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Existence and Ulam-Hyers stability analysis for nonlinear Langevin equation featuring two fractional orders involving anti-periodic boundary conditions

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Abstract

This paper presents an investigation of the existence and the unique feature of solutions for nonlinear Langevin equations involving two fractional orders with anti-periodic boundary conditions (APBCs). As a result of employing some fixed point theorems like Schauder and contraction mapping principles, the existence and uniqueness of solutions are examined. On top of this, the stability within the scope of Ulam–Hyers of solutions to this problem is also considered. The distinctive features of the present study are its similar variant and the existence of derivatives of Caputo and Riemann in the problem structure. Finally, to illustrate the result of the study, an example is presente

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1 Introduction

The nonlinear Langevin equation (NLE) was first suggested by Paul Langevin in [24], a renowned Ferench physicist, in the 19th century. This scholar employed the Langevien equation for the purpose of preparing a thorough and accurate description of Brownian motion.

In the domains of oscillation, the NLE is used as a model to describe physical phenomena. Various important phenomena and theories in which this equation can be applied are anomalous transport [20] modeling the evacuation processes [22], photo-electron counting [40], self-organization in complex systems [14], protein dynamics [35], studying the fluid suspensions [16], deuteron cluster dynamics [36] and analyzing the stock market [7].

The generalized NLE (GNLE) may be considered as one of the important topics. In nineteenth century, this equation was determined by Kubo [23]. The main reason why this equation was offered is that the NLE would not succeed a convenient explanation of dynamics in a number of phenomena like anomalous diffuse processes or in case of systems like complicated media and fractal media.

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Expression of particle motion is another related example for GNLE, in case of distinguishing between the random fluctuation force and white noise [10].

Among other interesting topics, we can mention fractional nonlinear Langevin's equation (FNLE), which is a special model of the GNLE. The Langevin-Qarray equation in the late 19th century, was offered by Mainardi and associates [28, 29].

Including usages of the NLE, motor control system modeling [39], modeling gait variability [39], one may point to financial markets [34], financial markets single file diffusion [12], describing anomalous diffusion [21], and studies on Brownian fractional motion [17]. It should be noted that a number of researches including fractional derivatives have been conducted in the field of the FNLE. One of the first researches in this field is the paper written by Lutz [27] about Brownian motion involving the fraction derivative.

The extension of the NLE to two or three fractional-order derivatives has been studied by many researchers [2, 4, 11, 13, 15, 18, 25, 26, 31, 32, 37, 38, 41, 42, 43]. The discussion with help from theorems such as Krasnosel'skii fixed point (KFP), Schauder fixed point (SFP), Banach contraction principle (BCP) and Leray-Schauder type (LSN) on the existence and uniqueness of the solution of the FNLEs has been examined in most of the papers mentioned.

Recently a great degree of attention has been given to APBVPs that occur in the process of modeling a myriad of physical phenomena. Readers are referred to illustrations and details of APBCs in [3, 8, 9].

The study of Ulam stability for the fractional differential system (FDS) has been investigated by many authors. They have discussed various Ulam−Hyers stability problems for different kinds of FDS including Langevin equation by using many techniques; see [1, 5, 6], and the references therein.

In the present research work, the distinctive feature and the existence of the APBVPs of Langevin equation containing three distinct fractional orders are investigated:

$$\begin{cases} \mathcal{D}^{\alpha_2}\big(\mathcal{D}^{\alpha_1}+\lambda\big)x(t)=\sigma\big(t,\ {}^{RL}\mathcal{D}^{\alpha-1}x(t)\big), \quad 0<\alpha_1\le 1, \quad 1<\alpha_2\le 2, \quad t\in[0,1] \\ x(0)+x(1)=0, \\ \mathcal{D}^{\alpha_1}x(0)+\mathcal{D}^{\alpha_1}x(1)=0, \quad \mathcal{D}^{2\alpha_1}x(0)+\mathcal{D}^{2\alpha_1}x(1)=0, \end{cases} \tag{1.1}$$

where $\mathcal{D}^{\alpha_1}$ and ${}^{RL}\mathcal{D}^{\alpha-1}$ are the Caputo and Riemann-Liouville fractional derivative of order α_1, respectively, $\sigma : [0,1]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function and λ is a real number.

Moreover, $\mathcal{D}^{\alpha_1}x$ is the sequential fractional derivative suggested by Miller and Ross [30]

$$\begin{cases} \mathcal{D}^{\alpha_1}x=D^{\alpha_1}x, \\ \mathcal{D}^{k\alpha_1}x=\mathcal{D}^{\alpha_1}\mathcal{D}^{(k-1)\alpha_1}x, \end{cases} \tag{1.2}$$

The uniqueness and the existence of solutions for APBVPs (1.1) are examined by utilizing the SFP theorem and BCP.

2 Preliminaries

In this section, a number of familiar definitions and qualities of fractional calculus theory and a number of lemmas that are employed in the rest of the paper are described. Interested researchers can study [19] for more details.

Definition 2.1. [19] The Riemann-Liouville fractional integral of order $\rho>0$ for integrable function $z:(0,\infty)\to\Re$ is defined by

$$\mathcal{I}^{\rho}z(\tau)=\int_0^{\tau}\frac{(\tau-s)^{\rho-1}}{\Gamma(\rho)}z(s)ds,$$

in case the integral exists.

Definition 2.2. [19] The Riemann-Liouville fractional derivative of order $\rho>0$ of a continuous function $z:(0,\infty)\to\Re$ is described by

$$\mathcal{D}^{\rho}z(\tau)=\frac{1}{\Gamma(n-\rho)}\left(\frac{d}{d\tau}\right)^{n}\int_0^{\tau}\frac{(\tau-s)^{n-\rho-1}}{\Gamma(\rho)}z(s)ds,$$

where $n=[\rho]+1$, on condition that a finite integral exists.

Definition 2.3. [19] The Caputo fractional derivative of order $\rho > 0$, of an integrable function $z : (0, \infty) \to \Re$ can be described by

$$\mathcal{D}^{\rho} z(\tau) = \mathcal{I}^{n-\rho} z^{(n)}(\tau) = \int_0^{\tau} \frac{(\tau - s)^{n-\rho-1}}{\Gamma(n-\rho)} z^{(n)}(s) ds,$$

where $n = [\rho] + 1$, provided that the integral exists and is finite.

