Int. J. Nonlinear Anal. Appl. In Press, 1–17
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2024.32857.4886

Ulam stability of $\wp$-mild solutions for ψ-Caputo-type fractional semilinear differential equations

Baihi Asmaa*, Kajouni Ahmed, Hilal Khalid

LMACS Laboratory, Sultan Moulay Slimane University, Beni Mellal, Morocco

(Communicated by Michael Th. Rassias)

Abstract

We study in this paper the existence and uniqueness of solutions to initial value problems for semilinear differential equations involving ψ-Caputo differential derivatives of an arbitrary $l \in (0, 1)$, using the fixed theorem. We do analyse further the M-L-U-H stability and the M-L-U-H-R stability. Then we conclude with an example to illustrate the result.

Keywords: Semilinear differential equations; ψ-Caputo differential derivatives; Mittag-Leffler-Ulam-Hyers stability
2020 MSC: 26A33, 34A08, 34A12, 3AD20, 47H08

1 Introduction

Recently, it has been proven that differential equations involving fractional derivatives have captured the interest of many mathematicians. This is because they can represent various phenomena in several scientific fields (the nonlinear oscillation of earthquakes, the fluid-dynamic traffic model, flow in porous, ...) and have proven to be effective models in areas such as physics, mechanics, biology, chemistry, control theory, and other domains. for example, see [10, 22, 16, 18, 19]-[31, 37, 39, 40, 48].

It is common to research the existence and uniqueness of solutions, as well as techniques for explicit and numerical solutions that are exact and stable. Commonly used techniques to demonstrate the existence and uniqueness of solutions include the fixed point theorem, upper-lower solutions, iterative approach, and numerical method.

In the study of fractional differential equations, Ulam-Hyers stability is a crucial concept that relates to the stability of solutions concerning changes in the initial data ([2, 3, 12, 20, 21, 24, 25, 52]).

Furthermore, Mittag-Leffler-Ulam-Hyers stability, generalized Mittag-Leffler-Ulam-Hyers stability, Mittag-Leffler-Ulam-Hyers–Rassias stability, and generalized Mittag-Leffler–Ulam–Hyers–Rassias stability are the four types of stabilities of the mild solution of the fractional evolution equation in Banach space that Wang and Zhou presented in [53].

There are two famous ways to define fractional integrals and derivatives: the Riemann-Liouville and the Caputo. In a paper by Almeida [7], he presents a generalized version of these derivatives by considering the Caputo fractional derivative of a function with respect to another function called ψ. He studied some useful properties of this new

*Corresponding author
Email addresses: asmaabaihi000@gmail.com (Baihi Asmaa), kajjouni@gmail.com (Kajouni Ahmed), hilalkhalid2005@yahoo.fr (Hilal Khalid)

Received: December 2023 *Accepted:* January 2024

definition of the fractional derivative. The advantage of this definition is that it allows for greater model accuracy by choosing an appropriate ψ function. For more information on ψ-Caputo and Caputo fractional derivatives, please refer to the following papers: [1, 7]-[9, 46]. Motivated by [49, 55] we consider the following evolution problem ψ-Caputo fractional semilinear differential equation:

$$\begin{cases} {}^C D^{l,\psi}_{0^+}\chi(\iota) = -\mathfrak{A}\chi(\iota) + \hbar(\iota, \chi(\iota), \Re\chi(\iota)), \iota \in \Lambda, \\ \chi(0) = \chi_0 \end{cases} \tag{1.1}$$

with $-\mathfrak{A}$ generates an analytic compact semigroup $(V(\iota))_{\iota\geq 0}$ of uniformly bounded linear operators on a Banach space $\mathcal{X}$, $\hbar : \Lambda \times \mathcal{X}_\wp \times \mathcal{X}_\wp \to \mathcal{X}$ is a given function and ${}^C D^{l,\psi}_{0^+}$ is the ψ-Caputo fractional derivative of order $l \in (0, 1)$, $\nu > 0$, $x_0 \in \mathcal{X}$. This derivative gives a more general framework to the results in the literature. The term $\Re\chi(\iota)$ which may be interpreted as a control on the system is defined by: $\Re\chi(\iota) = \int_0^\iota J(\iota, \tau)\chi(\tau)d\tau$, where $J \in C(S, \mathbb{R}^+)$, with $S = \{(\iota, \tau) \in \mathbb{R}^2 : 0 \leq \tau \leq \iota \leq \nu\}$.

The theory that the nonlinear components meet Lipschitz criteria or linear growth requirements was used to treat fractional differential equations in various previous publications. It can be difficult to verify these prerequisites at times.

In our research, we examined the existence and uniqueness of a mild solution for a fractional semilinear differential equation under certain unusual situations. We specifically looked at cases where the nonlinear term satisfies only a few local growth requirements (see conditions $(\mathcal{C}_2)$ and $(\mathcal{C}_3)$). These conditions are much weaker than Lipschitz's circumstances and linear growth conditions.

2 Preliminaries

We proceed by setting $\Lambda = [0, \nu]$. We denote by $\mathcal{X}$ a Banach space with norm $\|.\|$ and $-\mathfrak{A} : \mathcal{D}(\mathfrak{A}) \to \mathcal{X}$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $(V(\iota))_{\iota\geq 0}$. This means there exists $M > 1$ such that

$$\|V(\iota)\| \leq M_V.$$

To define the fractional power $\mathfrak{A}^\wp$ for $0 < \alpha < 1$ as a closed linear operator on its domain $\mathcal{D}(\mathfrak{A}^\wp)$ with inverse $\mathfrak{A}^{-\wp}$, we assume that $0 \in \rho(\mathfrak{A})$. we will present some basic properties in the theorem below.

Theorem 2.1. [43]

1. $\mathcal{X}_\wp = \mathcal{D}(\mathfrak{A}^\wp)$ is a Banach space with the norm $\|\chi\|_\wp = \|\mathfrak{A}^\wp\chi\|$ for each $\chi \in \mathcal{D}(\mathfrak{A}^\wp)$.
2. $V(\iota) : \mathcal{X} \to \mathcal{X}_\alpha$ for each $\iota > 0$.
3. $\mathfrak{A}^\wp V(\iota)\chi = V(\iota)\mathfrak{A}^\wp\chi$ for each $\chi \in \mathcal{D}(\mathfrak{A}^\wp)$ and $\iota \geq 0$.
4. For every $\iota > 0$, $\mathfrak{A}^\wp V(\iota)$ is bounded on $\mathcal{X}$ and there exist $C_\wp > 0$ such that
$$\|\mathfrak{A}^\wp V(\iota)\| \leq \frac{M_\wp}{\iota^\wp}.$$
5. $\mathfrak{A}^{-\wp}$ is bounded linear operator for $0 \leq \wp \leq 1$, there exists $C_\wp$ such that $\|\mathfrak{A}^{-\wp}\| \leq C_\wp$.
6. If $< \wp \leq \gamma$, then $\mathcal{D}(\mathfrak{A}^\gamma) \hookrightarrow \mathcal{D}(\mathfrak{A}^\wp)$.

Remark 2.2. [34] The restriction $V_\wp(\iota)$ of $V(\iota)$ to $\mathcal{X}_\wp$ is exactly the part of $V(\iota)$ in $\mathcal{X}_\wp$. Let $\chi \in \mathcal{X}_\wp$, $(V(\iota))_{\iota\geq 0}$ is a strongly continuous semigroup on $\mathcal{X}_\wp$ and $\|V_\wp(\iota)\| \leq \|V(\iota)\|$ for all $\iota \geq 0$.

Lemma 2.3. [34] $(V_\wp(\iota))_{\iota\geq 0}$ is an immediately compact semigroup in $\mathcal{X}_\wp$, hence it is immediately norm continuous.

The Banach space $C(\Lambda, \mathcal{X}_\wp)$ is denoted by $\mathcal{C}_\wp$ with $\wp \in (0, \nu)$ it supnorm:

$$\|\chi\|_\infty = \sup_{\iota\in\Lambda} \|\chi\|_\wp, \quad for\ \chi \in \mathcal{C}_\wp$$

We will also provide the required data and resources on ψ-fractional derivatives and ψ-fractional integrals,

Definition 2.4. [8] Let $l>0$, $\chi\in L^1(\Lambda,\mathbb{R})$ and $\psi\in C^n(\Lambda,\mathbb{R})$ such that $\psi'(\iota)>0$ for all $\iota\in\Lambda$. The ψ-Riemann Liouville fractional integral of order l of the function χ is given by

$$^{C}I_{0^+}^{l,\psi}\chi(\iota)=\frac{1}{\Gamma(l)}\int_0^{\iota}\psi'(\kappa)(\psi(\iota)-\psi(\kappa))^{l-1}\chi(\kappa)d\kappa. \tag{2.1}$$

Definition 2.5. [8] Let $l>0$, $\chi\in C^{n-1}(\Lambda,\mathbb{R})$ and $\psi\in C^n(D,\mathbb{R})$ such that $\psi'(\iota)>0$ for all $\iota\in\Lambda$. The ψ-Caputo fractional derivative of order l of the function χ is given by

$$^{C}D_{0^+}^{l,\psi}\chi(\iota)=\frac{1}{\Gamma(n-l)}\int_0^{\iota}\psi'(\kappa)(\psi(\iota)-\psi(\kappa))^{n-l-1}\chi_{[n]}^{\psi}(\kappa)d\kappa, \tag{2.2}$$

where

$$\chi_{[n]}^{\psi}(\kappa)=\left(\frac{1}{\psi'(\iota)}\frac{d}{d\kappa}\right)^n\chi(\kappa),\quad n=[l]+1,$$

and $[l]$ denotes the integer part of the real number l.

