^{\perp} \Big] \subseteq \big(\sigma^G_{K_G}\big)^{\perp} \cap \big(\sigma^H_{K_H}\big)^{\perp} \cap \Bigg(\bigcap_{k \in K_{GH}} \Big[\big(\partial_c G_k(\hat{x})\big)^{\perp} \cup \big(\partial_c H_k(\hat{x})\big)^{\perp} \Big] \Bigg). \tag{3.1}$$

" $\supseteq$ ": Conversely, suppose that

$$\nu \in \big(\sigma^G_{K_G}\big)^{\perp} \cap \big(\sigma^H_{K_H}\big)^{\perp} \cap \Bigg(\bigcap_{k \in K_{GH}} \Big[\big(\partial_c G_k(\hat{x})\big)^{\perp} \cup \big(\partial_c H_k(\hat{x})\big)^{\perp} \Big] \Bigg), \tag{3.2}$$

is given. Let

$$\tilde{K_*} := \Big\{k \in K_{GH} \mid \nu \in \big(\partial_c G_k(\hat{x})\big)^{\perp}\Big\}.$$

Thus, $K_{GH} \setminus \tilde{K_*} \subseteq \Big\{k \in K_{GH} \mid \nu \in \big(\partial_c H_k(\hat{x})\big)^{\perp}\Big\}$, and so

$$\nu \in \Big(\bigcap_{k \in \tilde{K_*}} \big(\partial_c G_k(\hat{x})\big)^{\perp}\Big) \cap \Big(\bigcap_{k \in K_{GH} \setminus \tilde{K_*}} \big(\partial_c H_k(\hat{x})\big)^{\perp}\Big).$$

This inclusion and (3.2) deduce that

$$\begin{aligned}
\nu &\in \big(\sigma^G_{K_G}\big)^{\perp} \cap \big(\sigma^H_{K_H}\big)^{\perp} \cap \Big(\bigcap_{k \in \tilde{K_*}} \big(\partial_c G_k(\hat{x})\big)^{\perp}\Big) \cap \Big(\bigcap_{k \in K_{GH} \setminus \tilde{K_*}} \big(\partial_c H_k(\hat{x})\big)^{\perp}\Big) \\
&= \big(\sigma^G_{K_G}\big)^{\perp} \cap \big(\sigma^H_{K_H}\big)^{\perp} \cap \Big(\bigcup_{k \in \tilde{K_*}} \partial_c G_k(\hat{x})\Big)^{\perp} \cap \Big(\bigcup_{k \in K_{GH} \setminus \tilde{K_*}} \partial_c H_k(\hat{x})\Big)^{\perp} \\
&= \big(\sigma^G_{K_G \cup \tilde{K_*}}\big)^{\perp} \cap \big(\sigma^H_{K_H \cup (K_{GH} \setminus \tilde{K_*})}\big)^{\perp} \\
&\subseteq \bigcup_{K_* \subseteq K_{GH}} \Big[\big(\sigma^G_{K_G \cup K_*}\big)^{\perp} \cap \big(\sigma^H_{K_H \cup (K_{GH} \setminus K_*)}\big)^{\perp}\Big].
\end{aligned}$$

Since ν was chosen arbitrarily in (3.2), the converse inclusion of (3.1) holds, and the proof is complete. □

The following corollary shows that if the functions G_k and H_k are continuously differentiable, the GCQ$_\sharp$ coincides with the MPSC-GCQ presented in [11].

Corollary 3.4. If the functions G_k and H_k are continuously differentiable as $k \in K$, then

$$\begin{aligned}
&\bigcup_{K_* \subseteq K_{GH}} \Big[\big(\sigma^G_{K_G \cup K_*}\big)^{\perp} \cap \big(\sigma^H_{K_H \cup (K_{GH} \setminus K_*)}\big)^{\perp}\Big] = \\
&\Big\{\nu \in \mathbb{R}^n \mid \langle \nu, \nabla G_k(\hat{x})\rangle = 0,\ k \in K_G;\ \langle \nu, \nabla H_k(\hat{x})\rangle = 0,\ k \in K_H;\ \langle \nu, \nabla G_k(\hat{x})\rangle\langle \nu, \nabla H_k(\hat{x})\rangle = 0,\ k \in K_{GH}\Big\}.
\end{aligned}$$

Proof . The result is a direct consequence of Theorem 3.3, $\partial_c G_k(\hat{x}) = \{\nabla G_k(\hat{x})\}$ and $\partial_c H_k(\hat{x}) = \{\nabla H_k(\hat{x})\}$ for all $k \in K$. In fact

$$\begin{aligned}
\bigcap_{k \in K_{GH}} \Big[\big\{\nabla G_k(\hat{x})\big\}^{\perp} \cup \big\{\nabla H_k(\hat{x})\big\}^{\perp}\Big] &= \Big\{\nu \in \mathbb{R}^n \mid \langle \nu, \nabla G_k(\hat{x})\rangle = 0 \ \text{ or } \ \langle \nu, \nabla H_k(\hat{x})\rangle = 0,\ k \in K_{GH}\Big\} \\
&= \Big\{\nu \in \mathbb{R}^n \mid \langle \nu, \nabla G_k(\hat{x})\rangle\langle \nu, \nabla H_k(\hat{x})\rangle = 0,\ k \in K_{GH}\Big\}.
\end{aligned}$$

□

Note that for the checking of Abadie and Guignard types CQs, the calculation of the contingent cone of the feasible set is required, and this task is usually tricky (especially for (MP), which has a non-convex feasible set F). It is desirable to define some CQs in the Mangasarian-Fromovitz type, which are not only stronger than the Abadie and Guignard types CQs, but also their checking does not require the calculation of contingent cone.

