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Turán-type inequalities for certain class of meromorphic functions

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Abstract

In this study, a broader class of rational functions $r(u(z))$ of degree mn, where $u(z)$ is a polynomial of degree m is taken into consideration and obtain certain sharp compact generalization of well-known inequalities for rational functions.

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1 Introduction

Let $\mathbb{P}_n$ denote the set of all complex polynomials $P(z)=\sum\limits_{j=0}^{n} b_j z^j$ of degree at most n and $P'(z)$ its derivative. Then from a well known inequality due to Bernstein [2], we have

$$\max_{|z|=1}|P'(z)| \leq n \max_{|z|=1}|P(z)|. \tag{1.1}$$

Inequality (1.1) is best possible and equality holds if $P(z)$ has all zeros at the origin. The above inequality (1.1) was proved by Bernstein in 1912 and has been the starting point of a considerable literature in polynomial approximations and, over a period, various versions and generalizations of this inequality is produced by introducing restrictions on the multiplicity of zero at $z=0$, the modulus of largest root of $P(z)$, restrictions on coefficients etc. If we restrict ourselves to the class of polynomials having no zeros in $|z|<1$, then the above inequality (1.1) can be sharpened. In fact, Erdös conjectured and later Lax [5] proved that, if $P(z)\neq 0$ in $|z|<1$, then

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{2} \max_{|z|=1}|P(z)|, \tag{1.2}$$

whereas, if $P(z)$ has no zeros in $|z|>1$, then

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{2} \max_{|z|=1}|P(z)|. \tag{1.3}$$

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Inequality (1.3) is due to Turán [12]. Both the inequalities (1.2) and (1.3) are best possible and equality holds if $P(z)$ has all its zeros on $|z| = 1$. It is worth mentioning that different versions of the Turán's inequality have appeared in the literature in more generalized forms in which the underlying polynomial is replaced by more general class of functions. These inequalities have their own significance and importance in Approximation theory. For the latest research and development in this direction, one can see some papers (see [8]-[11]).

Jain [4] had used a parameter β and proved an interesting generalization of (1.3). More precisely, Jain [4] proved that if $P(z)$ is a polynomial of degree n and $P(z)$ has all its zeros in $|z| \leq 1$, then for every β with $|\beta| \leq 1$,

$$\max_{|z|=1} \left|zP'(z) + \frac{n\beta}{2}P(z)\right| \geq \frac{n}{2}\left\{1 + Re(\beta)\right\} \max_{|z|=1} |P(z)|. \tag{1.4}$$

Li et al. [7] gave a new perspective to the above inequalities (1.1)-(1.3), and extended them to rational functions with fixed poles. Essentially, in these inequalities they replaced the polynomial $P(z)$ by a rational function $r(z)$ with poles $a_1, a_2, ..., a_n$ all lying in $|z| > 1$, and z^n was replaced by a Blaschke product $B(z)$. Before proceeding towards their results, we first introduce the set of rational functions involved. For $a_j \in \mathbb{C}$ with $j = 1, 2, ..., n$, let

$$W(z) := \prod_{j=1}^{n} (z - a_j)$$

and

$$B(z) := \prod_{j=1}^{n} \left(\frac{1 - \overline{a}_j z}{z - a_j}\right), \ \mathbb{R}_n := \mathbb{R}_n(a_1, a_2, ..., a_n) = \left\{\frac{P(z)}{W(z)} : \ P \in \mathbb{P}_n\right\}.$$

Then $\mathbb{R}_n$ is the set of rational functions with poles $a_1, a_2, ..., a_n$ at most and with finite limit at ∞. Note that $B(z) \in \mathbb{R}_n$ and $|B(z)| = 1$ for $|z| = 1$. For $r(z) = \frac{P(z)}{W(z)} \in \mathbb{R}_n$, the conjugate transpose r^* of r is defined by $r^*(z) = B(z)\overline{r(\frac{1}{\overline{z}})}$. In the past few years several papers pertaining to Bernstein-type inequalities for rational functions have appeared in the study of rational approximations (see, e.g., [1], [3], [6] and [7]). For $r \in \mathbb{R}_n$, Li et al. [7] proved the following, similar to (1.1), inequality for rational functions:

$$|r'(z)| \leq |B'(z)| \max_{|z|=1} |r(z)|. \tag{1.5}$$

As extensions of (1.2) and (1.3) to rational functions, Li et al. [7] also showed that if $r \in \mathbb{R}_n$, and $r(z) \neq 0$ in $|z| < 1$, then for $|z| = 1$,

$$|r'(z)| \leq \frac{|B'(z)|}{2} \max_{|z|=1} |r(z)|, \tag{1.6}$$

whereas, if $r \in \mathbb{R}_n$, has exactly n zeros in $|z| \leq 1$, then for $|z| = 1$,

$$|r'(z)| \geq \frac{|B'(z)|}{2} \max_{|z|=1} |r(z)|. \tag{1.7}$$

We investigate a broader class of rational functions $r(u(z))$ in this study defined by

$$(r \circ u)(z) = r(u(z)) := \frac{P(u(z))}{W(u(z))},$$

where $u(z)$ is a polynomial of degree m and $r \in \mathbb{R}_n$, so that $r(u(z)) \in \mathbb{R}_{mn}$, and

$$W(u(z)) = \prod_{j=1}^{mn} \left(z - a_j\right).$$

Also, the Blashke product is given by

$$B(z) = \frac{W^*(u(z))}{W(u(z))} = \frac{\overline{W(u(\frac{1}{\overline{z}}))}}{W(u(z))} = \prod_{j=1}^{mn} \left(\frac{1 - \overline{a}_j z}{z - a_j}\right)$$

and provide some findings for the aforementioned class of rational functions $r(u(z))$ with restricted zeros which in turn generalizes as well as refines the above inequalities.

2 Main Results

From now on, we shall always assume that all poles $a_1, a_2, ..., a_n$ of $r(u(z))$ lie in $|z| > 1$. For the case when all poles are in $|z| < 1$, we can obtain analogous results with suitable modification.

Theorem 2.1. Let $r(u(z)) \in \mathbb{R}_{mn}$ and $r(u(z)) \neq 0$ in $|z| > 1$ except with s-fold zeros at the origin, then for every β with $|\beta| \leq 1$, $Re(\beta) \geq 0$ and for $|z| = 1$, we have

$$\left| zr'(u(z)).u'(z) + \frac{n}{2}Re(\beta)r(u(z)) \right| \geq \frac{1}{2}\left\{ |B'(z)| + s - m(n-n') + nRe(\beta) \right\} |r(u(z))|, \tag{2.1}$$

where mn' and mn are respectively the number of zeros and poles of $r(u(z))$.

For $u(z) = z$, Theorem 2.1 reduces to the following result.

Corollary 2.2. Let $r(z) \in \mathbb{R}_n$ and $r(z) \neq 0$ in $|z| > 1$ except with s-fold zeros at the origin, then for every β with $|\beta| \leq 1$, $Re(\beta) \geq 0$ and for $|z| = 1$, we have

$$\left| zr'(z) + \frac{n}{2}Re(\beta)r(z) \right| \geq \frac{1}{2}\left\{ |B'(z)| + s - (n-n') + nRe(\beta) \right\} |r(z)|. \tag{2.2}$$

We first discuss some consequences of Corollary 2.2. If we take $\alpha_j = \alpha$, $|\alpha| \geq 1$, for $j = 1, 2, ..., n$, then $W(z) = (z-\alpha)^n$ and $r(z) = \frac{P(z)}{(z-\alpha)^n}$, and hence we have

$$\begin{aligned} r'(z) &= \frac{(z-\alpha)^n P'(z) - n(z-\alpha)^{n-1}P(z)}{(z-\alpha)^{2n}} \\ &= -\left\{ \frac{nP'(z) - (z-\alpha)P'(z)}{(z-\alpha)^{n+1}} \right\} \\ &= \frac{-D_\alpha P(z)}{(z-\alpha)^{n+1}}, \end{aligned}$$

where $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ is a polynomial of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty} \left\{ \frac{D_\alpha P(z)}{\alpha} \right\} := P'(z).$$