Proposition 2.6. [8] Let $l>0$, $\chi\in C^{n-1}(\Lambda,\mathbb{R})$, then we have the following propositions

1. $^{C}D_{0^+}^{l,\psi}I_{0^+}^{l,\psi}\chi(\iota)=\chi(\iota)$.
2. $I_{0^+}^{l,\psi\,C}D_{0^+}^{l,\psi}\chi(\iota)=\chi(\iota)-\sum\limits_{k=0}^{n-1}\frac{\chi_{[k]}^{\psi}(0)}{k!}(\psi(\iota)-\psi(0))^k$.
3. $I_{0^+}^{l,\psi}$ is linear and bounded from $C(\Lambda,\mathbb{R})$ to $C(\Lambda,\mathbb{R})$.

Definition 2.7. [22] Let χ, $\psi:[0,\infty)\to\mathbb{R}$ be real valued functions such that $\psi(\iota)$ is continuous and $\psi'(\iota)>0$ on $[0,\infty)$. The generalized Laplace transform of χ is denoted by

$$\mathcal{L}_{\psi}\{\chi(\kappa)\}(\mu)=\int_0^{\infty}e^{-\mu(\psi(\kappa)-\psi(0))}\psi'(\kappa)\chi(\kappa)d\kappa. \tag{2.3}$$

for all κ.

Definition 2.8. [22] Let χ and σ be two functions that are piecewise continuous on Λ and $\psi(\iota)$ of exponential order. We define the generalized convolution of χ and σ by

$$(\chi*_{\psi}\sigma)(\iota)=\int_0^{\iota}\chi(\kappa)\sigma\left(\psi^{-1}(\psi(\iota)+\psi(0)-\psi(\kappa))\right)\psi'(\kappa)d\kappa.$$

Theorem 2.9. [22] Let $l>0$ and χ be a piecewise continuous function on Λ and $\psi(\kappa)$ of exponential order. Then

$$\mathcal{L}_{\psi}\{I_{0^+}^{l,\psi}\chi(\kappa)\}(\mu)=\frac{I_{0^+}^{l,\psi}\chi(\kappa)}{\mu^l}. \tag{2.4}$$

Theorem 2.10. [5, 48] Let χ_1,χ_2 be two integrable functions and $\hbar$ be a continuous function on Λ. Allow $\psi\in C(\Lambda,\mathbb{R})$ be an increasing function to the extent that $\psi'(\iota)>0$ for $\iota\in\Lambda$. Suppose that

1. χ_1 and χ_2 are nonnegative.
2. $\hbar$ is nonnegative and nondecreasing. In case

$$\chi_1(\iota)\le\chi_2(\iota)+\hbar(\iota)\int_0^{\iota}(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)\chi_1(\varsigma)d\varsigma,$$

subsequently

$$\chi_1(\iota)\le\chi_2(\iota)+\int_0^{\iota}\sum_{w=1}^{\infty}\frac{(\hbar(\iota)\Gamma(l))^w}{\Gamma(wl)}(\psi(\iota)-\psi(\varsigma))^{wl-1}\psi'(\varsigma)\chi_2(\varsigma)d\varsigma,$$

for all $\iota\in\Lambda$.

Corollary 2.11. Under the hypotheses of theorem 2.10, let χ_2 be nondecreasing function on Λ. Then we have

$$\chi_1(\iota)\le\chi_2(\iota)E_l[\hbar(\iota)\Gamma(l)(\psi(\iota)-\psi(0))^l],\quad\iota\in\Lambda$$

where

$$E_l(\varrho)=\sum_{w=0}^{\infty}\frac{\varrho^w}{\Gamma(wl+1)},$$

is the Mittag-Leffler function with one parameter for all $\varrho\in\mathbb{C}$ and $l>0$.

3 Representation of mild solution

In this part, we will first define the term "mild solution" for the problem (1.1) and demonstrate the following theorem and foundational lemma. To ensure that, we make the following assumptions.

($\mathcal{C}_1$) $V(\kappa),\ \kappa > 0$ is a compact analytic semigroup and $\lim\limits_{\kappa\to 0^+} V(\kappa) = I$ (the identity operator).

($\mathcal{C}_2$) there exists $\gamma \in [\wp, 1]$ such that the function $\hbar : \Lambda \times \mathcal{X}_\wp \times \mathcal{X}_\wp \to \mathcal{X}_\gamma$ satisfies the following properties:

(a) for all $(\chi, \varsigma) \in \mathcal{X}_\wp \times \mathcal{X}_\wp$, the function $\hbar(., \chi, \varsigma)$ is strongly measurable.

(b) for all $\iota \in \Lambda$, the function $\hbar(\iota, .) : \mathcal{X}_\wp \times \mathcal{X}_\wp \to \mathcal{X}_\gamma$ is continuous.

(c) for all $o > 0$, $\exists$ a functionn $I_o \in L^\infty(\Lambda, (0, \infty))$ such that

$$\sup\{\|\hbar(\iota, \chi, \varsigma)\|_\gamma :\ \|\chi\|_\wp \le o,\ \|\varsigma\|_\wp \le j^*\nu o\} \le I_o(\iota),\ \iota \in \Lambda,$$

and there exists a constant $\xi > 0$ such that

$$\lim_{o\to+\infty} \inf \frac{1}{o} \int_0^\iota (\psi(\iota) - \psi(\kappa))^{l-1} \psi'(\kappa) I_o(\kappa) d\kappa \le \xi < +\infty.$$

where $j^* := \max\{J(\iota, \kappa) :\ (\iota, \kappa) \in S\}$.

($\mathcal{C}_3$) $\hbar : \Lambda \times \mathcal{X}_\wp \times \mathcal{X}_\wp \to \mathcal{X}$ is continuous function and there exist functions; $\ell_1(\iota)$ and $\ell_2(\iota)$ such that

$$\|\hbar(\iota, \chi_2(\iota), \Re\chi_2(\iota)) - \hbar(\iota, \chi_1(\iota), \Re\chi_1(\iota))\|_\wp \le \ell_1(\iota) \|\chi_2 - \chi_1\|_\wp + \ell_2(\iota) \|\Re\chi_2 - \Re\chi_1\|_\wp,$$

$\iota \in \Lambda$, $\chi_2, \chi_1 \in \mathcal{X}_\wp$, denote $I_{0^+}^{l,\psi}\ell(\iota) = \sup\limits_{\nu\in\Lambda}\{I_{0^+}^{l,\psi}\ell_1, I_{0^+}^{l,\psi}\ell_2\}$.

According to the definition 2.5 and the proposition 2.6, it is necessary to rewrite the Cauchy problem in the equivalent integral equation

$$\chi(\iota) = \chi_0 + \frac{1}{\Gamma(l)} \int_0^\iota (\psi(\iota) - \psi(\kappa))^{l-1} \psi'(\kappa)(\mathfrak{A}\chi(\kappa) + \hbar(\kappa, \chi(\kappa), \Re\chi(\kappa)))d\kappa. \tag{3.1}$$

Lemma 3.1. If (3.1) holds, then we have

$$\chi(\iota) = \int_0^\infty \Phi_l(\varrho) V((\psi(\iota)-\psi(0))^l \varrho)\chi_0 d\varrho + l \int_0^\iota \int_0^\infty \varrho\Phi_l(\varrho)(\psi(\iota)-\psi(\kappa))^{l-1}\psi'(\kappa) V((\psi(\iota)-\psi(0))^l\varrho)\hbar(\kappa, \chi(\kappa), \Re\chi(\kappa)))d\varrho d\kappa, \tag{3.2}$$

where

$$\Phi_l(\varrho) = \frac{1}{l}\varrho^{-1-\frac{1}{l}}\omega_l\left(\varrho^{-\frac{1}{l}}\right) \ge 0, \quad \omega_l = \frac{1}{\pi}\sum_{p=1}^\infty \left((-1)^{p-1}\varrho^{-lp-1}\frac{\Gamma(pl+1)}{p!} sin(pl\pi)\right),\quad \varrho \in (0, \infty).$$

