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Perfect 2-coloring of the six regular graphs up to order 10

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Abstract

In this paper, we study perfect 2-coloring of the six regular graphs up to order 10. We first obtain all possible color parameter matrices and then examine them according to the limitation of our theorem.

Keywords: Regular graph, Parameter matrices, Perfect coloring
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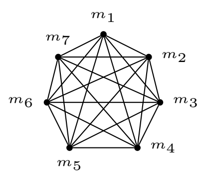
1 Introduction

A graph is called regular graph if degree of each vertex is equal. Also called k- regular if degree of each vertex in the graph is k, for some positive integer k.

Definition 1.1. For each graph Γ and each integer n, a mapping $f : V(\Gamma) \to \{1, ..., n\}$ is called a perfect n- coloring with matrix $M = (m_{ij})_{i,j\in\{1,...,n\}}$ if it is surjective, and for all i, j for every vertex of color i, the number of its neighbors of color j is equal to m_{ij}. The matrix M is called the parameter matrix of a perfect coloring. When $n = 2$, we denote the two colors by W and B representing white and black respectively.

Perfect 2-coloring of the cubic graphs of order less than or equal to 10 and perfect 3-coloring of the cubic graphs of order 10 were described; in [2, 5], respectively. Also, to read more about perfect coloring, refer to articles [1, 3, 4] We will show that all possible parameter matrices for six regular graphs with two colors. The six regular graphs up to order 10 are given in Fig $1 - 4$. We first give some results concerning necessary condition for the existence of perfect 2-coloring six regular graphs with a given parameter matrix $M = (m_{ij})_{i,j\in 1,2}$. The simplest condition for the existence of a perfect 2- colorings of a six regular graphs with the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $a + b = c + d = 6$.



Γ_1

Figure 1: The six regular graph of order 7

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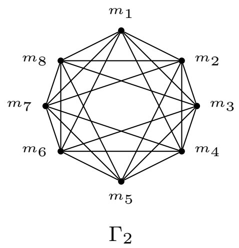

Figure 2: The six regular graph of order 8

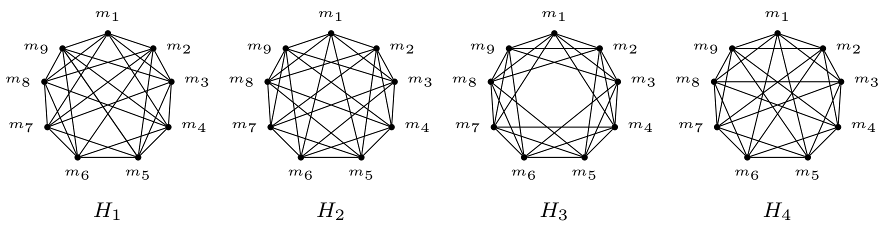

Figure 3: The six regular graphs of order 9

In Section 2, we present the preliminary theorems related to perfect 2-coloring. Our method consists of the following steps; First, obtain the eigenvalues of the adjacency matrix of the graph. Then, we show that the parameter matrices of perfect 2-coloring must have eigenvalues that are a subset of the eigenvalues of the adjacency matrix. In the next, we calculate the number of white vertices for the remaining parameter matrices and exclude those whose number is not an integer. Finally according to the parameter matrix and the number of white vertices, as well as the definition of perfect coloring, we assign a color to each vertex that satisfies the condition of the definition. If this is not possible, we conclude that the graph does not have perfect 2-coloring.

2 Preliminaries and analysis

In this section, first we have the following basic lemmas see [1, 5, 6, 7, 8, 9]. Then we calculate the parameter matrices for the perfect coloring of our graphs.

Lemma 2.1. Suppose that Γ is a k-regular graph and f is a perfect n- coloring with matrix $M = (m_{ij})_{i,j \in \{1,\ldots,n\}}$ in graph Γ. Then the sum of each row in matrix M is k.

Lemma 2.2. [10] If f is a perfect coloring of the graph Γ with n colors, then any eigenvalue of the parameter matrix is an eigenvalue subset of the adjacent matrix Γ.

Lemma 2.3. [6] If W is the set of white vertex in a perfect 2- coloring of a graph Γ with matrix $M = (m_{ij})_{i,j \in 1,2}$. Then $|W| = |V(\Gamma)| \frac{m_{21}}{m_{12}+m_{21}}$.

Lemma 2.4. Suppose that $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is a parameter matrix of a perfect 2- coloring of a k-regular graph. Then eigenvalues of the parameter matrix are k and $a - c$ such that we obviously have $a - c \neq k$.