Also, $W^*(z) = (1 - \overline{\alpha} z)^n$, which gives $B(z) := \prod_{j=1}^n \left(\frac{1-\overline{a}_j z}{z-a_j} \right)$. This implies $B'(z) = \frac{n(1-\overline{\alpha}z)^{n-1}(|\alpha|^2-1)}{(z-\alpha)^{n+1}}$. Using these observations in (2.2) and assuming $Re(\beta) \geq 0$, we get for $|z| = 1$,

$$\begin{aligned} \left| zD_\alpha P(z) + \frac{n}{2}Re(\beta)(\alpha - z)P(z) \right| &\geq \frac{1}{2}\left\{ \frac{n(|\alpha|^2-1)}{|\alpha|+1} + s(|\alpha|-1) - (n-n')(|\alpha|-1) + n(|\alpha|-1)Re(\beta) \right\} |P(z)| \\ &= \frac{(|\alpha|-1)}{2}\left\{ s + n' + nRe(\beta) \right\} |P(z)|. \end{aligned}$$

Corollary 2.3. Let $P(z)$ be a polynomial of degree n having no zeros in $|z| > 1$ except with s-fold zeros at the origin, then for every real or complex number α with $|\alpha| \geq 1$ and $Re(\beta) \geq 0$, we have

$$|zD_\alpha P(z) + \frac{n}{2}Re(\beta)(\alpha - z)P(z)| \geq \frac{(|\alpha| - z)}{2}\left\{ |B'(z)| + s - (n-n') + nRe(\beta) \right\} |P(z)|. \tag{2.3}$$

Dividing two sides of inequality (2.3) by $|\alpha|$ and letting $|\alpha| \to \infty$, we have the following result.

Corollary 2.4. Let $P(z)$ be a polynomial of degree n having no zeros in $|z| > 1$ except with s-fold zeros at the origin, then for $Re(\beta) \geq 0$, we have

$$|zP'(z) + \frac{n}{2}Re(\beta)P(z)| \geq \frac{1}{2}\left\{ |B'(z)| + s - (n-n') + nRe(\beta) \right\} |P(z)|. \tag{2.4}$$

We demonstrate a more broad result rather than establishing Theorem 2.1.

Theorem 2.5. Let $r(u(z)) \in \mathbb{R}_{mn}$ and $r(u(z)) \neq 0$ in $|z| > 1$ except with s-fold zeros at the origin, then for every β with $|\beta| \leq 1$, $0 \leq \zeta < 1$, $Re(\beta) \geq 0$ and for $|z| = 1$, we have

$$\begin{aligned} &\left|zr'(u(z)).u'(z) + \frac{n}{2}Re(\beta)r(u(z))\right| + \zeta m(rou,1)\left(s + \frac{n}{2}Re(\beta)\right) \\ &\geq \frac{1}{2}\left\{|B'(z)| + s - m(n-n') + nRe(\beta)\right\}\left(|r(u(z))| + \zeta m(rou,1)\right), \end{aligned} \tag{2.5}$$

where mn' and mn are respectively the number of zeros and poles of $r(u(z))$ and $m(rou, 1) = \min_{|z|=1} |r(u(z))|$.

Remark 2.6. It is important to mention that bound obtained from Theorem 2.5 is optimal when $\zeta = 1$. However, the parameter ζ plays a vital role for making Theorem 2.5 more general and to get different bounds from it by simply giving different values to it from 0 to 1 and without changing the hypothesis of the Theorem. For example, for ζ= 0 (without assuming that $r(u(z))$ has a zero on $|z| = 1$), we obtain Theorem 2.1. If we take $s = 0$ in (2.5), we get the following result.

Corollary 2.7. Let $r(u(z)) \in \mathbb{R}_{mn}$ and $r(u(z)) \neq 0$ in $|z| > 1$, then for every β with $|\beta| \leq 1$, $0 \leq \zeta < 1$, $Re(\beta) \geq 0$ and for $|z| = 1$, we have

$$\left|zr'(u(z)).u'(z) + \frac{n}{2}Re(\beta)r(u(z))\right| + \zeta m(rou,1)\frac{n}{2}Re(\beta) \geq \frac{1}{2}\left\{|B'(z)| - m(n-n') + nRe(\beta)\right\}\left(|r(u(z))| + \zeta m(rou,1)\right), \tag{2.6}$$

where $m(rou, 1) = \min_{|z|=1} |r(u(z))|$.

Now we prove the following result which provides generalization to inequality (1.7).