Definition 3.5. We say that (MP) satisfies the

(i): first Cottle CQ, denoted by FCCQ, at $\hat{x}$, if

$$\big(\sigma^g_{J(\hat{x})}\big)^{\prec} \cap \big(\sigma^G_{K_G}\big)^{\perp} \cap \big(\sigma^H_{K_H}\big)^{\perp} \neq \emptyset.$$

(ii): second Cottle CQ, denoted by SCCQ, at $\hat{x}$, if

$$\big(\sigma^g_{J(\hat{x})}\big)^{\prec} \cap \big(\sigma^G_{K_G \cup K_{GH}}\big)^{\perp} \cap \big(\sigma^H_{K_H \cup K_{GH}}\big)^{\perp} \neq \emptyset.$$

(iii): $\mathrm{CCQ}_\sharp$ at $\hat{x}$, if

$$\left(\sigma^{g}_{J(\hat{x})}\right)^{\prec} \cap \bigcup_{K_*\subseteq K_{GH}} \left[\left(\sigma^{G}_{K_G\cup K_*}\right)^{\perp} \cap \left(\sigma^{H}_{K_H\cup(K_{GH}\setminus K_*)}\right)^{\perp}\right] \neq \emptyset.$$

(iv): CCQ_{K_*} at $\hat{x}$, if

$$\left(\sigma^{g}_{J(\hat{x})}\right)^{\prec} \cap \left(\sigma^{G}_{K_G\cup K_*}\right)^{\perp} \cap \left(\sigma^{H}_{K_H\cup(K_{GH}\setminus K_*)}\right)^{\perp} \neq \emptyset.$$

At first glance, the question arises why we did not choose the names "first Cottle CQ" and "second Cottle CQ", respectively "strong Cottle CQ" and "weak Cottle CQ". The following theorem, which specifies the relationships between the defined CQs, shows that if we named them like that, the implications between Cottle CQs would be the opposite of the implications between Abadie and Guignard CQs; and this would not only cause confusion but also be incompatible with the choice of "weak" and "strong" names.

Theorem 3.6. Let K_* be an arbitrary subset of K_{GH}. Then, the following implications hold between the defined CQs at $\hat{x}$:

$$\begin{array}{ccccc}
 & & & & \mathrm{SCCQ} \\
 & & & & \downarrow \\
 & & \mathrm{ACQ}_S & & \mathrm{CCQ}_{K_*} \\
 & \swarrow & \downarrow & & \downarrow \\
\mathrm{GCQ}_S & & \mathrm{ACQ}_\sharp & \overset{(\dagger)}{\longleftarrow} & \mathrm{CCQ}_\sharp \\
\downarrow & \swarrow & \downarrow & & \downarrow \\
\mathrm{GCQ}_\sharp & & \mathrm{ACQ}_{K_*} & & \mathrm{FCCQ} \\
\downarrow & \swarrow & \downarrow & & \\
\mathrm{GCQ}_{K_*} & & \mathrm{ACQ}_W & & \\
\downarrow & \swarrow & & & \\
\mathrm{GCQ}_W & & & &
\end{array} \tag{3.3}$$

where the implication that is marked with $(\dagger)$ holds when the G_k functions as $k \in K_G \cup K_{GH}$ and the H_k functions as $k \in K_H \cup K_{GH}$ are ∂_c-pseudolinear at $\hat{x}$.

Proof . The implications $\mathrm{ACQ}_\square \Longrightarrow \mathrm{GCQ}_\square$, for $\square \in \{S, W, \sharp, K_*\}$, are true by $\Gamma(\digamma, \hat{x}) \subseteq \overline{cone}\big(\Gamma(\digamma, \hat{x})\big)$. Also, the implications $\square_S \Longrightarrow \square_\sharp \Longrightarrow \square_{K_*} \Longrightarrow \square_W$, for $\square \in \{\mathrm{ACQ}, \mathrm{GCQ}\}$, and $\mathrm{SCCQ} \Longrightarrow \mathrm{CCQ}_{K_*} \Longrightarrow \mathrm{CCQ}_\sharp \Longrightarrow \mathrm{FCCQ}$ hold by the following chain of inclusions:

$$\begin{aligned}
\left(\sigma^{G}_{K_G\cup K_{GH}}\right)^{\perp} \cap \left(\sigma^{H}_{K_H\cup K_{GH}}\right)^{\perp} \quad &\subseteq \quad \left(\sigma^{G}_{K_G\cup K_*}\right)^{\perp} \cap \left(\sigma^{H}_{K_H\cup(K_{GH}\setminus K_*)}\right)^{\perp} \\
&\subseteq \quad \bigcup_{K_*\subseteq K_{GH}} \left[\left(\sigma^{G}_{K_G\cup K_*}\right)^{\perp} \cap \left(\sigma^{H}_{K_H\cup(K_{GH}\setminus K_*)}\right)^{\perp}\right] \\
&\subseteq \quad \left(\sigma^{G}_{K_G}\right)^{\perp} \cap \left(\sigma^{H}_{K_H}\right)^{\perp}.
\end{aligned}$$

We observe that the above inclusions are written by the following obvious relation, which is based on (2.2):

$$\Lambda_1 \subseteq \Lambda_2 \implies \sigma^{\psi}_{\Lambda_1} \subseteq \sigma^{\psi}_{\Lambda_2} \implies \left(\sigma^{\psi}_{\Lambda_2}\right)^{\perp} \subseteq \left(\sigma^{\psi}_{\Lambda_1}\right)^{\perp}, \quad \forall \Lambda_1, \Lambda_2 \subseteq K,\ \forall \psi \in \{G, H\}.$$

We are going to prove the implication $\mathrm{CCQ}_\sharp \overset{(\dagger)}{\Longrightarrow} \mathrm{ACQ}_\sharp$. Put

$$\theta(x) := \max_{j\in J} g_j(x), \qquad \forall x \in \mathbb{R}^n.$$

Suppose that $\mathrm{CCQ}_\sharp$ holds at $\hat{x}$. So, we can choose

$$\nu \in \left(\sigma^{g}_{J(\hat{x})}\right)^{\prec} \cap \bigcup_{K_*\subseteq K_{GH}} \left[\left(\sigma^{G}_{K_G\cup K_*}\right)^{\perp} \cap \left(\sigma^{H}_{K_H\cup(K_{GH}\setminus K_*)}\right)^{\perp}\right].$$

Thus, there exists a $K_* \subseteq K_{GH}$ such that $\nu \in \left(\sigma^{g}_{J(\hat{x})}\right)^{\prec} \cap \left(\sigma^{G}_{K_G\cup K_*}\right)^{\perp} \cap \left(\sigma^{H}_{K_H\cup(K_{GH}\setminus K_*)}\right)^{\perp}$, and hence by (2.4), (2.2), and (2.7) we obtain that

$$\nu \in \Big(conv\big(\sigma^{g}_{J(\hat{x})}\big)\Big)^{\prec} \cap \left(\sigma^{G}_{K_G\cup K_*}\right)^{\perp} \cap \left(\sigma^{H}_{K_H\cup(K_{GH}\setminus K_*)}\right)^{\perp} \subseteq \left(\partial_c\theta(x_0)\right)^{\prec} \cap \left(\sigma^{G}_{K_G\cup K_*}\right)^{\perp} \cap \left(\sigma^{H}_{K_H\cup(K_{GH}\setminus K_*)}\right)^{\perp}.$$