Proof . Let $\mu > 0$. Applying (2.3) and (2.4) to (3.1), we get

$$X(\mu) = \frac{\chi_0}{\mu} + \frac{1}{\mu^l}(\mathfrak{A}X(\mu) + H(\mu)),$$

where

$$X(\mu) = \int_0^\infty e^{-\mu(\psi(\kappa)-\psi(0))}\psi'(\kappa)\chi(\kappa)d\kappa \quad \text{and} \quad H(\mu) = \int_0^\infty e^{-\mu(\psi(\kappa)-\psi(0))}\psi'(\kappa)\hbar(\kappa, \chi(\kappa), \Re\chi(\kappa)))d\kappa.$$

This means that

$$X(\mu) = \mu^{l-1}(\mu^l I - \mathfrak{A})^{-1}\chi_0 + (\mu^l I - \mathfrak{A})^{-1}H(\mu) = \mu^{l-1}\int_0^\infty e^{-\mu^l\varsigma}V(\varsigma)\chi_0 d\varsigma + \int_0^\infty e^{-\mu^l\varsigma}V(\varsigma)H(\mu)d\varsigma.$$

By putting $\varsigma = \kappa^l$, $d\varsigma = l\kappa^{l-1}d\kappa$, we get

$$X(\mu) = l\int_0^\infty (\mu\kappa)^{l-1}e^{-(\mu\kappa)^l}V(\kappa^l)\chi_0 d\kappa + l\int_0^\infty \kappa^{l-1}e^{-(\mu\kappa)^l}V(\kappa^l)H(\mu)d\kappa,$$

next, we alter κ to $\psi(\iota)-\psi(0)$;

$$\begin{aligned}X(\mu)=&l\int_0^\infty \mu^{l-1}(\psi(\iota)-\psi(0))^{l-1}\psi'(\iota)e^{-(\mu(\psi(\iota)-\psi(0)))^l}V((\psi(\iota)-\psi(0))^l)\chi_0dt\\&+l\int_0^\infty(\psi(\iota)-\psi(0))^{l-1}\psi'(\iota)e^{-(\mu(\psi(\iota)-\psi(0)))^l}V((\psi(\iota)-\psi(0))^l)H(\mu)d\iota\\=&\int_0^\infty\frac{-1}{\mu}\frac{d}{d\iota}\left[e^{-(\mu(\psi(\iota)-\psi(0)))^l}\right]V((\psi(\iota)-\psi(0))^l)\chi_0d\iota\\&+\int_0^\infty\int_0^\infty l(\psi(\iota)-\psi(0))^{l-1}\psi'(\varsigma)\psi'(\iota)e^{-(\mu(\psi(\iota)-\psi(0)))^l}V((\psi(\iota)-\psi(0))^l)e^{-\mu(\psi(\iota)-\psi(0))}H(\mu)d\varsigma d\iota.\end{aligned}$$

Using the fact that $\int_0^\infty e^{-\mu\varrho}\omega_l(\varrho)d\varrho=e^{-\mu^l}$, where $l\in(0,\infty)$:

$$\begin{aligned}X(\mu)=&\int_0^\infty\int_0^\infty\varrho\omega_l(\varrho)e^{-\mu(\psi(\iota)-\psi(0))}\psi'(\iota)V((\psi(\iota)-\psi(0))^l)\chi_0d\varrho d\iota\\&+\int_0^\infty\int_0^\infty\int_0^\infty l(\psi(\iota)-\psi(0))^{l-1}\psi'(\varsigma)\psi'(\iota)\omega_l(\varrho)e^{-\mu(\psi(\iota)-\psi(0))\varrho}V((\psi(\iota)-\psi(0))^l)\times e^{-\mu(\psi(\iota)-\psi(0))}H(\mu)d\varrho d\varsigma d\iota\\=&\int_0^\infty e^{-\mu(\psi(\iota)-\psi(0))}\psi'(\iota)\left(\int_0^\infty\omega_l(\varrho)V\left(\frac{(\psi(\iota)-\psi(0))^l}{\varrho^l}\right)\chi_0d\varrho\right)d\iota\\&+\int_0^\infty\int_0^\infty\int_0^\infty l\psi'(\varsigma)\psi'(\iota)e^{-\mu(\psi(\iota)+\psi(\varsigma)-2\psi(0))}\frac{(\psi(\iota)-\psi(0))^{l-1}}{\varrho^l}\omega_l(\varrho)\times V\left(\frac{(\psi(\iota)-\psi(0))^l}{\varrho^l}\right)H(\mu)d\varrho d\varsigma d\iota\\=&\int_0^\infty e^{-\mu(\psi(\iota)-\psi(0))}\psi'(\iota)\left(\int_0^\infty\omega_l(\varrho)V\left(\frac{(\psi(\iota)-\psi(0))^l}{\varrho^l}\right)\chi_0d\varrho\right)d\iota\\&+\int_0^\infty\int_\iota^\infty\int_0^\infty l\psi'(\kappa)\psi'(\iota)e^{-\mu(\psi(\kappa)-\psi(0))}\frac{(\psi(\iota)-\psi(0))^{l-1}}{\varrho^l}\omega_l(\varrho)V\left(\frac{(\psi(\iota)-\psi(0))^l}{\varrho^l}\right)\\&\times\hbar\left(\psi^{-1}(\psi(\kappa)-\psi(\iota)+\psi(0)),\chi(\psi^{-1}(\psi(\kappa)-\psi(\iota)+\psi(0))),\Re\chi(\psi^{-1}(\psi(\kappa)-\psi(\iota)+\psi(0)))\right)d\varrho d\kappa d\iota\\=&\int_0^\infty e^{-\mu(\psi(\iota)-\psi(0))}\psi'(\iota)\left(\int_0^\infty\omega_l(\varrho)V\left(\frac{(\psi(\iota)-\psi(0))^l}{\varrho^l}\right)\chi_0d\varrho\right)d\iota+\int_0^\infty e^{-\mu(\psi(\kappa)-\psi(0))}\\&\times\left(\int_0^\kappa\int_0^\infty l\psi'(\varsigma)\frac{(\psi(\kappa)-\psi(\varsigma))^{l-1}}{\varrho^l}\omega_l(\varrho)V\left(\frac{(\psi(\kappa)-\psi(\varsigma))^l}{\varrho^l}\right)\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))d\varrho d\varsigma\right)\psi'(\kappa)d\kappa.\end{aligned}$$

By using the inverse Laplace transform, we can accomplish the following:

$$\begin{aligned}X(\mu)=&\int_0^\infty V\left(\frac{(\psi(\iota)-\psi(\varsigma))^l}{\varrho^l}\right)\omega_l(\varrho)\chi_0d\varrho\\&+l\int_0^\iota\int_0^\infty V\left(\frac{(\psi(\iota)-\psi(\varsigma))^l}{\varrho^l}\right)\times\frac{(\psi(\iota)-\psi(\varsigma))^{l-1}}{\varrho^l}\omega_l(\varrho)\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))\psi'(\varsigma)d\varrho d\varsigma\\=&\int_0^\infty V\left((\psi(\iota)-\psi(0))^l\varrho\right)\Phi_l(\varrho)\chi_0d\varrho\\&+l\int_0^\iota\int_0^\infty\varrho V\left((\psi(\iota)-\psi(0))^l\varrho\right)\times(\psi(\iota)-\psi(\varsigma))^{l-1}\Phi_l(\varrho)\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))\psi'(\varsigma)d\varrho d\varsigma.\end{aligned}$$

For $\chi\in\mathcal{X}$ and $0<l<1$, two families $\{X_\psi^l(\iota,\varsigma):0\le\varsigma\le\iota\le\nu\}$ and $\{Y_\psi^l(\iota,\varsigma):0\le\varsigma\le\iota\le\nu\}$ of operators are as follows:

$$X_\psi^l(\iota,\varsigma)\chi=\int_0^\infty\Phi_l(\varrho)V((\psi(\iota)-\psi(\varsigma))^l\varrho)\chi d\varrho:\mathcal{X}\times\mathcal{X}\to\mathcal{X}_\wp$$

and

$$Y_\psi^l(\iota,\varsigma)\chi=l\int_0^\infty\varrho\Phi_l(\varrho)V((\psi(\iota)-\psi(\varsigma))^l\varrho)\chi d\varrho:\mathcal{X}\times\mathcal{X}\to\mathcal{X}_\wp$$

respectively. □

Lemma 3.2. [49] The operators $X_\psi^l(\iota,\varsigma)$ and $Y_\psi^l(\iota,\varsigma)$ meet the following requirements:

1. for all $\iota \geq \varsigma \geq 0$ and $\chi \in \mathcal{X}_\wp$, the operators $X^l_\psi(\iota,\varsigma)$ and $Y^l_\psi(\iota,\varsigma)$ are bounded linear operators, i.e
$$\left\|X^l_\psi(\iota,\varsigma)\chi\right\|_\wp \leq M_V \left\|\chi\right\|_\wp \quad \text{and} \quad \left\|Y^l_\psi(\iota,\varsigma)\chi\right\|_\wp \leq \frac{M_V}{\Gamma(l)} \left\|\chi\right\|_\wp .$$
2. The operators $X^l_\psi(\iota,\varsigma)$ and $Y^l_\psi(\iota,\varsigma)$ are strongly continuous for all $\iota \geq \varsigma \geq 0$. That is, $\chi \in \mathcal{X}_\wp$ and for all $0 \geq \varsigma \geq \iota_1 \geq \iota_2$, we have the following:
$$\left\|X^l_\psi(\iota_2,\varsigma)\chi - X^l_\psi(\iota_1,\varsigma)\chi\right\|_\wp \to 0 \ \ \textit{and} \ \ \left\|Y^l_\psi(\iota_2,\varsigma)\chi - Y^l_\psi(\iota_1,\varsigma)\chi\right\|_\wp \to 0$$
as $\iota_2 \to \iota_1$.
3. The operators $X^l_\psi(\iota,\varsigma)$ and $Y^l_\psi(\iota,\varsigma)$ are compact operators, for all $\iota,\varsigma > 0$.
4. If $X^l_\psi(\iota,\varsigma)$ and $Y^l_\psi(\iota,\varsigma)$ are strongly continuous compact semigroups of linear bounded operators.
5. If $0 < l < 1$, then
$${}^C D^{l,\psi}_{0^+}\{X^l_\psi(\iota,0)\chi_0\} = \mathfrak{A}\{X^l_\psi(\iota,0)\chi_0\}$$
and
$${}^C D^{l,\psi}_{0^+}\{P^\iota_0\{w(\iota)\}\} + w(\iota) = \mathfrak{A}\{P^\iota_0\{w(\iota)\}\} + w(\iota),$$
where
$$P^\iota_0\{\chi(\iota)\} := \int_{\iota_1}^{\iota_2} (\psi(\iota_2) - \psi(\varsigma))^{l-1}\psi'(\varsigma) Y^l_\psi(\iota_2,\varsigma)\chi(\varsigma)d\varsigma,$$
such that $\iota_1, \iota_2 \in \Lambda$; $\chi, w \in \mathcal{C}_\wp$.

Definition 3.3. A solution $\chi(.;\chi_0;u) \in \mathcal{C}_\wp$ is called an $\wp$-mild solution of (1.1) if it satisfies:
$$\chi(t) = X^l_\psi(\iota,0)\chi_0 + \int_0^\iota (\psi(\iota) - \psi(\varsigma))^{l-1}\psi'(\varsigma) Y^l_\psi(\iota,\varsigma)\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))d\varsigma \tag{3.3}$$

4 Existence and unicity result

Theorem 4.1. Suppose that the conditions $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$ are met and
$$\frac{M_V \xi}{\Gamma(l)} C^{\gamma-\wp} < 1 \tag{4.1}$$
then the fractional semilinear equation (1.1) has a mild solution on Λ.

Proof . We define the function $H_\epsilon : \mathcal{C}_\wp \to \mathcal{C}_\wp$ such as
$$(H_\epsilon\chi)(\iota) = X^l_\psi(\iota,0)\chi_0 + \int_0^\iota (\psi(\iota) - \psi(\varsigma))^{l-1}\psi'(\varsigma) Y^l_\psi(\iota,\varsigma)\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))d\varsigma.$$

Define, $\mho_o = \{\chi \in \mathcal{C}_\wp : \ \chi(0) = \chi_0; \ \|\chi\|_\wp \leq o\}$. Then, $\mho_o$ is a closed, convex, and bounded subset of the Banach space $\mathcal{C}_\wp$, it is necessary to prove that, the operator $H_\epsilon : \mathcal{C}_\wp \to \mathcal{C}_\wp$ has a fixed point. We will now proceed step by step:

Step 1. For an arbitrary $\vartheta > 0$, there exists a positive constant $o = o(\vartheta)$ such that $H_\vartheta(\mho_{o(\vartheta)}) \subset \mho_{o(\vartheta)}$. Let $\vartheta > 0$, there exists $\chi \in \mho_o$, then for $\iota_o \in \Lambda$; such that $\left\|H_\vartheta(\mho_{o(\vartheta)})\right\|_\wp > o$.

Depending on Lemma 3.2 and condition $(\mathcal{C}_2)$, here's what we have:
$$\begin{aligned}
o < \|(H_\vartheta\chi)(\iota_o)\|_\wp &\leq \left\|X^l_\psi(\iota_o,0)\chi_0 + \int_0^{\iota_o} (\psi(\iota_o) - \psi(\varsigma))^{l-1}\psi'(\varsigma) Y^l_\psi(\iota_o,\varsigma)\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))d\varsigma\right\|_\wp \\
&\leq \left\|X^l_\psi(\iota_o,0)\chi_0\right\|_\wp + \left\|\int_0^{\iota_o} (\psi(\iota_o) - \psi(\varsigma))^{l-1}\psi'(\varsigma) Y^l_\psi(\iota_o,\varsigma)\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))d\varsigma\right\|_\wp \\
&\leq M_V \|\chi_0\|_\wp + \int_0^{\iota_o} (\psi(\iota_o) - \psi(\varsigma))^{l-1}\psi'(\varsigma) \left\|\mathfrak{A}^{\wp-\gamma} Y^l_\psi(\iota_o,\varsigma)\mathfrak{A}^{\gamma}\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))\right\| ds \\
&\leq M_V \|\chi_0\|_\wp + \frac{M_V C^{\gamma-\wp}}{\Gamma(l)} \int_0^{\iota_o} (\psi(\iota_o) - \psi(\varsigma))^{l-1}\psi'(\varsigma) I_o(\varsigma)d\varsigma \\
&\leq M_V \|\chi_0\|_\wp + \frac{M_V C^{\gamma-\wp}}{\Gamma(l)} \times \xi.
\end{aligned}$$

Dividing to both side by o and taking the lower limit as $o \to \infty$, we obtain

$$1 \leq \lim_{o\to\infty} \inf \frac{M_V \|\chi_0\|_\wp}{o} + \lim_{o\to\infty} \inf \frac{\frac{M_V C^{\gamma-\wp}}{\Gamma(l)} \times \xi}{o} = \frac{M_V \xi}{\Gamma(l)} C^{\gamma-\wp} < 1,$$

which leads us to a contradiction. Therefore, $H_\vartheta(\mho_{o(\vartheta)}) \subset \mho_{o(\vartheta)}$ for some $o(\vartheta) > 0$.

Step 2. We show that H_ϑ is continuous. Let (χ_n) be a sequence of $\mho_o$ so that $\chi_n \to \chi$ in $\mho_o$. The function $\hbar$ is continuous on $\Lambda \times \mathcal{X}_\wp \times \mathcal{X}_\wp$, then we can say that

$$\hbar(\varsigma, \chi_n(\varsigma), \Re\chi_n(\varsigma)) \longrightarrow \hbar(\varsigma, \chi(\varsigma), \Re\chi(\varsigma)), \quad as \quad n \to \infty.$$

We have now, for $\iota \in \Lambda$

$$\begin{aligned}\|(H_\vartheta\chi)(\iota) - (H_\vartheta\chi_n)(\iota)\|_\wp &= \int_0^\iota \psi'(\varsigma)(\psi(\iota) - \psi(\varsigma))^{l-1} \left\|\mathfrak{A}^{\wp-\gamma} Y_\psi^l(\iota,\varsigma)\mathfrak{A}^\gamma[\hbar(\varsigma, \chi_n(\varsigma), \Re\chi_n(\varsigma)) - \hbar(\varsigma, \chi(\varsigma), \Re\chi(\varsigma))]\right\| d\varsigma \\ &\leq \frac{M_V C^{\gamma-\wp}}{\Gamma(l)} \int_0^\iota \psi'(\varsigma)(\psi(\iota) - \psi(\varsigma))^{l-1} \|\hbar(\varsigma, \chi_n(\varsigma), \Re\chi_n(\varsigma)) - \hbar(\varsigma, \chi(\varsigma), \Re\chi(\varsigma))\|_\gamma \, d\varsigma\end{aligned}$$

for $\iota \in \Lambda$. Using the fact that

$$\|\hbar(\varsigma, \chi_n(\varsigma), \Re\chi_n(\varsigma)) - \hbar(\varsigma, \chi(\varsigma), \Re\chi(\varsigma))\|_\gamma \leq 2I_o(\varsigma) \text{ for } \varsigma \in \Lambda;$$

and for each $\iota \in \Lambda$ since $\hbar$ conforms to $(\mathcal{C}_2)$, using the Lebesgue Dominated Convergence Theorem ones proves that

$$\frac{M_V C^{\gamma-\alpha}}{\Gamma(l)} \int_0^\iota \psi'(\varsigma)(\psi(\iota) - \psi(\varsigma))^{l-1} \|\hbar(\varsigma, \chi_n(\varsigma), \Re\chi_n(\varsigma)) - \hbar(\varsigma, \chi(\varsigma), \Re\chi(\varsigma))\|_\gamma \, d\varsigma \longrightarrow 0.$$

Then we can say that

$$\lim_{\iota\to\infty} \|H_\vartheta\chi_n - H_\vartheta\chi\|_\wp = 0.$$

In other terms H_ϑ is continuous.