Theorem 2.8. Let $r(u(z)) \in \mathbb{R}_{mn}$, and assume $r(u(z))$ has all its zeros in $|z| \leq 1$. Then for every β with $|\beta| \leq 1$ and $|z| = 1$,

$$\left|\frac{r'(u(z)).u'(z)}{B'(z)} + \frac{\beta}{2}\frac{r(u(z))}{B(z)}\right| \geq \frac{1}{2}(1-|\beta|)|r(u(z))|. \tag{2.7}$$

If we take $\beta = 0$ in (2.7), we get the following result.

Corollary 2.9. Let $r(u(z)) \in \mathbb{R}_{mn}$, and assume $r(u(z))$ has all its zeros in $|z| \leq 1$. Then for $|z| = 1$,

$$\left|r'(u(z)).u'(z)\right| \geq \frac{|B'(z)|}{2}|r(u(z))|. \tag{2.8}$$

For $u(z) = z$, (2.8) reduces to (1.7). Next, we prove a more improved result. The above inequality (2.7) will be a consequence from the more fundamental inequality presented by the following theorem.

Theorem 2.10. Let $r(u(z)) \in \mathbb{R}_{mn}$, and assume $r(u(z))$ has all its zeros in $|z| \leq 1$. Then for every β with $|\beta| \leq 1$ and $|z| = 1$,

$$\left|\frac{r'(u(z))u'(z)}{B'(z)} + \frac{\beta}{2}\frac{r(u(z))}{B(z)}\right| \geq \frac{1}{2}\left\{(1-|\beta|)|r(u(z))| + \left(\left|1+\frac{\beta}{2}\right| - \left|\frac{\beta}{2}\right|\right)m(rou,1)\right\}, \tag{2.9}$$

where $m(rou, 1) = \min_{|z|=1} |r(u(z))|$.

Remark 2.11. Theorem 2.10 is a refinement of Theorem 2.8, it can be easily seen by observing that $|1 + \frac{\beta}{2}| \geq |\frac{\beta}{2}|$ for $|\beta| \leq 1$.

3 Auxiliary results

For the proofs of our main results, we need the following lemmas.

Lemma 3.1. If $r(u(z)) \in \mathbb{R}_{mn}$ having all its zeros in $|z| \leq 1$ except with s-fold zeros at the origin, then for $r(u(z)) \neq 0$ and $|z| = 1$, we have

$$Re\left(\frac{zr'(u(z)).u'(z)}{r(u(z))}\right) \geq \frac{1}{2}\left\{|B'(z) + s - m(n-n')|\right\},$$

where mn' and mn are respectively the number of zeros and poles of $r(u(z))$.

Proof . Since $r(u(z)) = \frac{P(u(z))}{W(u(z))} \in \mathbb{R}_{mn}$, where $P(u(z))$ has mn' zeros in $|z| \leq 1$ with a zero of multiplicity s at the origin, we can write

$$r(u(z)) = \frac{z^s \prod_{i=1}^{mn'-s}(z-b_i)}{\prod_{j=1}^{mn}(z-a_j)},$$

where $|b_i| \leq 1,\ i = 1, 2, 3..., mn' - s$. This gives,

$$\frac{z(r(u(z)))'}{r(u(z))} = s + \sum_{i=1}^{mn'-s} \frac{z}{z-b_i} - \sum_{j=1}^{mn} \frac{z}{z-a_j}. \tag{3.1}$$

Since all the zeros of $P(u(z))$ lie in $|z| \leq 1$, for $|z| = 1$ with $z \neq b_i,\ i = 1, 2, 3, ...mn' - s$, we have

$$\left|\frac{z}{z-b_i}\right| \geq \left|\frac{z}{z-b_i} - 1\right|. \tag{3.2}$$

Using the fact that $Re(z) \geq \frac{1}{2}$ if and only if $|z| > |z-1|$, we get from (3.2), $Re\left(\frac{z}{z-b_i}\right) \geq \frac{1}{2},\ i = 1, 2, 3...mn' - s$. Hence from (3.1), we get

$$\begin{aligned} Re\left(\frac{zr'(u(z)).u'(z)}{r(u(z))}\right) \geq & s + \sum_{i=1}^{mn'-1} \frac{1}{2} - \sum_{j=1}^{mn} Re\left(\frac{z}{z-a_j}\right) \\ =& s + \sum_{j=1}^{mn-s} Re\left(\frac{1}{2} - \frac{z}{z-a_j}\right) - \frac{1}{2}[mn - mn' + s] \\ =& s + \sum_{j=1}^{mn-s} \frac{|a_j|^2 - 1}{2|z-a_j|^2} - \frac{1}{2}[mn - mn' + s] \\ =& \frac{1}{2}\left\{|B'(z) + s - m(n-n')|\right\}. \end{aligned}$$