The last inclusion and (2.9) imply that for each $t > 0$ one has

$$\begin{cases} \langle \xi_\theta, \nu \rangle < 0, & \forall \xi_\theta \in \partial_c \theta(\hat{x}) \\ \langle \xi_G, \nu \rangle = 0, & \forall \xi_G \in \sigma^G_{K_G \cup K_*} \\ \langle \xi_H, \nu \rangle = 0, & \forall \xi_H \in \sigma^H_{K_H \cup (K_{GH} \setminus K_*)} \end{cases} \Longrightarrow \begin{cases} \theta^\circ(\hat{x}; \nu) = \max\left\{\langle \xi_\theta, \nu \rangle \mid \xi_\theta \in \partial_c \theta(\hat{x})\right\} < 0 & \\ \langle \xi_G, \hat{x} + t\nu - \hat{x} \rangle = 0, & \forall \xi_G \in \sigma^G_{K_G \cup K_*} \\ \langle \xi_H, \hat{x} + t\nu - \hat{x} \rangle = 0, & \forall \xi_H \in \sigma^H_{K_H \cup (K_{GH} \setminus K_*)}. \end{cases}$$

This relation, the definition of Clarke directional derivative of θ, and ∂_c-pseudolinearity of G_k, for $k \in K_G \cup K_{GH}$, and H_k, for $k \in K_H \cup K_{GH}$, conclude that we can find a scalar $\delta > 0$ such that

$$\begin{cases} \theta(\hat{x} + \varepsilon\nu) < \theta(\hat{x}) \leq 0, & \forall \varepsilon \in (0, \delta] \\ G_k(\hat{x} + t\nu) = G_k(\hat{x}) = 0, & \forall k \in K_G \cup K_*, \ \forall t > 0 \\ H_k(\hat{x} + t\nu) = H_k(\hat{x}) = 0, & \forall k \in K_H \cup (K_{GH} \setminus K_*), \ \forall t > 0. \end{cases}$$

Therefore, for each $\varepsilon \in (0, \delta]$ we have

$$\begin{cases} \theta(\hat{x} + \varepsilon\nu) < 0, & \\ G_k(\hat{x} + \varepsilon\nu) = 0, & \forall k \in K_G \\ H_k(\hat{x} + \varepsilon\nu) =, & \forall k \in K_H \\ G_k(\hat{x} + \varepsilon\nu) = 0 \text{ or } H_k(\hat{x} + \varepsilon\nu) = 0, & \forall k \in K_{GH} \end{cases} \Longrightarrow \begin{cases} g_j(\hat{x} + \varepsilon\nu) \leq \theta(\hat{x} + \varepsilon\nu) < 0, & \forall j \in J \\ G_k(\hat{x} + \varepsilon\nu) = 0, & \forall k \in K_G \\ H_k(\hat{x} + \varepsilon\nu) =, & \forall k \in K_H \\ G_k(\hat{x} + \varepsilon\nu) H_k(\hat{x} + \varepsilon\nu) = 0, & \forall k \in K_{GH} \end{cases}.$$

So, $\hat{x} + \varepsilon\nu \in F$, for all $\varepsilon \in (0, \delta]$, and hence $\nu \in \Gamma(F, \hat{x})$. Since ν was chosen arbitrarily, we obtain that

$$\left(\sigma^g_{J(\hat{x})}\right)^{\prec} \cap \bigcup_{K_* \subseteq K_{GH}} \left[\left(\sigma^G_{K_G \cup K_*}\right)^\perp \cap \left(\sigma^H_{K_H \cup (K_{GH} \setminus K_*)}\right)^\perp\right] \subseteq \Gamma(F, \hat{x}).$$

This inclusion, (2.3), and the closedness of $\bigcup_{K_* \subseteq K_{GH}} \left[\left(\sigma^G_{K_G \cup K_*}\right)^\perp \cap \left(\sigma^H_{K_H \cup (K_{GH} \setminus K_*)}\right)^\perp\right]$ and $\Gamma(F, \hat{x})$ conclude that

$$\begin{aligned} \left(\sigma^g_{J(\hat{x})}\right)^{\preceq} \cap \bigcup_{K_* \subseteq K_{GH}} \left[\left(\sigma^G_{K_G \cup K_*}\right)^\perp \cap \left(\sigma^H_{K_H \cup (K_{GH} \setminus K_*)}\right)^\perp\right] = \\ \overline{\left(\sigma^g_{J(\hat{x})}\right)^{\prec} \cap \bigcup_{K_* \subseteq K_{GH}} \left[\left(\sigma^G_{K_G \cup K_*}\right)^\perp \cap \left(\sigma^H_{K_H \cup (K_{GH} \setminus K_*)}\right)^\perp\right]} \subseteq \\ \overline{\Gamma(F, \hat{x})} = \Gamma(F, \hat{x}). \end{aligned}$$

The proof is complete. □

The following example shows valuable contents in the analysis of Diagram (3.3).

Example 3.7. Consider the following problem

$$\begin{aligned} (Q): \qquad & \min \quad \left(x_1 + 3|x_2| \ , \ x_2^2 + x_1|x_2|\right) \\ & s.t. \quad x_1 \geq 0, \\ & \qquad\ \ x_1^2 x_2 = 0. \end{aligned}$$

This problem has the form of (MP) by

$$g_1(x_1, x_2) := -x_1, \qquad G_1(x_1, x_2) := x_1^2, \qquad H_1(x_1, x_2) := x_2.$$