Step 3. To prove that H_ϑ is compact. we have to show that $\varphi(\iota) = \{(H_\vartheta\chi)(\iota) : \chi \in \mho_o\}$ is relatively compact in $\mathcal{X}_\wp$, for all $\iota \in \Lambda$. It goes without saying that $\{(H_\vartheta\chi)(0) : \chi \in \mho_o\}$ is compact. Let $\iota \in (0, \nu]$, $\forall g > 0$, we define the set $\varphi_\kappa(\iota) = \{(H_\vartheta^{\kappa,g}\chi)(\iota) : \chi \in \mho_o\}$:

$$\begin{aligned}(H_\vartheta^{\kappa,g}\chi)(\iota) = &\int_g^\infty \Phi_l(\varrho)V((\psi(\iota) - \psi(0))^l\varrho)\chi_0 d\varrho \\ &+ l\int_0^{\iota-g}\int_g^\infty \varrho\Phi_l(\varrho)(\psi(\iota) - \psi(\varsigma))^{l-1}\psi'(\varsigma)V((\psi(\iota) - \psi(0))^l\varrho)\hbar(\varsigma, \chi(\varsigma), \Re\chi(\varsigma))d\varrho d\varsigma \\ = &\int_g^\infty \Phi_l(\varrho)V((\psi(\iota) - \psi(0))^l\varrho - \kappa^l g + \kappa^l g)\chi_0 d\varrho \\ &+ l\int_0^{\iota-g}\int_g^\infty \varrho\Phi_l(\varrho)(\psi(\iota) - \psi(\varsigma))^{l-1}\psi'(\varsigma)V((\psi(\iota) - \psi(0))^l\varrho - \kappa^l g + \kappa^l g)\hbar(\varsigma, \chi(\varsigma), \Re\chi(\varsigma))d\varrho d\varsigma \\ = &\int_g^\infty \Phi_l(\varrho)[V(\kappa^l g)V((\psi(\iota) - \psi(0))^l\varrho - \kappa^l g)]\chi_0 d\varrho \\ &+ l\int_0^{\iota-g}\int_g^\infty \varrho\Phi_l(\varrho)(\psi(\iota) - \psi(\varsigma))^{l-1}\psi'(s)[V(\kappa^l g)V((\psi(\iota) - \psi(0))^l\varrho - \kappa^l g)]\hbar(\varsigma, \chi(\varsigma), \Re\chi(\varsigma))d\varrho ds \\ = &V(\kappa^l g)\int_g^\infty \Phi_l(\varrho)[V((\psi(\iota) - \psi(0))^l\varrho - \kappa^l g)]\chi_0 d\varrho \\ &+ lV(\kappa^l g)\int_0^{\iota-g}\int_g^\infty \varrho\Phi_l(\varrho)(\psi(\iota) - \psi(\varsigma))^{l-1}\psi'(\varsigma)[V((\psi(\iota) - \psi(0))^l\varrho - \kappa^l g)]\hbar(\varsigma, \chi(\varsigma), \Re\chi(\varsigma))d\varrho d\varsigma.\end{aligned}$$

Thus, by the compactnesss of $V(\kappa^l g)$ for $\kappa^l g > 0$ and the boundedness of $\int_g^\infty \Phi_l(\varrho)[V((\psi(\iota)-\psi(0))^l\varrho-\kappa^l g)]\chi_0 d\varrho + l\int_0^{\iota-g}\int_g^\infty \varrho\Phi_l(\varrho)(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)[V((\psi(\iota)-\psi(0))^l\varrho-\kappa^l g)] \times \hbar(\varsigma, \chi(\varsigma), \Re\chi(\varsigma))d\varrho d\varsigma$ on $\mho_o$, it is obvious that $\varphi_\kappa(\iota)$

is a relatively compact set in $\mathcal{X}_\wp$. In addition,

$$
\begin{aligned}
\|(H_\vartheta\chi)(\iota)-(H_\vartheta^{\kappa,g}\chi)(\iota)\|_\wp =& \left\|\int_0^g \Phi_l(\varrho)V((\psi(\iota)-\psi(0))^l\varrho)\chi_0 d\varrho\right\|_\wp \\
&+ l\left\|\int_0^{\iota-g}\int_g^\infty \varrho\Phi_l(\varrho)(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)V((\psi(\iota)-\psi(0))^l\varrho)\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))d\varrho d\varsigma\right\|_\wp \\
=& \left\|V((\psi(\iota)-\psi(0))^l\varrho)\chi_0\right\|_\wp \int_0^g \Phi_l(\varrho)d\varrho \\
&+ q\left\|\int_{\iota-\kappa}^t\int_g^\infty \varrho\Phi_l(\varrho)(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)V((\psi(\iota)-\psi(0))^l\varrho)\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))d\varrho d\varsigma\right\|_\wp \\
=& M_V\|\chi_0\|_\alpha \int_0^g \Phi_l(\varrho)d\varrho + lM_V(C_{\gamma-\wp}+\|I_o\|_{L^\infty})\int_{\iota-\kappa}^\iota (\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)\int_0^g \varrho\Phi_l(\varrho)d\varrho d\varsigma.
\end{aligned}
$$

In other words, we can tell that there are relatively compact sets arbitrarily close to the set $\varphi(\iota)$, $\forall\iota\in\Lambda$. Thus, $\varphi(\iota)$, is relatively compact in $\mathcal{X}_\wp$. Since it is compact at $\iota=0$. We get the relative compactness of $\varphi(\iota)$ in $\mathcal{X}_\wp$, $\forall\iota\in\Lambda$.
Step 4. For $0<\iota_2\le\iota_1\le\nu$; we have

$$
\begin{aligned}
&\|(H\chi)(\iota_2)-(H\chi)(\iota_1)\|_\wp \le \left\|X_\psi^l(\iota_2,0)\chi_0 - X_\psi^l(\iota_1,0)\chi_0\right\|_\wp + \left\|\int_{\iota_1}^{\iota_2}\psi'(\varsigma)(\psi(\iota_2)-\psi(\varsigma))^{l-1}Y_\psi^l(\iota_2,\varsigma)\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))d\varsigma\right\|_\wp \\
&+\left\|\int_0^{\iota_1}\psi'(\varsigma)[(\psi(\iota_2)-\psi(\varsigma))^{l-1}-(\psi(\iota_1)-\psi(\varsigma))^{l-1}]Y_\psi^l(\iota_2,\varsigma)\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))d\varsigma\right\|_\wp \\
&+\left\|\int_0^{\iota_1}\psi'(\varsigma)(\psi(\iota_2)-\psi(\varsigma))^{l-1}[Y_\psi^l(\iota_2,\varsigma)-Y_\psi^l(\iota_1,\varsigma)]\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))d\varsigma\right\|_\wp \\
&\le \left\|X_\psi^l(\iota_2,0)\chi_0\right\|_\wp + \left\|X_\psi^l(\iota_1,0)\chi_0\right\|_\wp + \int_{\iota_1}^{\iota_2}\psi'(\varsigma)(\psi(\iota_2)-\psi(\varsigma))^{l-1}\left\|Y_\psi^l(\iota_2,\varsigma)\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))d\varsigma\right\|_\wp \\
&+\int_0^{\iota_1}\psi'(\varsigma)[(\psi(\iota_2)-\psi(\varsigma))^{l-1}-(\psi(\iota_1)-\psi(\varsigma))^{l-1}]\left\|Y_\psi^l(\iota_2,\varsigma)\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))d\varsigma\right\|_\wp \\
&+\left\|\int_0^{\iota_1-\varepsilon}\psi'(\varsigma)(\psi(\iota_2)-\psi(\varsigma))^{l-1}[Y_\psi^l(\iota_2,\varsigma)-Y_\psi^l(\iota_1,\varsigma)]\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))d\varsigma\right\|_\wp \\
&+\left\|\int_{\iota_1-\varepsilon}^{\iota_1}\psi'(\varsigma)(\psi(\iota_2)-\psi(\varsigma))^{l-1}[Y_\psi^l(\iota_2,\varsigma)-Y_\psi^l(\iota_1,\varsigma)]\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))d\varsigma\right\|_\wp \\
&\le M_V\|\chi_0\|_\wp + M_V\|\chi_0\|_\wp + \frac{(\psi(\iota_2)-\psi(\iota_1))^l}{l}\left\|\mathfrak{A}^{\wp-\gamma}Y_\psi^l(\iota_2,\varsigma)\mathfrak{A}^\gamma\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))d\varsigma\right\| \\
&+\frac{1}{l}[(\psi(\iota_2)-\psi(\iota_1))^l+(\psi(\iota_1)-\psi(0))^l-(\psi(\iota_2)-\psi(0))^l]\left\|\mathfrak{A}^{\wp-\gamma}Y_\psi^l(\iota_2,\varsigma)\mathfrak{A}^\gamma\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))d\varsigma\right\| \\
&+\frac{1}{l}[(\psi(\iota_1)-\psi(0))^l+(\psi(\iota_1)-\psi(\iota_1-\varepsilon))^l]\left\|\mathfrak{A}^{\wp-\gamma}[Y_\psi^l(\iota_2,\varsigma)-Y_\psi^l(\iota_1,\varsigma)]\mathfrak{A}^\gamma\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))d\varsigma\right\| \\
&+\frac{1}{l}(\psi(\iota_1)-\psi(\iota_1-\varepsilon))^l\left\|\mathfrak{A}^{\wp-\gamma}[Y_\psi^l(\iota_2,\varsigma)-Y_\psi^l(\iota_1,\varsigma)]\mathfrak{A}^\gamma\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))d\varsigma\right\| \\
&\le 2M_V\|\chi_0\|_\wp + \frac{(\psi(\iota_2)-\psi(\iota_1))^l M_V C_{\gamma-\wp}}{\Gamma(l+1)}\|I\|_{L^\infty} + \frac{M_V C_{\gamma-\wp}}{\Gamma(l+1)}[(\psi(\iota_2)-\psi(\iota_1))^l+(\psi(\iota_1)-\psi(0))^l-(\psi(\iota_2)-\psi(0))^l]\|I\|_{L^\infty} \\
&+\frac{M_V C_{\gamma-\wp}}{\Gamma(l+1)}[(\psi(\iota_1)-\psi(0))^l+(\psi(\iota_1)-\psi(\iota_1-\varepsilon))^l]\sup_{\varsigma\in[0,\iota_1-\varepsilon]}\left\|Y_\psi^l(\iota_2,\varsigma)-Y_\psi^l(\iota_1,\varsigma)\right\|\|I\|_{L^\infty} \\
&+2(\psi(\iota_1)-\psi(\iota_1-\varepsilon))^l\frac{M_V C_{\gamma-\wp}}{\Gamma(l+1)}\|I\|_{L^\infty}.
\end{aligned}
$$