This completes the proof of Lemma 3.1. □

Lemma 3.2. If $r(u(z)) \in \mathbb{R}_{mn}$ having all its zeros in $|z| \geq 1$ except with s-fold zeros at the origin, then for $r(u(z)) \neq 0$ and $|z| = 1$, we have

$$Re\left(\frac{zr'(u(z)).u'(z)}{r(u(z))}\right) \leq \frac{1}{2}\left\{|B'(z) + s - m(n-n')|\right\},$$

where mn' and mn are respectively the number of zeros and poles of $r(u(z))$.

Proof . The proof of Lemma 3.2 follows on the same lines as the proof of Lemma 3.1, so we omit the details. □

Lemma 3.3. Let $r(u(z)),\ s(u(z)) \in \mathbb{R}_{mn}$, and all the mn' zeros of $s(u(z))$ lie in $|z| \leq 1$ and for $|z| = 1$,

$$|r(u(z))| \leq |s(u(z))|.$$

Then for every $|\beta| \leq 1$ and $|z| = 1$,

$$|B(z)r'(u(z))u'(z) + \frac{\beta}{2}B'(z)r(u(z))| \leq |B(z)s'(u(z))u'(z) + \frac{\beta}{2}B'(z)s(u(z))|. \tag{3.3}$$

Proof . The proof of Lemma 3.3 follows on the same lines as that of given by Li ([6], Theorem 3.2). Hence, we omit the details. □

Lemma 3.4. Let $r(u(z)) \in \mathbb{R}_{mn}$, and all the mn' zeros of $r(u(z))$ lie in $|z| \leq 1$. Then for every $|\beta| \leq 1$ and $|z| = 1$,

$$|B(z)(r^*(u(z)))' + \frac{\beta}{2}B'(z)r^*(u(z))| \leq |B(z)r'(u(z))u'(z) + \frac{\beta}{2}B'(z)r(u(z))|. \tag{3.4}$$

Proof . Since $r^*(u(z)) = B(z)\overline{r(u(1/\bar{z}))}$, we have

$$|r^*(u(z))| = |r(u(z))| \text{ for } |z| = 1.$$

Also, $r(u(z))$ has all its zeros in $|z| \leq 1$, we apply Lemma 3.3 with $r(u(z))$ and $s(u(z))$ being replaced by $r^*(u(z))$ and $r(u(z))$ respectively to obtain the result. □

Lemma 3.5. Let $r(u(z)) \in \mathbb{R}_{mn}$, then for $|z| = 1$,

$$|(r^*(u(z)))'| + |(r(u(z)))'| \geq |B'(z)||r(u(z))|. \tag{3.5}$$

Proof . We have $r^*(u(z)) = B(z)\overline{r(u(1/\bar{z}))}$. Therefore,

$$(r^*(u(z)))' = B'(z)\overline{r(u(1/\bar{z}))} - \overline{r'(u(1/\bar{z})).u'(1/\bar{z})}\frac{1}{z^2}.$$

Hence for $|z| = 1$, we have

$$|(r^*(u(z)))'| = \left|\frac{zB'(z)}{B(z)}\overline{r(u(z)} - z\overline{r'(u(z))u'(z)}\right|.$$

Using the fact that $\frac{zB'(z)}{B(z)}$ is real, we get

$$\begin{aligned}|(r^*(u(z)))'| &= \left|\frac{zB'(z)}{B(z)}r(u(z)) - zr'(u(z))u'(z)\right| \\ &= \left|B'(z)r(u(z)) - r'(u(z))u'(z)B(z)\right| \\ &\geq |B'(z)r(u(z))| - |r'(u(z))||u'(z)||B(z)|.\end{aligned}$$