Lemma 2.4. [19] Let $\rho, \delta \geq 0$ and $n \in \mathbb{N}$, thus the relations given below are true.

i) ${}^{RL}\mathcal{D}^{\rho}\mathcal{I}^{\rho} x = x$;
ii) $\mathcal{I}^{\rho}\mathcal{I}^{\delta} x = \mathcal{I}^{\rho+\delta} x$;
iii) ${}^{RL}\mathcal{D}^{\rho}\mathcal{I}^{\delta} x = {}^{RL}\mathcal{D}^{\rho-\delta} x$ (if $\rho \geq \delta$);
iv) ${}^{RL}\mathcal{D}^{\rho}\mathcal{I}^{\delta} x = \mathcal{I}^{\delta-\rho} x$ (if $\delta \geq \rho$);

Lemma 2.5. Let $\rho > 0$, then the fractional differential equation $\mathcal{D}^{\rho} x(t) = 0$ has a general solution as:

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{m-1} t^{m-1},$$

for some $c_k \in \mathbb{R},\ k = 1, \cdots, m-1,\ m = [\rho] + 1$.

Lemma 2.6. Let $\rho > 0$, hence it results in

$$\mathcal{I}^{\rho}\mathcal{D}^{\rho} x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{m-1} t^{m-1},$$

where $c_k \in \mathbb{R},\ k = 1, \cdots, m-1,\ m = [\rho] + 1$.

Lemma 2.7. $x(t)$ is a solution of problem

$$\begin{cases} \mathcal{D}^{\alpha_2}\big(\mathcal{D}^{\alpha_1} + \lambda\big) x(t) = \phi(t), \quad 0 < \alpha_1 \leq 1, \quad 1 < \alpha_2 \leq 2, \quad t \in [0,1] \\ x(0) + x(1) = 0, \\ \mathcal{D}^{\alpha_1} x(0) + \mathcal{D}^{\alpha_1} x(1) = 0, \quad \mathcal{D}^{2\alpha_1} x(0) + \mathcal{D}^{2\alpha_1} x(1) = 0, \end{cases} \tag{2.1}$$

iff it is a solution of the non linear mixed Fredholm-Volterra integral equation

$$\begin{aligned} x(t) = & \int_0^t \frac{(t-s)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} \phi(s) ds - \lambda \int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} x(s) ds - \frac{1}{2} \int_0^1 \frac{(1-s)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} \phi(s) ds + \frac{\lambda}{2} \int_0^1 \frac{(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} x(s) ds \\ & + \left(\frac{1}{4\Gamma(\alpha_1+1)} - \frac{t^{\alpha_1}}{2\Gamma(\alpha_1+1)} \right) \int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} \phi(s) ds \\ & + \frac{\Gamma(2-\alpha_1)}{\Gamma(2+\alpha_1)} \left(\frac{1-\alpha_1}{4} + \frac{1+\alpha_1}{2} t^{\alpha_1} - t^{\alpha_1+1} \right) \int_0^1 \frac{(1-s)^{\alpha_2-\alpha_1-1}}{\Gamma(\alpha_2-\alpha_1)} \phi(s) ds. \end{aligned} \tag{2.2}$$

Proof . Let $x(t)$ be a solution of the problem (1.1). From (i) of Lemma 2.4, we have

$$\mathcal{D}^{\alpha_2}\big(\mathcal{D}^{\alpha_1} + \lambda\big) x(t) = \mathcal{D}^{\alpha_2}\mathcal{I}^{\alpha_2}\phi(t).$$

Then by the application of Lemma 2.6, we come to the equation below

$$\big(\mathcal{D}^{\alpha_1} + \lambda\big) x(t) - \mathcal{I}^{\alpha_2}\phi(t) = c_1 + c_2 t, \tag{2.3}$$

or equivalently,

$$\mathcal{D}^{\alpha_1}\Big[x(t) + \lambda \mathcal{I}^{\alpha_1} x(t) - \mathcal{I}^{\alpha_1+\alpha_2}\phi(t) - c_1 \frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)} - c_2 \frac{t^{\alpha_1+1}}{\Gamma(\alpha_1+2)}\Big].$$

By employing Lemma 2.6 one more time, the overall version of the problem (1.1) will be described as

$$x(t) = \mathcal{I}^{\alpha_1+\alpha_2}\phi(t) - \lambda \mathcal{I}^{\alpha_1} x(t) + c_0 + c_1 \frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)} + c_2 \frac{t^{\alpha_1+1}}{\Gamma(\alpha_1+2)}.$$

Through utilizing the anti-periodic boundary conditions for the problem (1.1), the following will be achieved.

$$c_0 = -\frac{1}{2}\int_0^1 \frac{(1-s)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}\phi(s)ds + \frac{\lambda}{2}\int_0^1 \frac{(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}x(s)ds$$
$$-\frac{1}{2\Gamma(\alpha_1+1)}\left(-\frac{1}{2}\int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\sigma(s)ds + \frac{\Gamma(2-\alpha_1)}{2}\int_0^1 \frac{(1-s)^{\alpha_2-\alpha_1-1}}{\Gamma(\alpha_2-\alpha_1)}\phi(s)ds\right)$$
$$-\frac{\Gamma(2-\alpha_1)}{2\Gamma(\alpha_1+2)}\int_0^1 \frac{(1-s)^{\alpha_2-\alpha_1-1}}{\Gamma(\alpha_2-\alpha_1)}\phi(s)ds$$

$$c_1 = -\frac{1}{2}\int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\phi(s)ds + \frac{\Gamma(2-\alpha_1)}{2}\int_0^1 \frac{(1-s)^{\alpha_2-\alpha_1-1}}{\Gamma(\alpha_2-\alpha_1)}\phi(s)ds$$

$$c_2 = -\Gamma(2-\alpha_1)\int_0^1 \frac{(1-s)^{\alpha_2-\alpha_1-1}}{\Gamma(\alpha_2-\alpha_1)}\phi(s)ds.$$

When we substitute the values of c_0, c_1 and c_2 in (2.3), the solution (2.2) will be obtained. To explain the results in another form, it can be said that it can easily be proved that, in case $x(t)$ is a solution of the integral equation (2.2) the $x(t)$ is also a solution of the problem (1.1). □

3 Main result

For the rest of the paper, the uniqueness and the existence of solutions for FLE will be examined. The space of continues functions $C[0,1]$ will be explained. Also the space E is defined as:

$$E = \left\{x \mid x \in C[0,1],\ {}^{RL}\mathcal{D}^{\alpha_1}x \in C[0,1]\right\}, \tag{3.1}$$

equipped with the norm

$$\|x\|_* = \max_{t\in[0,1]}|x(t)| + \max_{t\in[0,1]}|\ {}^{RL}\mathcal{D}^{\alpha_1}x(t)| = \|x\| + \|\ {}^{RL}\mathcal{D}^{\alpha_1}x\|. \tag{3.2}$$

Lemma 3.1. The space $(E, \|.\|_*)$ is a Banach space.