Lemma 2.5. Suppose f is a perfect 2-coloring with matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ of connected graph Γ. Then b and c are both opposite zero.

Lemma 2.6. If graph Γ has perfect 2-coloring with matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, then it also has a perfect 2-coloring with matrix $\left[\begin{smallmatrix} d & c \\ b & a \end{smallmatrix}\right]$.

Using the Lemma 2.2 and with the help of above lemmas we calculate all the parameter matrix perfect 2-coloring of the six regular graphs as following theorem:

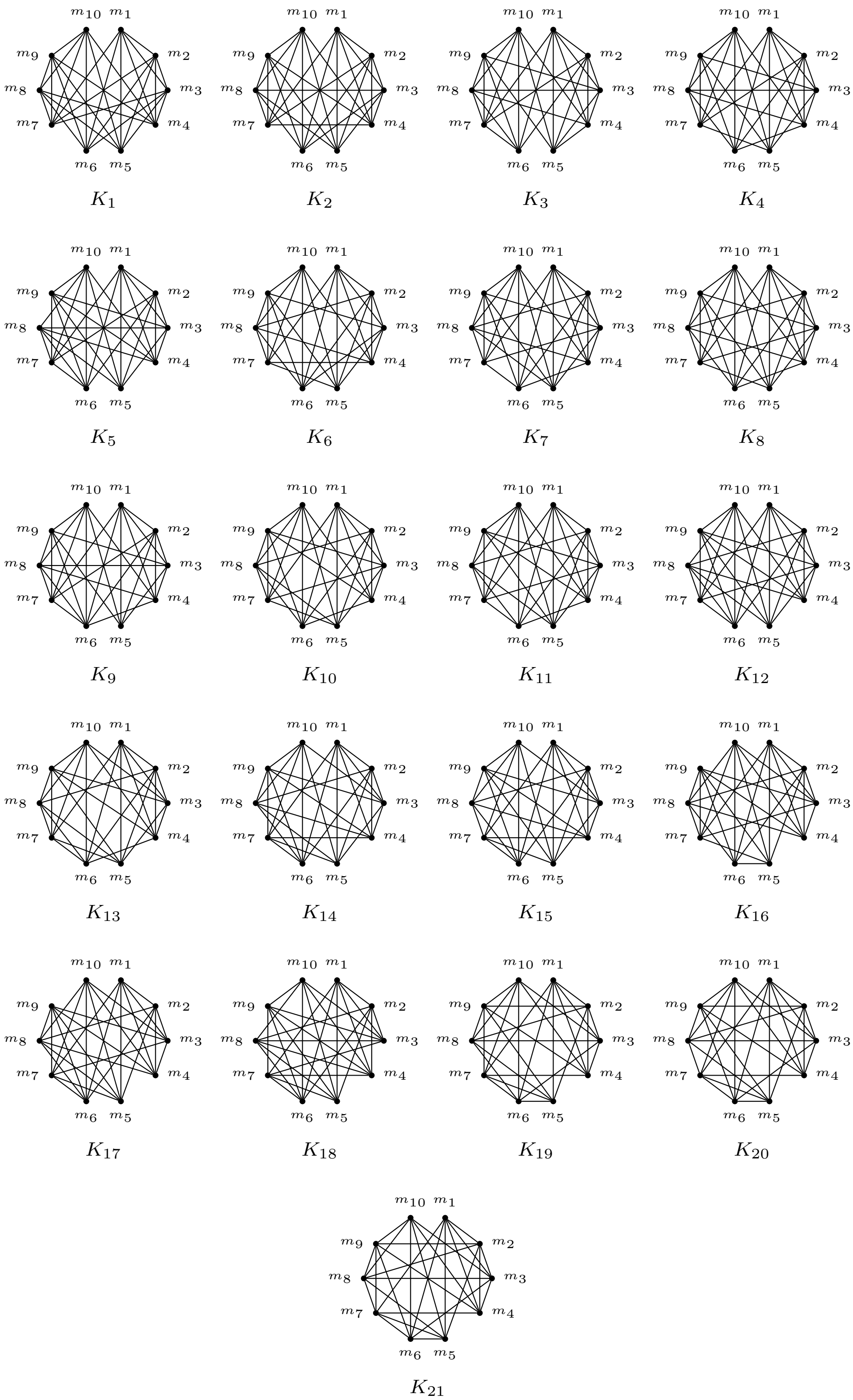

Figure 4: The six regular graphs of order 10

Theorem 2.7. If M is the parameter matrix corresponding to a perfect 2- coloring in connected six regular graph. Then M is one of the following possible matrices:

$$M_1 = \begin{bmatrix} 0 & 6 \\ 1 & 5 \end{bmatrix}, \ M_2 = \begin{bmatrix} 0 & 6 \\ 2 & 4 \end{bmatrix}, \ M_3 = \begin{bmatrix} 0 & 6 \\ 3 & 3 \end{bmatrix}, \ M_4 = \begin{bmatrix} 0 & 6 \\ 4 & 2 \end{bmatrix}, \ M_5 = \begin{bmatrix} 0 & 6 \\ 5 & 1 \end{bmatrix}, \ M_6 = \begin{bmatrix} 0 & 6 \\ 6 & 0 \end{bmatrix}, \ M_7 = \begin{bmatrix} 1 & 5 \\ 1 & 5 \end{bmatrix},$$

$$M_8 = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}, \ M_9 = \begin{bmatrix} 1 & 5 \\ 3 & 3 \end{bmatrix}, M_{10} = \begin{bmatrix} 1 & 5 \\ 4 & 2 \end{bmatrix}, M_{11} = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}, M_{12} = \begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix}, M_{13} = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}, M_{14} = \begin{bmatrix} 2 & 4 \\ 3 & 3 \end{bmatrix},$$

$$M_{15} = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}, M_{16} = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix}, M_{17} = \begin{bmatrix} 3 & 3 \\ 2 & 4 \end{bmatrix}, M_{18} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}, M_{19} = \begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix}, M_{20} = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}, M_{21} = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}.$$

Proof . Using Lemma 2.2, there are 49 possible matrix for six regular graphs. Using the previous three lemmas, only one of given 21 matrices can be possible parameter matrices. Using Lemma 2.6 it can be seen that the following matrices are equivalent in pairs and only their colors are shifted together.

$$\begin{bmatrix} 0 & 6 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 6 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 6 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 6 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 6 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 6 & 0 \end{bmatrix},$$

and some others. Therefore, only 21 given matrices can be perfect 2-coloring parameter matrices of six regular graphs. □

Using the above theorem, we show that only may be parameter matrices of six regular graphs up to order 10 as following:

Lemma 2.8. Suppose that Γ_1 is a six regular graph of order 7. Then Γ_1 has perfect 2-coloring with parameter matrices M_1, M_8, and M_{14}.

Proof . Using Theorem 2.7 of a parameter matrix corresponding to a perfect 2-coloring for the graph Γ_1 may be one of the 21 matrices given. Using the Lemma 2.2, we can eliminate 18 of these matrices, leaving only M_1, M_8, and M_{14} as possible candidates. By using Lemma 2.3 we can see the structures of graph Γ_1 that has perfect 2- coloring. Consider three maps as follows:

$F_1(m_1) = W$,

$F_1(m_2)= F_1(m_3)= F_1(m_4)= F_1(m_5)= F_1(m_6)= F_1(m_7) = B$,

$F_2(m_1) = F_2(m_2) = W$,

$F_2(m_3) = F_2(m_4) = F_2(m_5) = F_2(m_6) = F_2(m_7) = B$,

$F_3(m_1) = F_3(m_2) = F_3(m_7) = W$,

$F_3(m_3) = F_3(m_4) = F_3(m_5) = F_3(m_6) = B$.

It is clear that F_1, F_2, and F_3 are perfect 2-coloring with matrices M_1, M_8, and M_{14}, respectively. □

Lemma 2.9. Suppose that Γ_2 is a six regular graph of order 8. Then Γ_2 has perfect 2-coloring with parameter matrices M_2, M_{15}, and M_{18}.

Proof . Using Theorem 2.7 we know that a parameter matrix corresponding to a perfect 2-coloring for the graph Γ_2 must be one of the 21 matrices given. Using Lemmas 2.2 and 2.3, we can eliminate 17 of these matrices, leaving only M_2, M_9, M_{15}, and M_{18} as possible candidates. For example, by using Lemma 2.3, we can see that the number W of M_7 is not an integer, so M_7 cannot be a parameter matrix for Γ_2 . We can also show that M_9 cannot have a perfect 2-coloring. Using Lemma 2.3, we get the number of W and B, then we check the different cases. So we have:

$F_1(m_1) = F_1(m_5) = W$,

$F_1(m_2)= F_1(m_3)= F_1(m_4)= F_1(m_6)= F_1(m_7)= F_1(m_8) = B$,

$F_2(m_1) = F_2(m_3) = F_2(m_5) = F_2(m_7) = W$,

$F_2(m_2) = F_2(m_4) = F_2(m_6) = F_2(m_8) = B$,

$F_3(m_1) = F_3(m_2) = F_3(m_7) = F_3(m_8) = W$,

$F_3(m_2) = F_3(m_3) = F_3(m_4) = F_3(m_5) = F_3(m_6) = B$.