By the condition $(\mathcal{C}_1)$ and Lemma 3.2 we have proven that $\|(H\chi)(\iota_2)-(H\chi)(\iota_1)\|_\wp\to 0$ as $\iota_2\to\iota_1$. It is obvious to say that $\{H\chi),\chi\in\mho_o\}$ is a family of equicontinuous functions. By the Arzela-Ascoli Theorem, since $H(\mho_o)$ equicontinue, It is simple to infer that $H(\mho_o)$ is relatively compact in $\mathcal{C}_\wp$.

Furthermore, it is simple to determine that, H is continuous in $\mathcal{C}_\wp$, which means it is completely continuous on $\mathcal{C}_\wp$. Of course, this means that by Schauder's fixed point theorem H has a fixed point $\chi\in\mho_o$. This is the necessary confirmation to state that (1.1) has a mild solution on Λ. □

Remark 4.2. Using theorem 4.1, it is noted that if ψ is a bijection function, then (1.1) has a minimum of one mild solution, given that

$$\nu < \psi^{-1}\left(\left(\frac{\Gamma(l+1)}{M_V C_{\gamma-\wp}}\right)^{\frac{1}{l}} + \psi(0)\right).$$

Theorem 4.3. Assume the condition $(\mathcal{C}_3)$ holds. Then the problem (1.1) has a unique mild solution.

Proof . Let χ_1 and χ_2 be the solutions of the problem (1.1) in $\mho_o$. Then for each $w \in \{1, 2\}$, the solution χ_w satisfies

$$(H\chi_w)(\iota) = X^l_\psi(\iota, 0)\chi_0 + \int_0^\iota (\psi(\iota) - \psi(\varsigma))^{l-1}\psi'(\varsigma)Y^l_\psi(\iota, \varsigma)\hbar(\varsigma, \chi_w(\varsigma), \Re\chi_w(\varsigma))d\varsigma.$$

Then, for any $\iota \in \Lambda$, we have

$$\begin{aligned}
\|(H\chi_1)(\iota) - (H\chi_2)(\iota)\|_\wp \leq& \int_0^\iota (\psi(\iota) - \psi(\varsigma))^{l-1}\psi'(\varsigma) \left\|Y^l_\psi(\iota, \varsigma)[\hbar(\iota, \chi_1(\iota), \Re\chi_1(\iota)) - \hbar(\iota, \chi_2(\iota), \Re\chi_2(\iota))]\right\|_\wp d\varsigma \\
\leq& \frac{M_V}{\Gamma(l)} \int_0^\iota (\psi(\iota) - \psi(\varsigma))^{l-1}\psi'(\varsigma) \|\hbar(\iota, \chi_1(\iota), \Re\chi_1(\iota)) - \hbar(\iota, \chi_2(\iota), \Re\chi_2(\iota))\|_\wp d\varsigma \\
\leq& \frac{M_V}{\Gamma(l)} \int_0^\iota (\psi(\iota) - \psi(\varsigma))^{l-1}\psi'(\varsigma)[\ell_1(\varsigma) \|\chi_1(\varsigma) - \chi_2(\varsigma)\|_\wp + \ell_2(\varsigma) \|\Re\chi_1(\varsigma) - \Re\chi_2(\varsigma)\|_\wp]d\varsigma \\
\leq& M_V I^{l,\psi}_{0^+}\ell_1(\iota) \|\chi_1 - \chi_2\|_\wp + M_V j^* I^{l,\psi}_{0^+}\ell_2(\iota) \|\chi_1 - \chi_2\|_\wp \\
&\leq M_V I^{l,\psi}_{0^+}\ell(\iota) \|\chi_1 - \chi_2\|_\wp (1 + j^*) \\
\leq& \frac{M_V(1 + j^*)}{\Gamma(l)} \int_0^\iota (\psi(\iota) - \psi(\varsigma))^{l-1}\psi'(\varsigma)\ell(\varsigma) \|\chi_1(\varsigma) - \chi_2(\varsigma)\|_\wp d\varsigma. \\
\leq& \frac{\ell^* M_V(1 + j^*)}{\Gamma(l)} \int_0^\iota (\psi(\iota) - \psi(\varsigma))^{l-1}\psi'(\varsigma) \|\chi_1(\varsigma) - \chi_2(\varsigma)\|_\wp d\varsigma,
\end{aligned}$$

where $\ell^* = \sup_{\iota\in\Lambda} |\ell(\iota)|$. We get the wanted result by using the theorem 2.10: $\chi_1 \equiv \chi_2$. Hence there is only one unique solution for (1.1). □

Definition 4.4. [23] A function $\hbar$ satisfies the local Lipshitz condition in $\chi(\iota)$, uniformly in ι on bounded intervals if for every $\iota' \geq 0$ and $\kappa \geq 0$ there is a constant $\ell(cst, \iota')$ such that

$$\|\hbar(\iota, \chi_1) - \hbar(\iota, \chi_2)\|_\wp \leq \ell(cst, \iota') \|\chi_1 - \chi_2\|_\wp,$$

for all $\chi_1, \chi_2 \in \mathcal{X}_\wp$, and $\iota \in \Lambda$.

Theorem 4.5. Let $\hbar : \Lambda \times \mathcal{X}_\wp \times \mathcal{X}_\wp \to \mathcal{X}$ is continuous in ι for $\iota \geq 0$ and locally Lipschitz continuous in $\mathcal{X}_\wp$, uniformly in ι on bounded intervals. Then for every $\chi_0 \in \mathcal{X}_\wp$ there is a $\iota_{\max} < \infty$ such that

$$\begin{cases} {}^C D^{l,\psi}_{0^+}\chi(\iota) = -\mathfrak{A}\chi(\iota) + \hbar(\iota, \chi(\iota), \Re\chi(\iota)), \quad \iota > \iota_0, \\ \chi(\iota_0) = \chi_0 \end{cases}$$

has a unique mild solution χ on $[0, \iota_{\max})$.

Proof . We begin by demonstrating that for each $\iota > 0$, $\chi_0 \in \mathcal{X}_\wp$, the problem from above has a unique mild solution under the presumptions of our theorem, $\chi(\iota)$ on an interval $[\iota_0, \iota_1]$ whose length is restricted beneath by

$$\beta(\iota_0, \|\chi_0\|_\wp) = \min\{1, \frac{\|\chi_0\|_\wp \Gamma(l+1)}{j(\iota_0)\ell(j(\iota_0), \iota_0 + 1) + n(\iota_0)}\},$$

where $\ell(cst, \iota)$ is the local Lipschitz constant of $\hbar$ and $j(\iota_0) = 2 \|\chi_0\|_\wp M_V(\iota_0)$. Indeed let $\iota_1 = \iota_0 + \beta(\iota_0, \|\chi_0\|_\wp)$, the mapping $H : \mathcal{C}_\wp \to \mathcal{C}_\wp$ by

$$(H\chi)(\iota) = X^l_\psi(\iota, \iota_0)\chi_0 + \int_{\iota_0}^\iota (\psi(\iota) - \psi(\varsigma))^{l-1}\psi'(\varsigma)Y^l_\psi(\iota, \varsigma)\hbar(\varsigma, \chi(\varsigma), \Re\chi(\varsigma))d\varsigma.$$