Proof . Suppose $\{\mathfrak{u}_n\}_{n=1}^{\infty}$ be a Cauchy sequence in the space $(E, \|.\|_*)$. Thus, it is evident $\{\mathfrak{u}_n\}_{n=1}^{\infty}$ and $\{\ {}^{RL}\mathcal{D}^{\alpha_1}\mathfrak{u}_n\}_{n=1}^{\infty}$ are Cauchy sequences on $C[0,1]$. So $\{\mathfrak{u}_n\}_{n=1}^{\infty}$ and $\{\ {}^{RL}\mathcal{D}^{\alpha_1}\mathfrak{u}_n\}_{n=1}^{\infty}$ uniformly convergence to some v, w on $[0,1]$. It is sufficient that we prove $w = {}^{RL}\mathcal{D}^{\alpha_1}v$. Consider the following inequality:

$$\begin{aligned}|\mathfrak{I}^{\alpha_1}\left({}^{RL}\mathcal{D}^{\alpha_1}\mathfrak{u}_n(t)\right) - \mathfrak{I}^{\alpha_1}w(t)| &\le \int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha)}|\ {}^{RL}\mathcal{D}^{\alpha_1}\mathfrak{u}_n(s) - w(s)|ds \\ &\le \frac{1}{\Gamma(\alpha_1+1)}\max_{t\in[0,1]}|\ {}^{RL}\mathcal{D}^{\alpha_1}\mathfrak{u}_n(t) - w(t)|.\end{aligned}$$

For every $t \in [0,1]$, by the convergence of $\{\ {}^{RL}\mathcal{D}^{\alpha_1}\mathfrak{u}_n\}_{n=1}^{\infty}$, we have $\lim_{n\to\infty}\mathfrak{I}^{\alpha}\left({}^{RL}\mathcal{D}^{\alpha_1}u_n(t)\right) = \mathfrak{I}^{\alpha_1}w(t)$. uniformly. As well, by Lemmas 2.4 and 2.6, we have $\mathfrak{I}^{\alpha_1}\left({}^{RL}\mathcal{D}^{\alpha_1}\mathfrak{u}_n(t)\right) = \mathfrak{u}_n(t)$. Accordingly $v(t) = \mathfrak{I}^{\alpha_1}w(t)$. Lemma 2.6 concludes that it is the equivalent with $w = {}^{RL}\mathcal{D}^{\alpha_1}v$. This completes the proof. □ Problem (1.1) has become a fixed point problem with Lemma 2.7 as follow:

$$x = Tx,$$

where $T : E \to E$ is defined by

$$\begin{aligned}Tx(t) = &\int_0^t \frac{(t-s)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}x(s)\big)ds - \lambda\int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}x(s)ds - \frac{1}{2}\int_0^1 \frac{(1-s)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}x(s)\big)ds \\ &+ \frac{\lambda}{2}\int_0^1 \frac{(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}x(s)ds + \left(\frac{1}{4\Gamma(\alpha_1+1)} - \frac{t^{\alpha_1}}{2\Gamma(\alpha_1+1)}\right)\int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}x(s)\big)ds \\ &+ \frac{\Gamma(2-\alpha_1)}{\Gamma(2+\alpha_1)}\left(\frac{1-\alpha_1}{4} + \frac{1+\alpha_1}{2}t^{\alpha_1} - t^{\alpha_1+1}\right)\int_0^1 \frac{(1-s)^{\alpha_2-\alpha_1-1}}{\Gamma(\alpha_2-\alpha_1)}\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}x(s)\big)ds.\end{aligned} \tag{3.3}$$

It can be argued that the initial problem (1.1) has solutions only on condition that operator (3.3) contains fixed points.

Lemma 3.2. For $\alpha_1 > 0$, the following relation holds:

$$
\begin{aligned}
{}^{RL}\mathcal{D}^{\alpha_1} T x(t) = & \frac{1}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1} \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big) ds - \lambda x(t) - \frac{1}{2} \int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big) ds \\
& + \left(\frac{\Gamma(2-\alpha_1)}{2} - \Gamma(2-\alpha)\, t \right) \int_0^1 \frac{(1-s)^{\alpha_2-\alpha-1}}{\Gamma(\alpha_2-\alpha)} \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big) ds.
\end{aligned}
$$

Proof . From (3.3), we can write

$$
\begin{aligned}
Tx(t) = & \mathcal{I}^{\alpha_1+\alpha_2} \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big)(t) - \lambda \mathcal{I}^{\alpha_1} x(t) - \frac{1}{2} \int_0^1 \frac{(1-s)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big) ds + \frac{\lambda}{2} \int_0^1 \frac{(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} x(s) ds \\
& + \left(\frac{1}{4\Gamma(\alpha_1+1)} - \frac{t^{\alpha_1}}{2\Gamma(\alpha_1+1)} \right) \int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big) ds \\
& + \frac{\Gamma(2-\alpha_1)}{\Gamma(2+\alpha_1)} \left(\frac{1-\alpha_1}{4} + \frac{1+\alpha_1}{2} t^{\alpha_1} - t^{\alpha_1+1} \right) \int_0^1 \frac{(1-s)^{\alpha_2-\alpha_1-1}}{\Gamma(\alpha_2-\alpha_1)} \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big) ds. \qquad (3.4)
\end{aligned}
$$

Operating ${}^{RL}\mathcal{D}^{\alpha_1}$ on both side Eq.(3.4) and using Lemma 2.4, the proof is complete. □

In subsequent section, the following presupposition are required in order to prove the major results of this study.
Hypotheses:

(H1) $\sigma : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous function.

(H2) There exist a non negative function $\varphi \in L^1[0,1]$ such that $|\sigma(t,x)| \le \varphi(t) + a\,|x|$ where $a \in \mathbb{R}^+$.

(H3) $|\sigma(t,x) - \sigma(t,y)| \le K\,|x-y|$ where $K > 0$ is constant.