Considering $\hat{x} = 0_2 \in \digamma = (\mathbb{R}_+ \times \{0\}) \cup (\{0\} \times \mathbb{R})$, we have

$$K_{GH} = J(\hat{x}) = \{1\}, \quad K_G = K_H = \emptyset, \quad \sigma^g_{\{1\}} = \{(-1,0)\}, \quad \sigma^G_{\{1\}} = \{0_2\}, \quad \sigma^H_{\{1\}} = \{(0,1)\},$$

and so,

$$(\sigma^g_{\{1\}})^{\prec} = (0,+\infty) \times \mathbb{R}, \quad (\sigma^g_{\{1\}})^{\preceq} = \mathbb{R}_+ \times \mathbb{R}, \quad (\sigma^G_{\{1\}})^{\perp} = \mathbb{R} \times \mathbb{R}, \quad (\sigma^H_{\{1\}})^{\perp} = \mathbb{R} \times \{0\}.$$

Owing to

$$(\sigma^g_{J(\hat{x})})^{\preceq} \cap (\sigma^G_{K_G})^{\perp} \cap (\sigma^H_{K_H})^{\perp} = (\mathbb{R}_+ \times \mathbb{R}) \cap (\mathbb{R} \times \mathbb{R}) \cap (\mathbb{R} \times \mathbb{R}) = \mathbb{R}_+ \times \mathbb{R} \nsubseteq \digamma = \Gamma(\digamma, \hat{x}),$$

$$(\sigma^g_{J(\hat{x})})^{\preceq} \cap (\sigma^G_{K_G})^{\perp} \cap (\sigma^H_{K_H})^{\perp} = \mathbb{R}_+ \times \mathbb{R} = \overline{cone}(\Gamma(\digamma, \hat{x})),$$

we understand that ACQ_S fails, whereas GCQ_S holds at $\hat{x}$. So other introduced Guignard type CQs are satisfied at $\hat{x}$ (by (3.3)), and the inverse implication of $\text{ACQ}_S \Longrightarrow \text{GCQ}_S$ does not true. Also, since

$$(\sigma^g_{J(\hat{x})})^{\preceq} \cap (\sigma^G_{\emptyset})^{\perp} \cap (\sigma^H_{\{1\}})^{\perp} = (\mathbb{R}_+ \times \mathbb{R}) \cap (\mathbb{R} \times \mathbb{R}) \cap (\mathbb{R} \times \{0\}) = \mathbb{R}_+ \times \{0\} \subseteq \Gamma(\digamma, \hat{x}),$$

$$(\sigma^g_{J(\hat{x})})^{\preceq} \cap (\sigma^G_{\{1\}})^{\perp} \cap (\sigma^H_{\emptyset})^{\perp} = (\mathbb{R}_+ \times \mathbb{R}) \cap (\mathbb{R} \times \mathbb{R}) \cap (\mathbb{R} \times \mathbb{R}) = \mathbb{R}_+ \times \mathbb{R} \nsubseteq \Gamma(\digamma, \hat{x}),$$

$$(\sigma^g_{J(\hat{x})})^{\prec} \cap \bigcup_{K_* \subseteq K_{GH}} \left[(\sigma^G_{K_G \cup K_*})^{\perp} \cap (\sigma^H_{K_H \cup (K_{GH} \setminus K_*)})^{\perp} \right] =$$
$$(\sigma^g_{J(\hat{x})})^{\preceq} \cap \left[\left((\sigma^G_{\emptyset})^{\perp} \cap \sigma^H_{\{1\}})^{\perp} \right) \cap \left((\sigma^G_{\{1\}})^{\perp} \cap (\sigma^H_{\emptyset})^{\perp} \right) \right] = \mathbb{R}_+ \times \mathbb{R} \nsubseteq \Gamma(\digamma, \hat{x}),$$

unlike $\text{ACQ}_{\{1\}}$ and $\text{ACQ}_{\sharp}$, the $\text{ACQ}_{\emptyset}$ (and so, ACQ_W) holds at $\hat{x}$. According to

$$(\sigma^g_{J(\hat{x})})^{\prec} \cap (\sigma^G_{K_G \cup K_{GH}})^{\perp} \cap (\sigma^H_{K_H \cup K_{GH}})^{\perp} = ((0,+\infty) \times \mathbb{R}) \cap (\mathbb{R} \times \mathbb{R}) \cap (\mathbb{R} \times \{0\}) = (0,+\infty) \times \{0\} \neq \emptyset,$$

we see that SCCQ holds at $\hat{x}$. This shows that, in implication $\text{CCQ}_{\sharp} \Longrightarrow \text{ACQ}_{\sharp}$, the condition of ∂_c-pseudolinearity of G_k as $k \in K_G \cup K_{GH}$ and H_k as $k \in K_H \cup K_{GH}$ can not be removed. It should be noted that the function $H_1(x_1, x_2) = x_2^2$ is not ∂_c-pseudolinear at $\hat{x}$. As the last point of this section, we note that establishing different CQs for an MMPSC depends on the selection of functions G_k and H_k as $k \in K$. For example, if we consider the problem (Q), we can write it as (MP) where

$$g_1(x_1, x_2) := -x_1, \quad G_1(x_1, x_2) := x_1, \quad H_1(x_1, x_2) := x_1 x_2.$$

Thus, $K_{GH} = J(\hat{x}) = \{1\}$, $K_G = K_H = \emptyset$, and

$$\sigma^G_{\{1\}} = \{(1,0)\}, \quad \sigma^H_{\{1\}} = \{0_2\} \implies (\sigma^G_{\{1\}})^{\perp} = \{0\} \times \mathbb{R}, \quad (\sigma^H_{\{1\}})^{\perp} = \mathbb{R} \times \mathbb{R}.$$

It is easy to check that

$$(\sigma^g_{J(\hat{x})})^{\prec} \cap (\sigma^G_{K_G \cup K_{GH}})^{\perp} \cap (\sigma^H_{K_H \cup K_{GH}})^{\perp} = ((0,+\infty) \times \mathbb{R}) \cap (\{0\} \times \mathbb{R}) \cap (\mathbb{R} \times \mathbb{R}) = \emptyset,$$

$$(\sigma^g_{J(\hat{x})})^{\prec} \cap (\sigma^G_{\emptyset})^{\perp} \cap (\sigma^H_{\{1\}})^{\perp} = ((0,+\infty) \times \mathbb{R}) \cap (\mathbb{R} \times \mathbb{R}) \cap (\mathbb{R} \times \mathbb{R}) \neq \emptyset,$$

$$(\sigma^g_{J(\hat{x})})^{\preceq} \cap (\sigma^G_{\{1\}})^{\perp} \cap (\sigma^H_{\emptyset})^{\perp} = ((0,+\infty) \times \mathbb{R}) \cap (\{0\} \times \mathbb{R}) \cap (\mathbb{R} \times \mathbb{R}) = \emptyset.$$

The above relations show that SCCQ and $\text{CCQ}_{\{1\}}$ fail whereas $\text{CCQ}_{\emptyset}$ (and so, $\text{CCQ}_{\sharp}$ and FCCQ) holds at $\hat{x}$.