maps the sphere of radius $j(\iota_0)$ at 0 of $C([\iota_0,\iota_1],\mathcal{X}_\wp)$, into itself, then

$$\begin{aligned}
\|(H\chi)(\iota)\|_\wp \leq& \left\|X^l_\psi(\iota,\iota_0)\chi_0\right\|_\wp + \left\|\int_{\iota_0}^{\iota}(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)Y^l_\psi(\iota,\varsigma)\hbar(\iota,\chi_(\iota),\Re\chi(\iota))\right\|_\wp d\varsigma\\
\leq& M_V(\iota_0)\left\|\chi_0\right\|_\wp + \frac{M_V(\iota_0)}{\Gamma(l)}\int_{\iota_0}^{\iota}(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)\left\|\hbar(\iota,\chi_(\iota),\Re\chi(\iota))\right\|_\wp d\varsigma\\
\leq& M_V(\iota_0)\left\|\chi_0\right\|_\wp + \frac{M_V(\iota_0)}{\Gamma(l)}\int_{\iota_0}^{\iota}(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)\left[\left\|\hbar(\iota,\chi_(\iota),\Re\chi(\iota))-\hbar(\varsigma,0,0)\right\|_\wp + \|\hbar(\varsigma,0,0)|_\wp\right]d\varsigma\\
\leq& M_V(\iota_0)\left\|\chi_0\right\|_\wp\\
&+\frac{M_V(\iota_0)}{\Gamma(l)}\int_{\iota_0}^{\iota}(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)[\Big(\ell_1(cst_1,\iota')\left\|\chi(\varsigma)\right\|_\wp + \ell_2(cst_2,\iota')\left\|\Re\chi_(\varsigma)\right\|_\wp\Big) + \|\hbar(\varsigma,0,0)\|_\wp]d\varsigma\\
\leq& M_V(\iota_0)\left\|\chi_0\right\|_\wp + \frac{M_V(\iota_0)}{\Gamma(l)}\int_{\iota_0}^{\iota}(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)\left[\Big(\ell_1\left\|\chi_(\varsigma)\right\|_\wp + \ell_2\left\|\chi_(\varsigma)\right\|_\wp j^*\Big) + \|\hbar(\varsigma,0,0)\|_\wp\right]d\varsigma\\
\leq& M_V(\iota_0)\left\|\chi_0\right\|_\wp + \frac{M_V(\iota_0)(\psi(\iota)-\psi(\iota_0))^l}{\Gamma(l+1)}\left(\ell_1(j(\iota_0),\iota_0+1)j(\iota_0)+\ell_2(j(\iota_0),\iota_0+1)j(\iota_0)j^*\right)\\
&+\frac{M_V(\iota_0)}{\Gamma(l)}\int_{\iota_0}^{\iota}(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)\left\|\hbar(\varsigma,0,0)\right\|_\wp d\varsigma
\end{aligned}$$

and this is all due to the fact that $j(\iota_0)=2M_V(\iota_0)\left\|\chi_0\right\|_\wp$. Now, we put $n(\iota_0)=\max\{\|\hbar(\varsigma,0,0)\|_\wp : 0\leq\iota_0\leq\iota_0+1\}$,

$$\begin{aligned}
\|(H\chi)(\iota)\|_\wp \leq& M_V(\iota_0)\left\|\chi_0\right\|_\wp + M_V(\iota_0)j(\iota_0)\frac{\ell(j(\iota_0),\iota_0+1)(\psi(\iota)-\psi(\iota_0))^l}{\Gamma(l+1)} + \frac{M_V(\iota_0)}{\Gamma(l)}\int_{\iota_0}^{\iota}(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)n(\iota_0)d\varsigma\\
\leq& M_V(\iota_0)\left\|\chi_0\right\|_\wp + M_V(\iota_0)j(\iota_0)\frac{\ell(j(\iota_0),\iota_0+1)(\psi(\iota)-\psi(\iota_0))^l}{\Gamma(l+1)} + \frac{M_V(\iota_0)n(\iota_0)}{\Gamma(l)}(\psi(\iota)-\psi(\iota_0))^l\\
\leq& M_V(\iota_0)\left(\left\|\chi_0\right\|_\wp + \frac{(\psi(\iota)-\psi(\iota_0))^l}{\Gamma(l+1)}\left[j(\iota_0)\ell(j(\iota_0),\iota_0+1)+n(\iota_0)\right]\right)\\
\leq& 2M_V(\iota_0)n(\iota_0)=j(\iota_0)
\end{aligned}$$

whereby the last inequality is derived from the meaning of ι_1, in this sphere, H fulfills a uniform Lipschitz condition with constant $\ell=\ell(j(\iota_0),\iota_0+1)$ and hence, just like in Theorem's 4.1 proof, it has a single mild solution $\chi(\iota)$ in the sphere. The intended resolution of the mild solution is for our last problem on the interval $[\iota_1,\iota_2]$.

As a result of what we've just demonstrated if $\chi(\iota)$ is a mild solution of

$$\begin{cases}{}^CD^{l,\psi}_{0^+}\chi(\iota)=-\mathfrak{A}\chi(\iota)+\hbar(\iota,\chi(\iota),\Re\chi(\iota)), & \iota>\iota_0,\\ \chi(\iota_0)=\chi_0\end{cases}$$

on the interval $[0,\kappa]$. It can be extended to the interval $[0,\kappa+\beta]$ with $\beta>0$ by defining on $[\kappa,\kappa+\beta]$, $\chi(\iota)=\alpha(\iota)$ where $\alpha(\iota)$ is the solution of the integral equation

$$(H\chi)(\iota)=X^l_\psi(\kappa,\iota)\chi_0+\int_\kappa^\iota(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)Y^l_\psi(\iota,\varsigma)\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))d\varsigma,\quad \kappa\leq\iota\leq\kappa+\beta.$$

Additionally, β depends solely on $\|\chi(\kappa)\|$, $j(\kappa)$ and $n(\kappa)$. $[0,\iota_{\max}[$ should represent the maximal interval of existence of mild solution $\chi(\iota)$ of (1.1). Provided that $\iota_{\max}$ then $\lim\limits_{\iota\to\iota_{\max}}\|\chi(\iota)\|=\infty$, as there would be a sequence otherwise ι_w converge to $\iota_{\max}$ such that $\|\chi(\iota)\|\leq cst$ for all w.

This would suggest based on our recent proof that for each ιw close enough to $\iota_{\max}$, $\chi(\iota)$ specified on $[0,\iota_w]$ can reach $[0,\iota_w+\beta]$ in which $\beta>0$ is not dependent upon ι_w, therefore, it is possible to expand $\chi(\iota)$ beyond $\iota_{\max}$ conflict with the meaning of $\iota_{\max}$.

To illustrate the local mild solution's uniqueness $\chi(\iota)$ of (1.1) as we observe, if $\chi_1(\iota)$ is a mild solution of (1.1) thereafter, at each closed interval $[0,\iota_0]$ where $\chi(\iota)$ and $\chi_1(\iota)$ both exist and coincide according to the uniqueness argument provided after the Theorem 4.3's proof. Consequently, $\iota_{\max}$ is the same for both $\chi(\iota)$ and $\chi_1(\iota)$ and on $[0,\iota_{\max}[$: $\chi\equiv\chi_1$. □

5 Mittag-Leffler-Ulam-Hyers stability

For $\hbar : \Lambda \times \mathcal{X}_\wp \times \mathcal{X}_\wp \to \mathcal{X}$, $\varpi \in C(\Lambda, \mathbb{R}^+)$ and $\epsilon > 0$ we consider the equation

$$^{C}D_{0^+}^{l,\psi}\chi(\iota) = \mathfrak{A}\chi(\iota) + \hbar(\iota, \chi(\iota), \Re\chi(\iota)), \quad \iota \in \Lambda, \tag{5.1}$$

as well as the subsequent inequalities

$$\left| ^{C}D_{0^+}^{l,\psi}\chi(\iota) - \mathfrak{A}\chi(\iota) - \hbar(\iota, \chi(\iota), \Re\chi(\iota)) \right| \leq \epsilon, \;\; \iota \in \Lambda, \tag{5.2}$$

$$\left| ^{C}D_{0^+}^{l,\psi}\chi(\iota) - \mathfrak{A}\chi(\iota) - \hbar(\iota, \chi(\iota), \Re\chi(\iota)) \right| \leq \varpi(\iota), \;\; \iota \in \Lambda, \tag{5.3}$$

$$\left| ^{C}D_{0^+}^{l,\psi}\chi(\iota) - \mathfrak{A}\chi(\iota) - \hbar(\iota, \chi(\iota), \Re\chi(\iota)) \right| \leq \epsilon\varpi(\iota), \;\; \iota \in \Lambda. \tag{5.4}$$

Definition 5.1. [49] If there exists a real number $C > 0$ such that for each $\epsilon > 0$ and for each solution $\zeta \in C^1(\Lambda, \mathcal{X}_\wp)$ of inequality (5.2) there exists a mild solution $\chi \in \mathcal{C}_\wp$ of (5.1) with

$$|\zeta(\iota) - \chi(\iota)| \leq \epsilon C E_l(\iota), \quad \iota \in \Lambda,$$

we can say that (5.1) is M-L–U–H stable, concerning E_l

Definition 5.2. [49] If there exists a function $\vartheta \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\vartheta(0) = 0$, such that for each $\epsilon > 0$ and for each solution $\zeta \in C^1(\Lambda, \mathcal{X}_\wp)$ of inequality (5.2) there exists a mild solution $\chi \in \mathcal{C}_\wp$ of (5.1) with

$$|\zeta(\iota) - \chi(\iota)| \leq C\vartheta(\epsilon) E_l(\iota), \quad \iota \in \Lambda,$$

we can say that (5.1) is M-L–U–H stable, concerning E_l

Remark 5.3. Definition 5.1 implies definition 5.2.