It is clear that F_1, F_2, and F_3 are perfect 2-coloring with matrices M_2, M_{15}, and M_{18}, respectively. Now we show that Γ_2 has no perfect 2-coloring with parameter matrix M_9. To prove this claim, we assume the opposite that Γ_2 with matrix M_9 has perfect 2-coloring. Then have the following possibilities:

$F(m_1) = F(m_2) = F(m_5) = W,$

$F(m_3) = F(m_4) = F(m_6) = F(m_7) = F(m_8) = B.$

According to the parameter matrix M_9, each vertex with color W must have one neighbor with color W but vertex m_2 is two neighbors with W which is a contradiction with the first row matrix M_9. So graph Γ_2 has no perfect 2-coloring with M_9. □

Lemma 2.10. Suppose that H_1, H_2, H_3 and H_4 are six regular graphs of order 9. Then H_1 has no perfect 2-coloring but H_2, H_3, H_4 have perfect 2-colorings as shown in the Table 1.

Graphs	H_2	H_3	H_4
Matrices	M_3	M_{13}	M_3, M_{13}

Table 1

Proof . Using Lemmas 2.2 and 2.3, we know that H_1 can only have a perfect 2-coloring with M_3 or M_{10}, because the eigenvalues of the other matrices, such as M_{13},and M_{19} are not subsets of the adjacency matrix of H_1. Then we show that H_1 does not have a perfect 2-coloring with M_3. To prove the claim, we assume the opposite that H_1 has a perfect 2-coloring with M_3. According to M_3, each vertex with color W must have six neighbors with color B, and we know from Lemma 2.3 that the number of vertices with color W is three. If we assign $F(m_1) = W$, then we have:

$F(m_1) = F(m_5) = F(m_6) = W,$

$F(m_2) = F(m_3) = F(m_4) = F(m_7) = F(m_8) = F(m_9) = B.$

But this contradiction M_3, because $F(m_3) = B$ and $F(m_1)$ and $F(m_3)$ are adjacent. Therefore, H_1 does not have a perfect 2-coloring with M_3. The proof for M_{10} is similar to the above argument. Also using Lemmas 2.2 and 2.3, we know that H_2, and H_4 can only have a perfect 2-coloring with M_3, M_{10} or M_{13},and also graph H_3 can only have a perfect 2-coloring with M_{13}.Now we show that the graph H_2 has perfect 2- coloring with parameter matrices M_3. Consider mapping F as follows:

$F(m_1) = F(m_4) = F(m_7) = W,$

$F(m_2) = F(m_3) = F(m_5) = F(m_6) = F(m_8) = F(m_9) = B.$

It is clear that F is a perfect 2-coloring with the matrix M_3. The graph H_4 has perfect 2- coloring with parameter matrices M_3 and M_{13}. Consider mappings F_1 and F_2 as follows:

$F_1(m_1) = F_1(m_4) = F_1(m_7) = W,$

$F_1(m_2) = F_1(m_3) = F_1(m_5) = F_1(m_6) = F_1(m_8) = F_1(m_9) = B,$

$F_2(m_1) = F_2(m_5) = F_2(m_6) = W,$

$F_2(m_2) = F_2(m_3) = F_2(m_4) = F_2(m_7) = F_2(m_8) = F_2(m_9) = B.$

It is clear that F_1 is a perfect 2- coloring with the matrix M_3 and also F_2 with M_{13}. □

Lemma 2.11. Suppose that K be a six regular graph of order 10. Then K_1, K_2, K_4, K_5, K_6, K_8, K_9, K_{10}, K_{11}, K_{12}, K_{13}, K_{14}, K_{15}, and K_{18} has no perfect 2-coloring.