Notation. For simplification in proving the next theorems as well as continuing the work, we will introduce the following symbols:

$$
\begin{aligned}
K_1 &= \tfrac{3}{2\Gamma(\alpha_1+\alpha_2+1)} + \tfrac{1}{4\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)} + \tfrac{(7+3\alpha_1)\Gamma(2-\alpha_1)}{4\Gamma(2+\alpha_1)\Gamma(\alpha_2-\alpha_1+1)} + \tfrac{3}{2\Gamma(\alpha_2+1)} + \tfrac{3\Gamma(2-\alpha_1)}{2\Gamma(\alpha_2-\alpha_1+1)}, \\
K_2 &= \left(\tfrac{3}{2\Gamma(\alpha_1+1)} + 1 \right) \lambda, \\
K_3 &= \max\{a\,K_1, K2\}.
\end{aligned}
$$

Theorem 3.3. Supposing that hypotheses (H1) and (H2) are true, then there exists a solution for fractional boundary value problem (1.1)

Proof . The operator $T : E \to E$ can be defined as

$$
\begin{aligned}
Tx(t) = & \int_0^t \frac{(t-s)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big) ds - \lambda \int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} x(s) ds \\
& - \frac{1}{2} \int_0^1 \frac{(1-s)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big) ds + \frac{\lambda}{2} \int_0^1 \frac{(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} x(s) ds \qquad (3.5) \\
& + \left(\frac{1}{4\Gamma(\alpha_1+1)} - \frac{t^{\alpha_1}}{2\Gamma(\alpha_1+1)} \right) \int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big) ds \\
& + \frac{\Gamma(2-\alpha_1)}{\Gamma(2+\alpha_1)} \left(\frac{1-\alpha_1}{4} + \frac{1+\alpha_1}{2} t^{\alpha_1} - t^{\alpha_1+1} \right) \int_0^1 \frac{(1-s)^{\alpha_2-\alpha_1-1}}{\Gamma(\alpha_2-\alpha_1)} \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big) ds.
\end{aligned}
$$

Let $B_r = \{x \in E; \|x\|_* < r\}$ with $r \geq \frac{K_1\|\phi\|}{1-K_3}$. For any $x \in B_r$ we have

$$
\begin{aligned}
|Tx(t)| \leq & \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_0^t (t-s)^{\alpha_1+\alpha_2-1} \left|\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big)\right| ds + \frac{|\lambda|}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} |x(s)|\, ds \\
& + \frac{1}{2\Gamma(\alpha_1+\alpha_2)} \int_0^1 (1-s)^{\alpha_1+\alpha_2-1} \left|\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big)\right| ds + \frac{|\lambda|}{2\Gamma(\alpha_1)} \int_0^1 (1-s)^{\alpha_1-1} |x(s)|\, ds \\
& + \frac{|1-2t^{\alpha_1}|}{4\Gamma(\alpha_1+1)\Gamma(\alpha_2)} \int_0^1 (1-s)^{\alpha_2-1} \left|\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big)\right| ds \\
& + \frac{\Gamma(2-\alpha_1)}{\Gamma(2+\alpha_1)\Gamma(\alpha_2-\alpha_1)} \left|\frac{1-\alpha_1}{4} + \frac{1+\alpha_1}{2} t_1^{\alpha} - t^{\alpha_1+1}\right| \int_0^1 (1-s)^{\alpha_2-\alpha_1-1} \left|\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big)\right| ds \\
\leq & \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_0^t (t-s)^{\alpha_1+\alpha_2-1} \big(\phi(s) + a \left| {}^{RL}\mathcal{D}^{\alpha_1} x(s)\right| ds\big) + \frac{|\lambda|}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} |x(s)|\, ds \\
& + \frac{1}{2\Gamma(\alpha_1+\alpha_2)} \int_0^1 (1-s)^{\alpha_1+\alpha_2-1} \big(\varphi(s) + a \left| {}^{RL}\mathcal{D}^{\alpha_1} x(s)\right| ds\big) + \frac{|\lambda|}{2\Gamma(\alpha_1)} \int_0^1 (1-s)^{\alpha_1-1} |x(s)|\, ds \\
& + \frac{1}{4\Gamma(\alpha_1+1)\Gamma(\alpha_2)} \int_0^1 (1-s)^{\alpha_2-1} \big(\varphi(s) + a \left| {}^{RL}\mathcal{D}^{\alpha_1} x(s)\right|\big) ds \\
& + \frac{(7+3\alpha_1)\Gamma(2-\alpha_1)}{4\Gamma(2+\alpha_1)\Gamma(\alpha_2-\alpha_1)} \int_0^1 (1-s)^{\alpha_2-\alpha_1-1} \big(\varphi(s) + a \left| {}^{RL}\mathcal{D}^{\alpha_1} x(s)\right|\big) ds \\
\leq & \big(\|\varphi\| + a \left\| {}^{RL}\mathcal{D}^{\alpha_1} x(s)\right\|\big) \frac{t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{|\lambda|\, t_1^{\alpha}}{\Gamma(\alpha_1+1)} \|x\| \\
& + \big(\|\varphi\| + a \left\| {}^{RL}\mathcal{D}^{\alpha_1} x(s)\right\|\big) \frac{1}{2\Gamma(\alpha_2+\alpha_1+1)} + \frac{|\lambda|}{2\Gamma(\alpha_1+1)} \|x\| \\
& + \big(\|\varphi\| + a \left\| {}^{RL}\mathcal{D}^{\alpha_1} x(s)\right\|\big) \frac{3}{4\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)} + \big(\|\varphi\| + a \left\| {}^{RL}\mathcal{D}^{\alpha_1} x(s)\right\|\big) \frac{(7+3\alpha_1)\Gamma(2-\alpha_1)}{4\Gamma(2+\alpha_1)\Gamma(\alpha_2-\alpha_1+1)} \\
\leq & \left(\frac{3}{2\Gamma(\alpha_1+\alpha_2+1)} + \frac{1}{4\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)} + \frac{(7+3\alpha_1)\Gamma(2-\alpha_1)}{4\Gamma(2+\alpha_1)\Gamma(\alpha_2-\alpha_1+1)}\right) \|\varphi\| \\
& + \left(\frac{3}{2\Gamma(\alpha_1+\alpha_2+1)} + \frac{3}{4\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)} + \frac{(7+3\alpha_1)\Gamma(2-\alpha_1)}{4\Gamma(2+\alpha_1)\Gamma(\alpha_2-\alpha_1+1)}\right) a \left\| {}^{RL}\mathcal{D}^{\alpha_1} x\right\| + \frac{3\,|\lambda|}{2\Gamma(\alpha_1+1)} \|x\|,
\end{aligned}
$$