4 Stationarity Conditions

At the start of this section, we recall the following definition from [2].

Definition 4.1. A feasible point $\hat{x} \in \digamma$ is called

- the properly efficient solution to (MP) when there exist some scalars $\lambda_i > 0$ as $i \in I$ such that

$$\sum_{i\in I} \lambda_i f_i(\hat{x}) \leq \sum_{i\in I} \lambda_i f_i(x), \qquad \forall x \in \digamma.$$

- the weakly efficient solution to (MP) when there is no $x \in \digamma$ satisfying $f_i(x) < f_i(\hat{x})$, for all $i \in I$.

The following lemma is proved in step one of [7, Theorem 3.4].

Lemma 4.2. Let $\hat{x}$ be a weakly efficient solution of (MP). Then

$$\Big(\bigcup_{i\in I} \partial_c f_i(\hat{x})\Big)^{\prec} \cap \Gamma\,(S, \hat{x}) = \emptyset.$$

Moreover, if the f_i functions, for $i \in I$, are continuously differentiable at $\hat{x}$, then

$$\big\{\nabla f_i(\hat{x}) \mid i \in I\big\}^{\prec} \cap \overline{cone}\big(\Gamma\,(S, \hat{x})\big) = \emptyset.$$

The following two theorems present some first-order optimality conditions at weakly efficient solutions of (MP).

Theorem 4.3. Let $\hat{x}$ be a weakly efficient solution of (MP) such that ACQ_S (resp. ACQ_W) holds at $\hat{x}$. Moreover, assume that

$$cone\big(\sigma^g_{J(\hat{x})}\big) + span\big(\sigma^G_{K_G}\big) + span\big(\sigma^H_{K_H}\big),$$

$$\Big(resp.\ \ cone\big(\sigma^g_{J(\hat{x})}\big) + span\big(\sigma^G_{K_G\cup K_{GH}}\big) + span\big(\sigma^H_{K_H\cup K_{GH}}\big)\Big),$$

is a closed set. Then, there exist some nonnegative scalars $\alpha_i \geq 0$ as $i \in I$ and $\beta_j \geq 0$ as $j \in J(\hat{x})$, as well as nonnegative coefficients $\hat{\eta}_k$, $\hat{\mu}_k$, $\tilde{\eta}_k$ and $\tilde{\mu}_k$ as $k \in K$, satisfying

$$\begin{cases} 0_n \in \displaystyle\sum_{i\in I} \alpha_i \partial_c f_i(\hat{x}) + \sum_{j\in J(\hat{x})} \beta_j \partial_c g_j(\hat{x}) + \sum_{k\in K} \big[\hat{\eta}_k \partial_c G_k(\hat{x}) - \tilde{\eta}_k \partial_c G_k(\hat{x}) + \hat{\mu}_k \partial_c H_k(\hat{x}) - \tilde{\mu}_k \partial_c H_k(\hat{x})\big], \\ \displaystyle\sum_{i\in I} \alpha_i = 1, \end{cases} \tag{4.1}$$

and

$$\hat{\eta}_k = \tilde{\eta}_k = 0 \ \text{ for } k \in K_H \cup K_{GH}, \qquad \hat{\mu}_k = \tilde{\mu}_k = 0 \ \text{ for } k \in K_G \cup K_{GH}. \tag{4.2}$$

$$\Big(\,resp.\quad \hat{\eta}_k = \tilde{\eta}_k = 0 \ \text{ for } k \in K_H, \qquad \hat{\mu}_k = \tilde{\mu}_k = 0 \ \text{ for } k \in K_G\Big). \tag{4.3}$$

Proof . We prove the theorem for case ACQ_S. Because the proof of case ACQ_W is similar, we will not repeat it. Let

$$\mathfrak{B} := \bigcup_{i\in I} \partial_c f_i(\hat{x}) \qquad \text{and} \qquad \mathfrak{D} := \sigma^g_{J(\hat{x})} \cup \sigma^G_{K_G} \cup \big(-\sigma^G_{K_G}\big) \cup \sigma^H_{K_H} \cup \big(-\sigma^H_{K_H}\big).$$

We claim that

$$conv(\mathfrak{B}) \cap \big(-cone(\mathfrak{D})\big) \neq \emptyset. \tag{4.4}$$

Suppose, on the contrary, that (4.4) does not hold. Since

$$-cone\big(\mathfrak{D}\big) = -\Big[cone\big(\sigma^g_{J(\hat{x})}\big) + span\big(\sigma^G_{K_G}\big) + span\big(\sigma^H_{K_H}\big)\Big], \tag{4.5}$$

$-cone\big(\mathfrak{D}\big)$ is a closed convex cone (by assumption), and since $conv\big(\mathfrak{B}\big)$ is a non-empty convex set, the well-known strongly separation theorem [15, Corollary 1.4.1] implies that there exists a vector $\nu \in \mathbb{R}^n$ such that

$$\begin{cases} \langle \nu, z\rangle < 0, & \forall z \in conv\big(\mathfrak{B}\big) \\ \langle \nu, w\rangle \geq 0, & \forall w \in -cone\big(\mathfrak{D}\big) \end{cases} \implies \nu \in conv\big(\mathfrak{B}\big)^{\prec} \cap cone\big(\mathfrak{D}\big)^{\preceq} = \mathfrak{B}^{\prec} \cap \mathfrak{D}^{\preceq},$$

where the last equality holds by (2.4). This inclusion, the fact that $\mathfrak{D}^{\preceq} = \big(\sigma^g_{J(\hat{x})}\big)^{\preceq} \cap \big(\sigma^G_{K_G}\big)^{\perp} \cap \big(\sigma^H_{K_H}\big)^{\perp}$, and ACQ$_S$ assumption at $\hat{x}$ imply $\nu \in \mathfrak{B}^{\prec} \cap \Gamma(\digamma, \hat{x})$, which contradicts Lemma 4.2. This contradiction shows that the claimed (4.4) holds, and hence