Equivalently,

$$|(r^*(u(z)))'| + |(r(u(z)))'| \geq |B'(z)||r(u(z))|.$$

This completes the proof of Lemma 3.5. □

4 Proofs of the Theorems

Proof of Theorem 2.1 Since $r(u(z)) \in \mathbb{R}_{mn}$, where $r(u(z))$ has all its zeros in $|z| \leq 1$ with s-fold zeros at the origin, therefore we have for $0 \leq \theta < 2\pi$,

$$Re\left(\frac{zr'(u(z)).u'(z)}{r(u(z))} + \frac{n\beta}{2}\right)\Bigg|_{z=e^{i\theta}} = Re\left(\frac{zr'(u(z)).u'(z)}{r(u(z))}\right)\Bigg|_{z=e^{i\theta}} + \frac{n}{2}Re(\beta). \tag{4.1}$$

Since $r(u(z)) \in \mathbb{R}_{mn}$ has all its zeros in $|z| \leq 1$ with s-fold zeros at the origin, therefore applying Lemma 3.1 on right hand side of (4.1), we have

$$Re\left(\frac{zr'(u(z)).u'(z)}{r(u(z))} + \frac{n\beta}{2}\right)\Bigg|_{z=e^{i\theta}} \geq \frac{1}{2}\left\{|B'(e^{i\theta})| + s - m(n-n') + nRe(\beta)\right\},$$

for the points $e^{i\theta}, 0 \leq \theta < 2\pi$, other than the zeros of $r(u(z))$. Hence, we have

$$\left|\frac{zr'(u(e^{i\theta})).u'(e^{i\theta})}{r(u(e^{i\theta}))} + \frac{n\beta}{2}\right| \geq \frac{1}{2}\left\{|B'(e^{i\theta})| + s - m(n-n') + nRe(\beta)\right\}, \tag{4.2}$$

for the points $e^{i\theta}, 0 \leq \theta < 2\pi$, other than the zeros of $r(u(z))$. Since (4.2) is true for the points $e^{i\theta},\ 0 \leq \theta < 2\pi$, which are the zeros of $r(u(z))$ also, it follows that

$$|zr'(u(z))u'(z) + \frac{n}{2}\beta r(u(z))| \geq \frac{1}{2}\left\{|B'(z)| + s - m(n-n') + nRe(\beta)\right\}|r(u(z))|,$$

for $|z| = 1$ and for every β with $|\beta| \leq 1$. This completes the proof of Theorem 2.1.

Proof of Theorem 2.5 In case $r(u(z))$ has a zero on $|z| = 1$, then $m(rou, 1) = \min_{|z|=1} |r(u(z))| = 0$, and the result in this case follows from Theorem 2.1. Henceforth, we suppose that $r(u(z))$ has all its zeros in $|z| < 1$, so that $m(rou, 1) > 0$. Now $m(rou, 1) \leq |r(u(z))|$ for $|z| = 1$. If μ is any complex number such that $|\mu| < 1$, then

$$|m(rou, 1)\mu z^s| < |r(u(z))| \quad \text{for } |z| = 1.$$

Since all the zeros of $r(u(z))$ lie in $|z| < 1$, it follows by Rouché's theorem that all the zeros of $T(u(z)) = r(u(z)) + \mu m(rou, 1)z^s$ also lie in $|z| < 1$, with s-fold zeros at the origin. Applying Lemma 3.1 to the rational function $T(z) = r(u(z)) + \mu m(rou, 1)z^s$, for any β with $|\beta| \leq 1$ and $|z| = 1$, we get

$$\begin{aligned} Re\left(\frac{zT'(u(z))u'(z)}{T(u(z))} + \frac{n\beta}{2}\right) =& Re\left(\frac{zT'(u(z))u'(z)}{T(u(z))}\right) + \frac{n}{2}Re(\beta) \\ \geq& \frac{1}{2}\left\{|B'(z)| + s - m(n-n') + nRe(\beta)\right\}, \end{aligned} \tag{4.3}$$

which implies

$$\left|zT'(u(z))u'(z) + \frac{n}{2}\beta T(u(z))\right| \geq \frac{1}{2}\left\{|B'(z)| + s - m(n-n') + nRe(\beta)\right\}|T(u(z))|,$$