and

$$
\begin{aligned}
\left| {}^{RL}\mathcal{D}^{\alpha_1}\, Tx(t)\right| \leq & \frac{1}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1} \left|\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big)\right| ds + |\lambda|\,|x(t)| + \frac{1}{2\Gamma(\alpha_2)} \int_0^1 (1-s)^{\alpha_2-1} \left|\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big)\right| ds \\
& + \left(\frac{\Gamma(2-\alpha_1)}{2\Gamma(\alpha_2-\alpha_1)} + \frac{\Gamma(2-\alpha_1)}{\Gamma(\alpha_2-\alpha_1)} t\right) \int_0^1 (1-s)^{\alpha_2-\alpha_1-1} \left|\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big)\right| ds \\
\leq & \big(\|\varphi\| + a \left\| {}^{RL}\mathcal{D}^{\alpha_1} x\right\|\big) \frac{t_2^{\alpha}}{\Gamma(\alpha_2+1)} + |\lambda|\,\|x\| \\
& + \big(\|\phi\| + a \left\| {}^{RL}\mathcal{D}^{\alpha_1} x\right\|\big) \frac{1}{2\Gamma(\alpha_2+1)} + \big(\|\phi\| + a \left\| {}^{RL}\mathcal{D}^{\alpha_1} x\right\|\big) \frac{\Gamma(2-\alpha_1)}{2\Gamma(\alpha_2-\alpha_1+1)} \left(\frac{1}{2} + t\right) \\
\leq & \left(\frac{3}{2\Gamma(\alpha_2+1)} + \frac{3\Gamma(2-\alpha_1)}{2\Gamma(\alpha_2-\alpha_1+1)}\right) \|\phi\| + |\lambda|\,\|x\| + \left(\frac{3}{2\Gamma(\alpha_2+1)} + \frac{3\Gamma(2-\alpha_1)}{2\Gamma(\alpha_2-\alpha_1+1)}\right) a \left\| {}^{RL}\mathcal{D}^{\alpha_1} x\right\|.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\|Tx\|_* = & \max_{t\in[0,1]} |T\, x(t)| + \max_{t\in[0,1]} \left| {}^{RL}\mathcal{D}^{\alpha_1} T\, x(t)\right| \\
\leq & K_1 \|\phi\| + K_2 \|x\| + aK_1 \left\| {}^{RL}\mathcal{D}^{\alpha_1} x\right\| \\
\leq & K_1 \|\phi\| + K_3 \big(\|x\| + \left\| {}^{RL}\mathcal{D}^{\alpha_1} x\right\|\big) = K_1 \|\phi\| + K_3 \|x\|_*,
\end{aligned}
$$

which means that $Tx \in B_r$. We have verified that operator T is completely continuous.

Let $M_1 = \sup_{t\in[0,1], z\in B_r} |\sigma(t,z)|$. For any $z \in B_r$ and $t_1,\, t_2 \in [0,1]$, such that $t_1 < t_2$ we have

$$
\begin{aligned}
|T\,x(t_2) - T\,x(t_2)| = \Bigg| & \int_0^{t_2} \frac{(t_2-s)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big) ds - \lambda \int_0^{t_2} \frac{(t_2-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} x\big(s\big) ds \\
& - \frac{t_2^{\alpha_1}}{2\Gamma(\alpha_1+1)} \int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big) ds \\
& + \frac{\Gamma(2-\alpha_1)}{\Gamma(2+\alpha)} \left(\frac{1+\alpha}{2} t_2^{\alpha} - t_2^{\alpha+1} \right) \int_0^1 \frac{(1-s)^{\alpha_2-\alpha-1}}{\Gamma(\alpha_2-\alpha)} \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big) ds \\
& - \int_0^{t_1} (t_1-s)^{\alpha+\alpha_2-1} \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big) ds + \lambda \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} x\big(s\big) ds \\
& + \frac{t_1^{\alpha}}{2\Gamma(\alpha+1)} \int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big) ds \\
& - \frac{\Gamma(2-\alpha)}{\Gamma(2+\alpha)} \left(\frac{1+\alpha}{2} t_1^{\alpha} - t_1^{\alpha+1} \right) \int_0^1 \frac{(1-s)^{\alpha_2-\alpha-1}}{\Gamma(\alpha_2-\alpha)} \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big) ds \Bigg| \\
\le \Bigg| & \int_0^{t_1} \frac{(t_2-s)^{\alpha+\alpha_2-1} - (t_1-s)^{\alpha+\alpha_2-1}}{\Gamma(\alpha+\alpha_2)} \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big) ds \\
& + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha+\alpha_2-1}}{\Gamma(\alpha+\alpha_2)} \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big) ds - \lambda \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} x\big(s\big) ds \\
& - \lambda \int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} x\big(s\big) ds - \frac{t_2^{\alpha} - t_1^{\alpha}}{2\Gamma(\alpha+1)} \int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big) ds \\
& + \frac{\Gamma(2-\alpha)}{\Gamma(2+\alpha)} \left(\frac{1+\alpha}{2} \left(t_2^{\alpha} - t_1^{\alpha}\right) \right) \int_0^1 \frac{(1-s)^{\alpha_2-\alpha-1}}{\Gamma(\alpha_2-\alpha)} \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big) ds \\
& - \frac{\Gamma(2-\alpha)}{\Gamma(2+\alpha)} \left(t_2^{\alpha+1} - t_1^{\alpha+1}\right) \int_0^1 \frac{(1-s)^{\alpha_2-\alpha-1}}{\Gamma(\alpha_2-\alpha)} \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big) ds \Bigg| \\
\le & \frac{M_1}{\Gamma(\alpha+\alpha_2+1)} \left(t_2^{\alpha+\alpha_2} - t_1^{\alpha+\alpha_2}\right) + \frac{2M_1}{\Gamma(\alpha+\alpha_2+1)} (t_2-t_1)^{\alpha+\alpha_2} \\
& + \frac{|\lambda|\, r}{\Gamma(\alpha+1)} \left(t_2^{\alpha} - t_1^{\alpha}\right) + \frac{2\,|\lambda|\, r}{\Gamma(\alpha+1)} (t_2-t_1)^{\alpha} + \frac{M_1}{2\Gamma(\alpha+1)\Gamma(\alpha_2+1)} \left(t_2^{\alpha} - t_1^{\alpha}\right) \\
& + M_1 \frac{(1+\alpha)\Gamma(2-\alpha)}{2\Gamma(2+\alpha)\Gamma(\alpha_2-\alpha+1)} \left(t_2^{\alpha} - t_1^{\alpha}\right) + M_1 \frac{\Gamma(2-\alpha)}{\Gamma(2+\alpha)\Gamma(\alpha_2-\alpha+1)} \left(t_2^{\alpha+1} - t_1^{\alpha+1}\right).
\end{aligned}
$$