$$0_n \in conv(\mathfrak{B}) + cone(\mathfrak{D}) = conv(\mathfrak{B}) + cone\big(\sigma^g_{J(\hat{x})}\big) + span\big(\sigma^G_{K_G}\big) + span\big(\sigma^H_{K_H}\big),$$

by (4.5). From this inclusion and (2.5)-(2.6), we get

$$\begin{cases} 0_n \in \sum\limits_{i\in I} \alpha_i \partial_c f_i(\hat{x}) + \sum\limits_{j\in J(\hat{x})} \beta_j \partial_c g_j(\hat{x}) + \sum\limits_{k\in K_G} \big[\hat{\eta}_k \partial_c G_k(\hat{x}) - \tilde{\eta}_k \partial_c G_k(\hat{x})\big] + \sum\limits_{k\in K_H} \big[\hat{\mu}_k \partial_c H_k(\hat{x}) - \tilde{\mu}_k \partial_c H_k(\hat{x})\big], \\ \sum\limits_{i\in I} \alpha_i = 1, \end{cases} \tag{4.6}$$

for some nonnegative scalars α_i as $i \in I$, β_j as $j \in J(\hat{x})$, $\hat{\eta}_k$ and $\tilde{\eta}_k$ as $k \in K_G$, and $\hat{\mu}_k$ and $\tilde{\mu}_k$ as $k \in K_H$. Putting $\hat{\eta}_k = \tilde{\eta}_k = 0$ as $k \in K_H \cup K_{GH}$ and $\hat{\mu}_k = \tilde{\mu}_k = 0$ as $k \in K_G \cup K_{GH}$, we conclude (4.1) and (4.2), and the proof is complete. □

Theorem 4.4. Let $\hat{x}$ be a weakly efficient solution of (MP) such that ACQ$_\sharp$ or ACQ$_{K_*}$, for some $K_* \subseteq K_{GH}$, holds at $\hat{x}$. Moreover, assume that the following cone is closed:

$$cone\big(\sigma^g_{J(\hat{x})}\big) + span\big(\sigma^G_{K_G\cup K_*}\big) + span\big(\sigma^H_{K_H\cup(K_{GH}\setminus K_*)}\big).$$

Then, there exist some nonnegative scalars $\alpha_i \geq 0$ as $i \in I$ and $\beta_j \geq 0$ as $j \in J(\hat{x})$, as well as nonnegative coefficients $\hat{\eta}_k$, $\hat{\mu}_k$, $\tilde{\eta}_k$ and $\tilde{\mu}_k$ as $k \in K$, satisfying (4.1) and

$$\hat{\eta}_k = \tilde{\eta}_k = 0 \;\text{ for } k \in K_H, \quad \hat{\mu}_k = \tilde{\mu}_k = 0 \;\text{ for } k \in K_G, \quad \hat{\eta}_k\hat{\mu}_k = \tilde{\eta}_k\hat{\mu}_k = \hat{\eta}_k\tilde{\mu}_k = \tilde{\eta}_k\tilde{\mu}_k = 0 \;\text{ for } k \in K_{GH}. \tag{4.7}$$

Proof . According to Theorem 3.6, it is enough to prove the theorem only for the case that ACQ$_{K_*}$ holds. Suppose that ACQ$_{K_*}$ is satisfied at $\hat{x}$. Repeating the proof of inclusion (4.6), we obtain that

$$0_n \in \sum_{i\in I} \alpha_i \partial_c f_i(\hat{x}) + \sum_{j\in J(\hat{x})} \beta_j \partial_c g_j(\hat{x}) + \sum_{k\in K_G\cup K_*} \big[\hat{\eta}_k \partial_c G_k(\hat{x}) - \tilde{\eta}_k \partial_c G_k(\hat{x})\big] + \sum_{k\in K_H\cup(K_{GH}\setminus K_*)} \big[\hat{\mu}_k \partial_c H_k(\hat{x}) - \tilde{\mu}_k \partial_c H_k(\hat{x})\big],$$

for some nonnegative scalars α_i as $i \in I$, β_j as $j \in J(\hat{x})$, $\hat{\eta}_k$ and $\tilde{\eta}_k$ as $k \in K_G \cup K_*$, and $\hat{\mu}_k$ and $\tilde{\mu}_k$ as $k \in K_H \cup (K_{GH} \setminus K_*)$. Put $\hat{\eta}_k = \tilde{\eta}_k = 0$ as $k \in K_H \cup (K_{GH} \setminus K_*)$ and $\hat{\mu}_k = \tilde{\mu}_k = 0$ as $k \in K_G \cup K_*$. Thus, (4.1) holds, furthermore considering $K_{GH} = K_* \cup (K_{GH} \setminus K_*)$, we conclude that for each $k \in K_{GH}$ one has $\hat{\eta}_k = \tilde{\eta}_k = 0$ or $\hat{\mu}_k = \tilde{\mu}_k = 0$. So, $\hat{\eta}_k\hat{\mu}_k = \tilde{\eta}_k\hat{\mu}_k = \hat{\eta}_k\tilde{\mu}_k = \tilde{\eta}_k\tilde{\mu}_k = 0$ for all $k \in K_{GH}$, and the result is proved. □

It is worth mentioning that when $m = 1$, conditions (4.1) & (4.2) were named the strong stationarity condition for (MP) in [3]. Thus, we call them the strong stationarity condition (SSC, in short). Also, motivated by [3], the conditions (4.1) & (4.3) and (4.1) & (4.7) are respectively named the weak stationarity condition (WSC) and the Mordukhovich stationarity condition (MSC). It is clear that the following implications hold at the weakly efficient solution $\hat{x}$ for (MP):

$$\text{SSC} \implies \text{MSC} \implies \text{WSC}.$$

Remark 4.5. As mentioned in [3], the restrictive assumption in Theorems 4.3 and 4.4 is the closedness of corresponding considered cones. Let us mention some critical conditions that ensure the closedness of these cones.

- **(i):** If g_j, G_k, and H_k are continuously differentiable as $j \in J(\hat{x})$ and $k \in K$, their Clarke subdifferentials contain single element, and so, the closedness condition of these cones automatically holds by Theorem 2.1(i).
- **(ii):** Whenever all appearing functions $g_j, G_k, H_k : \mathbb{R} \to \mathbb{R}$ are piecewise affine, their Clarke subdifferentials are (unions of) points and polyhedrons, and hence, the considered cones are finitely generated and naturally closed.
- **(iii):** According to the compactness of Clarke subdifferential, and using Theorem 2.1(ii), for each case, we can find a condition that implies the closedness of the considered cone.