or

$$\begin{aligned} &\left|zr'(u(z))u'(z) + \frac{n}{2}Re(\beta)r(u(z)) + \frac{n}{2}Re(\beta)\mu z^s m(rou, 1) + \mu s z^s m(rou, 1)\right| \\ &\qquad \geq \frac{1}{2}\left\{|B'(z)| + s - m(n-n') + nRe(\beta)\right\}\left|r(u(z)) + \mu z^s m(rou, 1)\right|, \end{aligned}$$

or

$$\begin{aligned} &\left|zr'(u(z))u'(z) + \frac{n}{2}Re(\beta)r(u(z)) + \left(s + \frac{n}{2}Re(\beta)\right)\mu z^s m(rou, 1)\right| \\ &\qquad \geq \frac{1}{2}\left\{|B'(z)| + s - m(n-n') + nRe(\beta)\right\}\left|r(u(z)) + \mu z^s m(rou, 1)\right|. \end{aligned} \tag{4.4}$$

Now, choosing the argument of μ suitably on the Right hand side of (4.4) such that

$$|r(u(z)) + \mu z^s m(rou,1)| = |r(u(z))| + |\mu| m(rou,1) \quad \text{for } |z|=1,$$

we obtain from (4.4) that

$$\begin{aligned}\left|zr'(u(z))u'(z) + \frac{n}{2}Re(\beta)r(u(z))\right| + |\mu| m(rou,1)\left(s + \frac{n}{2}Re(\beta)\right) \\ \geq \frac{1}{2}\left\{|B'(z)| + s - m(n-n') + nRe(\beta)\right\}\left(|r(u(z))| + |\mu| m(rou,1)\right).\end{aligned}$$

Equivalently,

$$\begin{aligned}\left|zr'(u(z))u'(z) + \frac{n}{2}Re(\beta)r(u(z))\right| + \zeta m(rou,1)\left(s + \frac{n}{2}Re(\beta)\right) \\ \geq \frac{1}{2}\left\{|B'(z)| + s - m(n-n') + nRe(\beta)\right\}\left(|r(u(z))| + \zeta m(rou,1)\right).\end{aligned}$$

This completes the proof of Theorem 2.5.

Proof of Theorem 2.8 Since $r(u(z)) \in \mathbb{R}_{mn}$, then by Lemma 3.5 we have for $|z|=1$,

$$|(r^*(u(z)))'| + |(r(u(z)))'| \geq |B'(z)||r(u(z))|. \tag{4.5}$$

For any $|\beta| \leq 1$, we have

$$\begin{aligned}&\left|B(z)r'(u(z))u'(z) + \frac{\beta}{2}B'(z)r(u(z))\right| + \left|B(z)(r^*(u(z)))' + \frac{\beta}{2}B'(z)r^*(u(z))\right| \\ &\geq |B(z)|\,|r'(u(z))u'(z)| + |B(z)||(r^*(u(z)))'| - \left|\frac{\beta}{2}\right||B'(z)||r(u(z))| - \left|\frac{\beta}{2}\right||B'(z)||r^*(u(z))|,\end{aligned}$$

which gives by using (4.5) and the fact that $|r(u(z))| = |r^*(u(z))|$ for $|z|=1$,

$$\begin{aligned}\left|B(z)r'(u(z))u'(z) + \frac{\beta}{2}B'(z)r(u(z))\right| + \left|B(z)(r^*(u(z)))' + \frac{\beta}{2}B'(z)r^*(u(z))\right| \\ \geq |r'(u(z))u'(z)| + |(r^*(u(z)))'| - |\beta||B'(z)||r(u(z))| \\ \geq |B'(z)||r(u(z))| - |\beta||B'(z)||r(u(z))|.\end{aligned} \tag{4.6}$$

Now, by Lemma 3.4, we have for $|z|=1$,

$$\left|B(z)r'(u(z))u'(z) + \frac{\beta}{2}B'(z)r(u(z))\right| \geq \left|B(z)(r^*(u(z)))' + \frac{\beta}{2}B'(z)r^*(u(z))\right|. \tag{4.7}$$

On combining inequalities (4.6) and (4.7), we get

$$\left|B(z)r'(u(z))u'(z) + \frac{\beta}{2}B'(z)r(u(z))\right| \geq \frac{|B'(z)|}{2}(1-|\beta|)|r(u(z))|, \tag{4.8}$$

for $|z|=1$ and $|\beta| \leq 1$. Since $|B'(z)| \neq 0$ and $|B(z)|=1$ for $|z|=1$, we get from (4.8), that

$$\left|\frac{r'(u(z))u'(z)}{B'(z)} + \frac{\beta}{2}\frac{r(u(z))}{B(z)}\right| \geq \frac{1}{2}(1-|\beta|)|r(u(z))|,$$

for $|z|=1$ and $|\beta| \leq 1$. This completes the proof of Theorem 2.8.