and

$$
\begin{aligned}
& \left| {}^{RL}\mathcal{D}^{\alpha_1}\, Tx(t_2) - {}^{RL}\mathcal{D}^{\alpha_1}\, Tx(t_1) \right| \\
= & \Bigg| \int_0^{t_2} \frac{(t_2-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big) ds - \lambda x(t_2) - \Gamma(2-\alpha_1) t_2 \int_0^1 \frac{(1-s)^{\alpha_2-\alpha_1-1}}{\Gamma(\alpha_2-\alpha_1)} \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big) ds \\
& - \int_0^{t_1} \frac{(t_1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big) ds + \lambda x(t_1) + \Gamma(2-\alpha_1) t_1 \int_0^1 \frac{(1-s)^{\alpha_2-\alpha_1-1}}{\Gamma(\alpha_2-\alpha_1)} \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big) ds \Bigg| \\
\le & \int_0^{t_1} \frac{(t_2-s)^{\alpha_2-1} - (t_1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} \left|\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big)\right| ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} \left|\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big)\right| ds - |\lambda x(t_2)| \\
& + |\lambda x(t_1)| + \Gamma(2-\alpha_1)(t_2-t_1) \int_0^1 \frac{(1-s)^{\alpha_2-\alpha_1-1}}{\Gamma(\alpha_2-\alpha_1)} \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1} x(s)\big) ds \\
\le & \frac{M_1}{\Gamma(\alpha_2+1)} \left(t_2^{\alpha_2} - t_1^{\alpha_2}\right) + \frac{2M_1}{\Gamma(\alpha_2+1)} (t_2-t_1)^{\alpha_2} + 2\,|\lambda|\, r + M_1 \frac{\Gamma(2-\alpha_1)}{\Gamma(\alpha_2-\alpha_1+1)} (t_2-t_1).
\end{aligned}
$$

We see that the functions $t_2^{\alpha_1+\alpha_2} - t_1^{\alpha_1+\alpha_2}$, $(t_2-t_1)^{\alpha_1+\alpha_2}$, $(t_2-t_1)^{\alpha_1}$, $t_2^{\alpha_1} - t_1^{\alpha_1}$, $t_2^{\alpha_1+1} - t_1^{\alpha_1+1}$, $(t_2-t_1)^{\alpha_2}$, $t_2^{\alpha_2} - t_1^{\alpha_2}$ and $t_2 - t_1$ are continuous and uniform on $[0,\ 1]$.

Consequently, the Arzela Ascoli theorem suggests that $T(B_r)$ is compact. Therefore, the operator $T : B_r \to B_r$ is completely continuous and the Schauder FP theorem illustrates that there exists a solution for problem (1.1). □

Theorem 3.4. Under assumptions (H1) and (H3) problem (1.1) have an uniqueness solution, provided that

$$w = \max\{KK_1, K_2\} < 1.$$

Proof . By condition (H3), for any $x, y \in E$, $t \in [0,\ 1]$, we have

$$\begin{aligned}
&|Tx(t) - Ty(t)| \le \int_0^t \frac{(t-s)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} \left|\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}x(s)\big) - \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}y(s)\big)\right| ds + |\lambda| \int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} |x(s)-y(s)|\, ds \\
&+ \frac{1}{2}\int_0^1 \frac{(1-s)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} \left|\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}x(s)\big) - \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}y(s)\big)\right| ds + \frac{|\lambda|}{2}\int_0^1 \frac{(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} |x(s)-y(s)|\, ds \\
&+ \left(\frac{1}{4\Gamma(\alpha_1+1)} - \frac{t^{\alpha_1}}{2\Gamma(\alpha_1+1)}\right)\int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} \left|\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}x(s)\big) - \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}y(s)\big)\right| ds \\
&+ \frac{\Gamma(2-\alpha_1)}{\Gamma(2+\alpha_1)}\left(\frac{1-\alpha_1}{4} + \frac{1+\alpha_1}{2}t^{\alpha_1} - t^{\alpha_1+1}\right)\int_0^1 \frac{(1-s)^{\alpha_2-\alpha_1-1}}{\Gamma(\alpha_2-\alpha_1)} \times \left|\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}x(s)\big) - \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}y(s)\big)\right| ds \\
\le& K\left\| {}^{RL}\mathcal{D}^{\alpha_1}x - {}^{RL}\mathcal{D}^{\alpha_1}y\right\| \frac{t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + |\lambda|\,\|x-y\| \frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{K}{2\Gamma(\alpha_1+\alpha_2+1)}\left\| {}^{RL}\mathcal{D}^{\alpha_1}x - {}^{RL}\mathcal{D}^{\alpha_1}y\right\| \\
&+ \frac{|\lambda|}{2\Gamma(\alpha_1+1)}\|x-y\| + \frac{K}{\Gamma(\alpha_2+1)}\left(\frac{1}{4\Gamma(\alpha_1+1)} - \frac{t^{\alpha_1}}{2\Gamma(\alpha_1+1)}\right)\left\| {}^{RL}\mathcal{D}^{\alpha_1}x - {}^{RL}\mathcal{D}^{\alpha_1}y\right\| \\
&+ \frac{K\Gamma(2-\alpha_1)}{\Gamma(\alpha_2-\alpha_1+1)\Gamma(2+\alpha_1)}\left(\frac{1-\alpha_1}{4} + \frac{1+\alpha_1}{2}t^{\alpha_1} - t^{\alpha_1+1}\right)\left\| {}^{RL}\mathcal{D}^{\alpha_1}x - {}^{RL}\mathcal{D}^{\alpha_1}y\right\| \\
\le& \left(\frac{3}{2\Gamma(\alpha_1+\alpha_2+1)} + \frac{3}{4\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)} + \frac{(7+3\alpha_1)\Gamma(2-\alpha_1)}{4\Gamma(2+\alpha_1)\Gamma(\alpha_2-\alpha_1+1)}\right) K\left\| {}^{RL}\mathcal{D}^{\alpha_1}x - {}^{RL}\mathcal{D}^{\alpha_1}y\right\| + \frac{3|\lambda|\,\|x-y\|}{2\Gamma(\alpha_1+1)},
\end{aligned}$$