The following example shows we can not replace the ACQ$_S$ with GCQ$_S$ in Theorem 4.3.

Example 4.6. Consider the following problem

$$\begin{array}{lll}(Q_1): & \min & -x_2+|x_1-x_2| \\ & s.t. & x_1\geq 0, \\ & & x_2 \geq 0, \\ & & x_1x_2=0.\end{array}$$

We can formalize this problem as (MP) with the following data:

$$g_1(x_1,x_2):=-x_1,\quad g_2(x_1,x_2):=-x_2,\quad G_1(x_1,x_2):=x_1,\quad H_1(x_1,x_2):=x_2.$$

We observe that $\digamma=(\{0\}\times\mathbb{R}_+)\cup(\mathbb{R}_+\times\{0\})$ and $\hat{x}:=0_2$ is an optimal solution of problem (Q_1). It is easy to see that

$$J(\hat{x})=\{1,2\},\quad K_{GH}=\{1\},\quad K_G=K_H=\emptyset,\quad \sigma^g_{\{1,2\}}=\{(-1,0),(0,-1)\},\quad \sigma^G_{\{1\}}=\{(1,0)\},\quad \sigma^H_{\{1\}}=\{(0,1)\},$$

and so,

$$\big(\sigma^g_{\{1,2\}}\big)^{\preceq}=\mathbb{R}_+\times\mathbb{R}_+,\quad \big(\sigma^G_{\{1\}}\big)^{\perp}=\{0\}\times\mathbb{R},\quad \big(\sigma^H_{\{1\}}\big)^{\perp}=\mathbb{R}\times\{0\}.$$

Since

$$\big(\sigma^g_{J(\hat{x})}\big)^{\preceq}\cap\big(\sigma^G_{K_G}\big)^{\perp}\cap\big(\sigma^H_{K_H}\big)^{\perp}=\big(\mathbb{R}_+\times\mathbb{R}_+\big)\cap\big(\{0\}\times\mathbb{R}\big)\cap\big(\mathbb{R}\times\{0\}\big)=\mathbb{R}_+\times\mathbb{R}_+\nsubseteq\digamma=\Gamma(\digamma,\hat{x}),$$

$$\big(\sigma^g_{J(\hat{x})}\big)^{\preceq}\cap\big(\sigma^G_{K_G}\big)^{\perp}\cap\big(\sigma^H_{K_H}\big)^{\perp}=\mathbb{R}_+\times\mathbb{R}_+=\overline{cone}\big(\Gamma(\digamma,\hat{x})\big),$$

the ACQ$_S$ fails, whereas GCQ$_S$ holds at $\hat{x}$. Note that $cone\big(\sigma^g_{J(\hat{x})}\big)+span\big(\sigma^G_{K_G}\big)+span\big(\sigma^H_{K_H}\big)=\mathbb{R}_+\times\mathbb{R}_+$, which is closed. Since

$$\partial_c f_1(\hat{x})=\Big\{(\rho,-1-\rho)\mid\rho\in[-1,1]\Big\},$$

it is easy to check that the below SSC does not hold for any non-negative scalars $\alpha_1,\beta_1,\beta_2,\hat{\eta}_1,\tilde{\eta}_1,\hat{\mu}_1$ and $\tilde{\mu}_1$:

$$\begin{cases} 0_2=\alpha_1(\rho,-1-\rho)+\beta_1(-1,0)+\beta_2(0,-1)+\hat{\eta}_1(1,0)-\tilde{\eta}_1(1,0)+\hat{\mu}_1(0,1)-\tilde{\mu}_1(0,1), \\ \\ \rho\in[-1,1],\quad \alpha_1=1,\quad \hat{\eta}_1=\tilde{\eta}_1=\hat{\mu}_1=\tilde{\mu}_1=0.\end{cases}$$

In fact, since $\beta_1(-1,0)+\beta_2(0,-1)=(-\beta_1,-\beta_2)$ has two non-negative components and $(\rho,-1-\rho)$ has at least one negative component for $\rho\in[-1,1]$, their sum can not be equal to zero.

The following Theorem shows that Theorems 4.3 and 4.4 have a simpler forms for smooth (MP), under Guignard type-CQs. As mentioned in Remark 4.5, in this case, the assumption of the closedness of cones will automatically be true. Note that when $m=1$, the following theorem coincides with SSC, WSC, and MSC, presented in [11].

Theorem 4.7. Assume that $\hat{x}$ is a weakly efficient solution for smooth (MP), and GCQ$_W$ or GCQ$_\sharp$ or GCQ$_{K_*}$ or GCQ$_S$ holds. Then, there exist some nonnegative scalars α_i and β_j for $(i,j)\in I\times J(\hat{x})$, as well as real coefficients η_k and μ_k for $k\in K$, satisfying

$$\begin{cases} \displaystyle\sum_{i\in I}\alpha_i\nabla f_i(\hat{x})+\sum_{j\in J(\hat{x})}\beta_j\nabla g_j(\hat{x})+\sum_{k\in K}\Big[\eta_k\nabla G_k(\hat{x})+\mu_k\nabla H_k(\hat{x})\Big]=0_n, \\ \\ \displaystyle\sum_{i\in I}\alpha_i=1.\end{cases}$$

Moreover, if GCQ$_W$ holds, we have

$$\eta_k=0\ \text{ for } k\in K_H,\qquad \mu_k=0\ \text{ for } k\in K_G,$$

and if GCQ$_\sharp$ or GCQ$_{K_*}$ holds, we have

$$\eta_k=0\ \text{ for } k\in K_H,\qquad \mu_k=0\ \text{ for } k\in K_G,\qquad \eta_k\mu_k=0\ \text{ for } k\in K_{GH},$$

and if GCQ$_S$ holds, we have

$$\eta_k=0\ \text{ for } k\in K_H\cup K_{GH},\qquad \mu_k=0\ \text{ for } k\in K_G\cup K_{GH}.$$