Proof of Theorem 2.10 In case $r(u(z))$ has some zeros on $|z|=1$, then $\min\limits_{|z|=1}|r(u(z))| = 0$ and the result follows by Theorem 2.8 in this case. Henceforth, we assume that all the zeros of $r(u(z))$ lie in $|z|<1$. Let $m(rou,1) = \min_{|z|=1}|r(u(z))|$. Clearly $m(rou,1) > 0$ and we have $|\lambda m(rou,1)| < |r(u(z))|$ on $|z|=1$ for any λ with $|\lambda|<1$.

By Rouché's theorem, the rational function $V(z) = r(u(z)) + \lambda m(rou, 1)$ has all its zeros in $|z| < 1$. Let $W(z) = B(z)\overline{V(1/\bar{z})} = r^*(u(z)) + \bar{\lambda} m(rou, 1)B(z)$, then $|W(z)| = |V(z)|$ for $|z| = 1$. On applying Lemma 3.4, we get for any β with $|\beta| \leq 1$ and $|z| = 1$,

$$\left|B(z)\Big((r^*(u(z)))' + \bar{\lambda}B'(z)m(rou, 1)\Big) + \frac{\beta}{2}B'(z)\Big(r^*(u(z)) + \bar{\lambda}B(z)m(rou, 1)\Big)\right| \\ \leq \left|B(z)r'(u(z)) + \frac{\beta}{2}B'(z)\Big(r(u(z)) + \lambda m(rou, 1)\Big)\right|. \tag{4.9}$$

Implying that,

$$\left|B(z)(r^*(u(z)))' + \frac{\beta}{2}B'(z)r^*(u(z)) + \bar{\lambda}\Big(1 + \frac{\beta}{2}\Big)B(z)B'(z)m(rou, 1)\right| \\ \leq \left|B(z)r'(u(z))u'(z) + \frac{\beta}{2}B'(z)r(u(z))\right| + \left|\frac{\beta}{2}\right| |\lambda| m(rou, 1)|B'(z)| \tag{4.10}$$

for $|z| = 1, |\beta| \leq 1$ and $|\lambda| < 1$. Choosing the arguments of λ in the left hand side of (4.10) such that

$$\left|B(z)(r^*(u(z)))' + \frac{\beta}{2}B'(z)r^*(u(z)) + \bar{\lambda}\Big(1 + \frac{\beta}{2}\Big)B(z)B'(z)m(rou, 1)\right| \\ = \left|B(z)(r^*(u(z)))' + \frac{\beta}{2}B'(z)r^*(u(z))\right| + |\lambda| m(rou, 1)\left|1 + \frac{\beta}{2}\right| |B(z)B'(z)|. \tag{4.11}$$

Hence, from (4.10) we get by using (4.11) and $|B(z)| = 1$ for $|z| = 1$,

$$\left|B(z)r'(u(z))u'(z) + \frac{\beta}{2}B'(z)r(u(z))\right| \geq \left|B(z)(r^*(u(z)))' + \frac{\beta}{2}B'(z)r^*(u(z))\right| + |\lambda||B'(z)|\left\{\left|1 + \frac{\beta}{2}\right| - \left|\frac{\beta}{2}\right|\right\}m(rou, 1). \tag{4.12}$$

Letting $|\lambda| \to 1$ in (4.12) and adding $|B(z)r'(u(z))u'(z) + \frac{\beta}{2}B'(z)r(u(z))|$ on both sides of it and using (4.6), we get the required assertion and this completes the proof of Theorem 2.10.

5 Conclusions

Certain Turán-type estimates for the modulus of the derivative of rational functions are obtained. The obtained results produce many inequalities for polynomials and polar derivatives as special cases.

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