Proof . Using Lemma 2.2, we know that K_1 can only have a perfect 2-coloring with M_4, M_{11}, M_{18}, or M_{20}, because the eigenvalues of the other matrices, such as M_1 and M_2 are not subsets of the eigenvalues of the adjacency matrix of K_1. Then we show that K_1 does not have a perfect 2-coloring with M_4. To prove the claim, we assume the opposite that K_1 has a perfect 2- coloring with M_4. According to M_4, each vertex with color W must have six neighbors with color B, and we know from Lemma 2.3 that the number of vertices with color W is four. If we assign $F(m_9) = W$, then we have:

$F(m_1) = F(m_8) = F(m_9) = F(m_{10}) = W,$

$F(m_2) = F(m_3) = F(m_4) = F(m_5) = F(m_6) = F(m_7) = B.$

But this contradiction M_4, because $F(m_4) = B$ and $F(m_9)$ and $F(m_4)$ are adjacent. Therefore, K_1 does not have a perfect 2-coloring with M_4. The proof for M_{11}, M_{18}, and M_{20} is similar to the above argument. Also, using Lemma 2.2, we know that K_2 can only have a perfect 2-coloring with M_{17}, or M_{18}, because the eigenvalues of the other matrices, such as M_1 and M_2 are not subsets of the eigenvalues of the adjacency matrix of K_2. Then we show that K_2 does not have a perfect 2-coloring with M_{18}. To prove the claim, we assume the opposite that K_2 has a perfect 2-coloring with M_{18}. According to M_{18}, each vertex with color W must have three neighbors with color B, and we know from Lemma 2.3 that the number of vertices with color W is five. If we assign $F(m_2) = B$, then we have:

$F(m_1) = F(m_3) = F(m_5) = F(m_7) = F(m_9) = W$,

$F(m_2) = F(m_4) = F(m_6) = F(m_8) = F(m_{10}) = B$.

But this contradiction M_{18}, because $F(m_2) = B$ and $F(m_2)$is only with $F(m_4)$ and $F(m_6)$ which are B adjacent. Therefore, K_2 does not have a perfect 2-coloring with M_{18}. The proof for M_{17} is similar to the above argument. □

Lemma 2.12. The 6-regular graphs of order 10 have perfect 2- colorings as shown in the Table 2.

Graphs	K_1	K_7	K_{16}	K_{17}	K_{19}	K_{20}	K_{21}
Matrices	M_{20}	M_{20}	M_{17}	M_{17}	M_{15},M_{17}	M_{17}	M_{17}

Table 2

Proof . Using Lemmas 2.1 and 2.2, we know that K_1 can only have a perfect 2-coloring with parameter matrix M_{20}. Consider mapping F as follows:

$F(m_1) = F(m_2) = F(m_3) = F(m_4) = F(m_5) = W$,

$F(m_6) = F(m_7) = F(m_8) = F(m_9) = F(m_{10}) = B$.

It is clear that F is a perfect 2- coloring with the matrix M_{20}. Also, K_7 can only have a perfect 2-coloring with parameter matrix M_{20}. Consider mapping F as follows:

$F(m_1) = F(m_2) = F(m_3) = F(m_4) = F(m_5) = W$,

$F(m_6) = F(m_7) = F(m_8) = F(m_9) = F(m_{10}) = B$.

It is clear that F is a perfect 2-coloring with the matrix M_{20}. The graph K_{16} can only have a perfect 2-coloring with parameter matrix M_{17}. Consider mappings F as follows:

$F(m_1) = F(m_2) = F(m_3) = F(m_4) = W$,

$F(m_5) = F(m_6) = F(m_7) = F(m_8) = F(m_9) = F(m_{10}) = B$.

It is clear that F is a perfect 2-coloring with the matrix M_{20}. For graph K_{17}, the proof is similar to the above proofs and K_{19} can only have a perfect 2-coloring with parameter matrices M_{15} and M_{17}. Consider mappings F_1 and F_2 as follows:

$F_1(m_3) = F_1(m_4) = F_1(m_5) = F_1(m_6) = F_1(m_9) = W$,

$F_1(m_1)F_1(m_2) = F_1(m_7) = F_1(m_8) = F_1(m_{10}) = B$,

$F_2(m_1) = F_2(m_2) = F_2(m_3) = F_2(m_4) = W$,

$F_2(m_5) = F_2(m_6) = F_2(m_7) = F_2(m_8) = F_2(m_9) = F_2(m_{10}) = B$.

It is clear that F_1 is a perfect 2- coloring with the matrix M_{15} and also F_2 with M_{17}. For other graph the proof is similar to the above proofs. □

We have the following result, by using the lemmas.

Theorem 2.13. All perfect 2-coloring of the six regular graphs up to order 10 as shown in the Table 3.

Graphs	Γ_1	Γ_2	H_2	H_3	H_4	K_3	K_7	K_{16}	K_{17}	K_{19}	K_{20}	K_{21}
Matrices	M_1, M_8, M_{14}	M_2, M_{15}, M_{18}	M_3	M_{13}	M_3, M_{13}	M_{20}	M_{20}	M_{17}	M_{17}	M_{15},M_{17}	M_{17}	M_{17}

Table 3

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