Proof . It is enough to prove the theorem only for the case that $\mathrm{GCQ}_\sharp$ holds. The proof in other cases is similar. Owing to $\partial_c\varphi(\hat{x}) = \{\nabla\varphi(\hat{x})\}$ for $\varphi \in \{f_i, g_j, G_k, H_k \mid (i,j,k) \in I \times J(\hat{x}) \times K\}$ and according to Theorem 2.1(i), we understand that the closedness condition in Theorem 4.4 is true. Thus, using Lemma 4.2, we can repeat the proof of Theorem 4.3 and get

$$0_n \in \sum_{i\in I}\alpha_i\{\nabla f_i(\hat{x})\} + \sum_{j\in J(\hat{x})}\beta_j\{\nabla g_j(\hat{x})\} + \sum_{k\in K}\Big[\hat{\eta}_k\{\nabla G_k(\hat{x})\} - \tilde{\eta}_k\{\nabla G_k(\hat{x})\} + \hat{\mu}_k\{\nabla H_k(\hat{x})\} - \tilde{\mu}_k\{\nabla H_k(\hat{x})\}\Big],$$

for some nonnegative coefficients $\alpha_i, \beta_j, \hat{\eta}_k, \tilde{\eta}_k, \hat{\mu}_k$ and $\tilde{\mu}_k$ satisfying in (4.7). Hence,

$$\sum_{i\in I}\alpha_i\nabla f_i(\hat{x}) + \sum_{j\in J(\hat{x})}\beta_j\nabla g_j(\hat{x}) + \sum_{k\in K}\Big[(\hat{\eta}_k - \tilde{\eta}_k)\nabla G_k(\hat{x}) + (\hat{\mu}_k - \tilde{\mu}_k)\nabla H_k(\hat{x})\Big] = 0_n.$$

Taking $\eta_k := \hat{\eta}_k - \tilde{\eta}_k \in \mathbb{R}$ and $\mu_k := \hat{\mu}_k - \tilde{\mu}_k \in \mathbb{R}$ for all $k \in K$, by (4.7) we have

$$\eta_k = 0 \ \text{ for } k \in K_H, \quad \mu_k = 0 \ \text{ for } k \in K_G, \quad \eta_k\mu_k = \hat{\eta}_k\hat{\mu}_k - \tilde{\eta}_k\hat{\mu}_k - \hat{\eta}_k\tilde{\mu}_k + \tilde{\eta}_k\tilde{\mu}_k = 0 \ \text{ for } k \in K_{GH},$$

and the proof is complete. □

It is worth mentioning that, in SSC, MSC, and WSC, we obtain nonnegative multipliers α_i as $i \in I$ associated with objective function f_i for $i \in I$, some of the multipliers may be equal to zero.We say that strict SSC (resp. strict MSC, and strict WSC), denoted by S-SSC (resp. S-MSC, and S-WSC), holds for (MP), when the multipliers α_i are positive for all components f_i of the objective function in SSC (resp. MSC, and WSC). As a consequence of Theorems 4.3 and 4.4, the following theorem is to derive the S-SSC, S-MSC, and S-WSC at the properly efficient solutions of (MP).

Theorem 4.8. Let $\hat{x}$ be a properly efficient solution of (MP) such that ACQ_S (resp. $\mathrm{ACQ}_\sharp$, and ACQ_W) holds at $\hat{x}$. If assume the corresponding cones, considered in Theorems 4.3 and 4.4, is closed, then S-SSC (resp. S-MSC, and S-WSC) is satisfied, i.e., the corresponding results of Theorems 4.3 and 4.4 hold with $\alpha_i > 0$ for all $i \in I$.

Proof . We only prove S-SSC under satisfying ACQ_S, and the proof for the other cases is similar. Since $\hat{x}$ is a properly efficient solution of (MP), we can find some scalars $\lambda_i > 0$ as $i \in I$ such that

$$\sum_{i\in I}\lambda_i f_i(\hat{x}) \le \sum_{i\in I}\lambda_i f_i(x), \qquad \forall x \in F.$$

This means $\hat{x}$ is an optimal solution of the following single-objective MPSC

$$\min \vartheta(x) \quad s.t. \quad x \in F,$$

in which $\vartheta(x) := \Big(\sum_{i\in I}\lambda_i f_i\Big)(x)$. Employing Theorem 4.3, there exist some nonnegative scalars $\beta_j, \hat{\eta}_k, \tilde{\eta}_k, \hat{\mu}_k$, and $\tilde{\mu}_k$ as $(j,k) \in J(\hat{x}) \times K$ satisfying (4.2) and

$$0_n \in \partial_c\vartheta(\hat{x}) + \sum_{j\in J(\hat{x})}\beta_j\partial_c g_j(\hat{x}) + \sum_{k\in K}\big[\hat{\eta}_k\partial_c G_k(\hat{x}) - \tilde{\eta}_k\partial_c G_k(\hat{x}) + \hat{\mu}_k\partial_c H_k(\hat{x}) - \tilde{\mu}_k\partial_c H_k(\hat{x})\big].$$

Since $\partial_c\vartheta(\hat{x}) \subseteq \sum_{i\in I}\lambda_i\partial_c f_i(\hat{x})$ by (2.8), the above inclusion implies that

$$0_n \in \sum_{i\in I}\lambda_i\partial_c f_i(\hat{x}) + \sum_{j\in J(\hat{x})}\beta_j\partial_c g_j(\hat{x}) + \sum_{k\in K}\big[\hat{\eta}_k\partial_c G_k(\hat{x}) - \tilde{\eta}_k\partial_c G_k(\hat{x}) + \hat{\mu}_k\partial_c H_k(\hat{x}) - \tilde{\mu}_k\partial_c H_k(\hat{x})\big],$$

for $\lambda_i > 0$ as $i \in I$, and the result is proved. □

Repeating the proof of Theorem 4.9, and using Theorem 4.7, we receive the following theorem.

Theorem 4.9. Let $\hat{x}$ be a properly efficient solution of smooth (MP) such that GCQ_S (resp. $\mathrm{GCQ}_\sharp$, and GCQ_W) holds at $\hat{x}$. If assuming the corresponding cones, considered in Theorem 4.7, is closed, then S-SSC (resp. S-MSC, and S-WSC) is satisfied.

Finally, we note that as shown in [11], any mathematical problems with vanishing constraints (MPVC, in brief) can be rewritten in the form of a MPSC. This issue concludes that the results obtained in this article are not only the generalization of the results of articles [11, 3, 4], but also the extension and rewriting of the results established in [8, 13, 16] and their references.

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