and

$$\begin{aligned}
&\left| {}^{RL}\mathcal{D}^{\alpha_1}Tx(t) - {}^{RL}\mathcal{D}^{\alpha_1}Ty(t)\right| = \Bigg|\int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}x(s)\big)ds - \lambda x(t) - \frac{1}{2}\int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}x(s)\big)ds \\
&+ \left(\frac{\Gamma(2-\alpha_1)}{2} + \Gamma(2-\alpha_1)t\right)\int_0^1 \frac{(1-s)^{\alpha_2-\alpha_1-1}}{\Gamma(\alpha_2-\alpha_1)}\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}x(s)\big)ds - \int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}x(s)\big)ds + \lambda y(t) \\
&+ \frac{1}{2}\int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}y(s)\big)ds - \left(\frac{\Gamma(2-\alpha_1)}{2} + \Gamma(2-\alpha_1)t\right)\int_0^1 \frac{(1-s)^{\alpha_2-\alpha_1-1}}{\Gamma(\alpha_2-\alpha_1)}\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}y(s)\big)ds\Bigg| \\
\le& \int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\left|\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}x(s)\big) - \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}y(s)\big)\right| ds + |\lambda|\,|x(t)-y(t)| \\
&+ \frac{1}{2}\int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\left|\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}x(s)\big) - \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}y(s)\big)\right| ds \\
&+ \left(\frac{\Gamma(2-\alpha_1)}{2} + \Gamma(2-\alpha_1)t\right)\int_0^1 \frac{(1-s)^{\alpha_2-\alpha_1-1}}{\Gamma(\alpha_2-\alpha_1)}\left|\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}x(s)\big) - \sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}y(s)\big)\right| ds \\
\le& K\left\| {}^{RL}\mathcal{D}^{\alpha_1}x - {}^{RL}\mathcal{D}^{\alpha_1}y\right\| \frac{t_1^{\alpha}}{\Gamma(\alpha_2+1)} + |\lambda|\,\|x-y\| + \frac{K}{2\Gamma(\alpha_2+1)}\left\| {}^{RL}\mathcal{D}^{\alpha_1}x - {}^{RL}\mathcal{D}^{\alpha_1}y\right\| \\
&+ \frac{1}{\Gamma(\alpha_2-\alpha_1+1)}\left(\frac{\Gamma(2-\alpha_1)}{2} + \Gamma(2-\alpha_1)t\right) K\left\| {}^{RL}\mathcal{D}^{\alpha_1}x - {}^{RL}\mathcal{D}^{\alpha_1}y\right\| \\
\le& \left(\frac{3}{2\Gamma(\alpha_2+1)} + \frac{3\Gamma(2-\alpha_1)}{2\Gamma(\alpha_2-\alpha_1+1)}\right) K\left\| {}^{RL}\mathcal{D}^{\alpha_1} - {}^{RL}\mathcal{D}^{\alpha_1}y\right\| + |\lambda|\,\|x-y\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|Tx(t)-Ty(t)\|_* &= \max_{t\in[0,1]}|Tx(t)-Ty(t)| + \max_{t\in[0,1]}\left|\ {}^{RL}\mathcal{D}^{\alpha_1}Tx(t) - \ {}^{RL}\mathcal{D}^{\alpha_1}Ty(t)\right| \\
&\le \left(\frac{3}{2\Gamma(\alpha_1+\alpha_2+1)} + \frac{3}{4\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)} + \frac{(7+3\alpha_1)\Gamma(2-\alpha_1)}{4\Gamma(2+\alpha_1)\Gamma(\alpha_2-\alpha_1+1)}\right)\times \\
&\qquad K\left\|\ {}^{RL}\mathcal{D}^{\alpha_1}x - \ {}^{RL}\mathcal{D}^{\alpha_1}y\right\| + \frac{3|\lambda|}{2\Gamma(\alpha_1+1)}\|x-y\| \\
&\quad + \left(\frac{3}{2\Gamma(\alpha_2+1)} + \frac{3\Gamma(2-\alpha_1)}{2\Gamma(\alpha_2-\alpha_1+1)}\right)K\left\|\ {}^{RL}\mathcal{D}^{\alpha_1}x - \ {}^{RL}\mathcal{D}^{\alpha_1}y\right\| + |\lambda|\,\|x-y\|. \\
&\le K_1K\left\|\ {}^{RL}\mathcal{D}^{\alpha_1}\,Tx(t) - \ {}^{RL}\mathcal{D}^{\alpha_1}\,Ty(t)\right\| + K_2\|x-y\| \\
&\le w\,\|x-y\|_*,
\end{aligned}$$

where $w=\max\{KK_1,\,K_2\}$. Since $w<1$, then the operator T is a contraction operator. Consequently, it is suggested by the contraction mapping principle that for problem (1.1), there is a unique solution. □

4 Hyers-Ulam stability

Here we describe some stability results. The said stability results are based on the Ulam−Hyers concept.

Proposition 4.1. The APBVPs of Langevin equation 1.1 has a unique solution given by

$$x(t)=\int_0^1\Psi_1(t,s)\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}x(s)\big)ds+\int_0^1\Psi_2(t,s)x(s)ds$$

in which

$$\Psi_1(\mathfrak{t},\mathfrak{s})=\begin{cases}\frac{(t-s)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}-\frac{(1-s)^{\alpha_1+\alpha_2-1}}{2\Gamma(\alpha_1+\alpha_2)}+\left(\frac{1}{4\Gamma(\alpha_1+1)}-\frac{t^{\alpha_1}}{2\Gamma(\alpha_1+1)}\right)\frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} \\ +\frac{\Gamma(2-\alpha_1)}{\Gamma(2+\alpha_1)}\left(\frac{1-\alpha_1}{4}+\frac{1+\alpha_1}{2}t^{\alpha_1}-t^{\alpha_1+1}\right)\frac{(1-s)^{\alpha_2-\alpha_1-1}}{\Gamma(\alpha_2-\alpha_1)}, & s\le t \\ -\frac{(1-s)^{\alpha_1+\alpha_2-1}}{2\Gamma(\alpha_1+\alpha_2)}+\left(\frac{1}{4\Gamma(\alpha_1+1)}-\frac{t^{\alpha_1}}{2\Gamma(\alpha_1+1)}\right)\frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} \\ +\frac{\Gamma(2-\alpha_1)}{\Gamma(2+\alpha_1)}\left(\frac{1-\alpha_1}{4}+\frac{1+\alpha_1}{2}t^{\alpha_1}-t^{\alpha_1+1}\right)\frac{(1-s)^{\alpha_2-\alpha_1-1}}{\Gamma(\alpha_2-\alpha_1)}, & t\le s\end{cases} \tag{4.1}$$

and

$$\Psi_2(\mathfrak{t},\mathfrak{s})=\begin{cases}-\lambda\frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}+\frac{\lambda}{2}\frac{(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}, & s\le t \\ \frac{\lambda}{2}\frac{(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}, & t\le s\end{cases} \tag{4.2}$$

Proof . From lemma 2.7, the proof is easy and we omit it. □
Remark. We consider a mapping χ independent of x, such that $|\chi(t)|\le\varepsilon$, for every, $t\in[0,1]$.

Theorem 4.2. The solution to the following perturbed problem

$$\mathcal{D}^{\alpha_2}\big(\mathcal{D}^{\alpha_1}+\lambda\big)x(t)=\sigma\big(t,\ {}^{RL}\mathcal{D}^{\alpha_1}x(t)\big)+\chi(t),\quad 0<\alpha_1\le1,\quad 1<\alpha_2\le2,\quad t\in[0,1] \tag{4.3}$$

satisfies the following relation

$$\left|x(t)-\int_0^1\Psi_1(t,s)\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}x(s)\big)ds-\int_0^1\Psi_2(t,s)x(s)ds\right|\le\Omega\varepsilon, \tag{4.4}$$

where

$$\Omega=\frac{3}{2\Gamma(\alpha_1+\alpha_2+1)}+\frac{3}{4\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}+\frac{2\Gamma(2-\alpha_1)}{2\Gamma(2+\alpha_1)\Gamma(\alpha_2-\alpha_1+1)}, \tag{4.5}$$

Proof . In view of lemma 2.7 and proposition 4.1, the solution is given by

$$x(t)=\int_0^1 \Psi_1(t,s)\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}x(s)\big)ds+\int_0^1 \Psi_2(t,s)x(s)ds+\int_0^1 \Psi_1(t,s)\chi(s)ds.$$

Further, we have for $t\in[0,1]$

$$\left|x(t)-\int_0^1 \Psi_1(t,s)\sigma\big(s,\ {}^{RL}\mathcal{D}^{\alpha_1}x(s)\big)ds-\int_0^1 \Psi_2(t,s)x(s)ds\right|\leq\int_0^1 |\Psi_1(t,s)|\,|\chi(s)|\,ds$$

$$\begin{aligned}
&\leq \varepsilon\int_0^1 \frac{(t-s)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}+\varepsilon\int_0^1 \frac{(t-s)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}\\
&\quad+\varepsilon\left(\frac{1}{4\Gamma\left(\alpha_1+1\right)}+\frac{t^{\alpha_1}}{2\Gamma\left(\alpha_1+1\right)}\right)\int_0^1\frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}ds\\
&\quad+\varepsilon\frac{\Gamma\left(2-\alpha_1\right)}{\Gamma\left(2+\alpha_1\right)}\left(\frac{1-\alpha_1}{4}+\frac{1+\alpha_1}{2}t^{\alpha_1}+t^{\alpha_1+1}\right)\int_0^1\frac{(1-s)^{\alpha_2-\alpha_1-1}}{\Gamma(\alpha_2-\alpha_1)}ds\\
&\leq\left(\frac{3}{2\Gamma(\alpha_1+\alpha_2+1)}+\frac{3}{4\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}+\frac{2\Gamma(2-\alpha_1)}{2\Gamma(2+\alpha_1)\Gamma(\alpha_2-\alpha_1+1)}\right)\varepsilon\\
&=\Omega\varepsilon.
\end{aligned}$$

□

5 An example

Example 5.1. Consider the following fractional Langevin problem

$$\begin{cases}\mathcal{D}^{\alpha_2}\big(\mathcal{D}^{\alpha_1}+\lambda\big)x(t)=\sigma\big(t,\ {}^{RL}\mathcal{D}^{\alpha_1}x(t)\big), \qquad t\in[0,1]\\ x(0)+x(1)=0,\\ \mathcal{D}^{\alpha_1}x(0)+\mathcal{D}^{\alpha_1}x(1)=0, \quad \mathcal{D}^{2\alpha_1}x(0)+\mathcal{D}^{2\alpha_1}x(1)=0,\end{cases} \tag{5.1}$$

where $\alpha_1=\frac{1}{2},\ \alpha_2=\frac{3}{2},\ \lambda=\frac{1}{10}$, and $\sigma\big(t,\ {}^{RL}\mathcal{D}^{\alpha_1}x(t)\big)=t^2+\frac{\sin t\ {}^{RL}\mathcal{D}^{\frac{1}{2}}x(t)}{4\sqrt{\pi}}$. It is easy to show that hypotheses (H1) hold. Also we gain:

$$\left|\sigma\big(t,\frac{1}{2}x(t)\big)\right|\leq t+\frac{1}{4\sqrt{\pi}}\left|\ {}^{RL}\mathcal{D}^{\frac{1}{2}}x(t)\right|,$$

i.e. hypothesis (H2) hold (with $\varphi(t)=t$ and $a=\frac{1}{4\sqrt{\pi}}$). Now, theorem 3.3 assure that system (5.1) has a solution.

We show that the hypothesis (H3) is established:

$$\begin{aligned}\left|\sigma\big(t,\ {}^{RL}\mathcal{D}^{\alpha_1}x(t)\big)-\sigma\big(t,\ {}^{RL}\mathcal{D}^{\alpha_1}y(t)\big)\right|&=\left|t^2+\frac{\sin t\ {}^{RL}\mathcal{D}^{\frac{1}{2}}x(t)}{4\sqrt{\pi}}-\left(t^2+\frac{\sin t\ {}^{RL}\mathcal{D}^{\frac{1}{2}}y(t)}{4\sqrt{\pi}}\right)\right|\\ &\leq\frac{1}{4\sqrt{\pi}}\left|\ {}^{RL}\mathcal{D}^{\frac{1}{2}}x(t)-\ {}^{RL}\mathcal{D}^{\frac{1}{2}}y(t)\right|\end{aligned}$$

Also, we have

$$w=\max\{KK_1,K_2\}=0.742051<1.$$

Now, Theorem3.4 yield, the problem (5.1) has a unique solution.

6 Conclusion

The present research work has dealt with nonlinear Langevin equations featuring two fractional orders. we investigated the existence, uniqueness, and and Ulam–Hyers stability of solutions for this novel class Langevin fractional differential equation. Through employing the Schauder fixed point theorem, it has been illustrated that there exists a solution for the problem (1.1). Following that, Banach fixed point theorem has been utilized in order to illustrate the uniqueness of the solution for the problem. Eventually there are new conditions, we have suggested a number of results without regarding any further presuppositions, in comparison with other similar studies.

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