Remark 5.4. A function $\chi \in C^1(\Lambda, \mathcal{X}_\wp)$ is a solution (5.2) if and only if there exists a function $\lambda \in C^1(\Lambda, \mathcal{X}_\wp)$ such that

1. $|\lambda(\iota)| \leq \epsilon$ for $\iota \in \Lambda$.
2. $^{C}D_{0^+}^{l,\psi}\chi(\iota) = \mathfrak{A}\chi(\iota) + \hbar(\iota, \chi(\iota), \Re\chi(\iota)) + \lambda(\iota)$, $\iota \in \Lambda$.

Remark 5.5. If $\zeta \in C^1(\Lambda, \mathcal{X}_\wp)$ is a solution (5.2), then ζ a solution of the following integral inequality

$$\left| \zeta(\iota) - X_\psi^l(\iota, 0)\zeta(0) - \int_0^\iota (\psi(\iota) - \psi(\varsigma))^{l-1}\psi'(\varsigma) Y_\psi^l(\iota, \varsigma)\hbar(\varsigma, \zeta(\varsigma), \Re\zeta(\varsigma))d\varsigma \right| \leq \epsilon \int_0^\iota (\psi(\iota) - \psi(\varsigma))^{l-1}\psi'(\varsigma) \left\| Y_\psi^l(\iota, \varsigma) \right\|_\wp d\varsigma.$$

Theorem 5.6. Assume that $\hbar : \Lambda \times \mathcal{X}_\wp \times \mathcal{X}_\wp \to \mathcal{X}$ and there exists $\ell_h > 0$ such that

$$|\hbar(\iota, \chi_1) - \hbar(\iota, \chi_2)| < \ell_h |\chi_1 - \chi_2|,$$

for all $\iota \in \Lambda$, and $\chi_1, \chi_2 \in \mathcal{X}$. Then (5.1) is M-L-U-H stable.

Proof . Let $\zeta \in C^1(\Lambda, \mathcal{X}_\wp)$ be a solution of (5.2). Let us denote by $\chi \in \mathcal{C}_\wp$ the unique solution of the semilinear problem

$$\begin{cases} ^{C}D_{0^+}^{l,\psi}\chi(\iota) = -\mathfrak{A}\chi(\iota) + \hbar(\iota, \chi(\iota), \Re\chi(\iota)), \iota \in \Lambda, \\ \chi(0) = \zeta(0) \end{cases} \tag{5.5}$$

We have

$$\chi(\iota) = X_\psi^l(\iota, 0)\zeta(0) + \int_0^\iota (\psi(\iota) - \psi(\varsigma))^{l-1}\psi'(\varsigma) Y_\psi^l(\iota, \varsigma)\hbar(\varsigma, \zeta(\varsigma), \Re\zeta(\varsigma))d\varsigma, \;\; \iota \in \Lambda.$$

Then we get

$$\begin{aligned}\left|\zeta(\iota)-X^l_\psi(\iota,0)\zeta(0)-\int_0^\iota(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)Y^l_\psi(\iota,\varsigma)\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))d\varsigma\right| \leq&\epsilon\int_0^\iota(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)\left\|Y^l_\psi(\iota,\varsigma)\right\|_\wp d\varsigma\\ \leq&\frac{M_V}{\Gamma(l)}\epsilon\int_0^\iota(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)d\varsigma\\ \leq&\frac{M_V}{\Gamma(l+1)}\epsilon(\psi(\iota)-\psi(0))^l.\end{aligned}$$

It follows that

$$\begin{aligned}|\zeta(\iota)-\chi(\iota)|\leq&\left|\zeta(\iota)-X^l_\psi(\iota,0)\zeta(0)-\int_0^\iota(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)Y^l_\psi(\iota,\varsigma)\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))d\varsigma\right|\\ \leq&\left|\zeta(\iota)-X^l_\psi(\iota,0)\zeta(0)-\int_0^\iota(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)Y^l_\psi(\iota,\varsigma)\hbar(\varsigma,\zeta(\varsigma),\Re\zeta(\varsigma))d\varsigma\right|\\ &+\int_0^\iota(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)\left\|Y^l_\psi(\iota,\varsigma)\right\|_\wp|\hbar(\varsigma,\zeta(\varsigma),\Re\zeta(\varsigma))-\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))|d\varsigma\\ \leq&\frac{M_V}{\Gamma(l+1)}\epsilon(\psi(\iota)-\psi(0))^l+\frac{M_V}{\Gamma(l)}\int_0^\iota(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)(\ell_{h1}|\zeta(\varsigma)-\chi(\varsigma)|+\ell_{h2}|\Re\zeta(\varsigma)-\Re\chi(\varsigma)|)d\varsigma\\ \leq&\frac{M_V}{\Gamma(l+1)}\epsilon(\psi(\iota)-\psi(0))^l+\frac{M_V}{\Gamma(l)}\int_0^\iota(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)(\ell_{h1}|\zeta(\varsigma)-\chi(\varsigma)|+\ell_{h2}|\zeta(\varsigma)-\chi(\varsigma)|j^*)d\varsigma\\ \leq&\frac{M_V}{\Gamma(l+1)}\epsilon(\psi(\iota)-\psi(0))^l+\frac{M_V}{\Gamma(l)}(1+j^*)\ell_h\int_0^\iota(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)|\zeta(\varsigma)-\chi(\varsigma)|d\varsigma,\end{aligned}$$

where $\ell_h=\sup_{\iota\in\Lambda}\{\ell_{h1},\ell_{h2}\}$. By corollary 2.11, we obtain

$$|\zeta(\iota)-\chi(\iota)|\leq\frac{M_V}{\Gamma(l+1)}\epsilon(\psi(\nu)-\psi(0))^l+E_l(M_V\ell_h(1+j^*)(\psi(\iota)-\psi(0))^l).$$

□

6 Mittag-Leffler-Ulam-Hyers-Rassias stability

Definition 6.1. [49] If there exists a real number $C_\varpi>0$, such that for each $\epsilon>0$ and for each solution $\zeta\in C^1(\Lambda,\mathcal{X}_\wp)$ of inequality (5.4) there exists a mild solution $\chi\in\mathcal{C}_\wp$ of (5.1) with

$$|\zeta(\iota)-\chi(\iota)|\leq\epsilon C_\varpi\varpi(\iota)E_l(\iota),\quad \iota\in\Lambda,$$

we can say that (5.1) is M-L–U–H-R stable, concerning ϖE_l

Definition 6.2. [49] If there exists a real number $C_\varpi>0$, such that for each solution $\zeta\in C^1(\Lambda,\mathcal{X}_\wp)$ of inequality (5.3) there exists a mild solution $\chi\in\mathcal{C}_\wp$ of (5.1) with

$$|\zeta(\iota)-\chi(\iota)|\leq C_\varpi\varpi(\iota)E_l(\iota),\quad \iota\in\Lambda,$$

we can say that (5.1) is M-L–U–H-R stable, concerning ϖE_l

Remark 6.3. Definition 6.1 implies definition 6.2.

Theorem 6.4. Suppose that the subsequent is true.

a) $\hbar:\Lambda\times\mathcal{X}_\wp\times\mathcal{X}_\wp\to\mathcal{X}$
b) $\ell_1(\iota)$ and $\ell_2(\iota)$ are nonnegative, non decreasing continuous functions defined on $\iota\in[0,\infty)$

$$|\hbar(\iota,\chi_2(\iota),\Re\chi_2(\iota))-\hbar(\iota,\chi_1(\iota),\Re\chi_1(\iota))|<\ell_1(\iota)|\chi_1-\chi_2|+\ell_2(\iota)|\Re\chi_1-\Re\chi_2|,$$

for all $\iota\in\Lambda$, and $\chi_1,\chi_2\in\mathcal{X}$.

c) The function $\varpi \in C([0,\infty),\mathbb{R}^+)$ is increasing and there exists $\alpha > 0$ such that

$$\int_0^\iota (\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)\left\|Y_\psi^l(\iota,\varsigma)\right\|_\wp d\varsigma \leq \alpha\varpi(\iota),$$

Then (5.1) is M-L-U-H-R stable with respect to ϖE_l.

Proof . Let $\zeta \in C^1(\Lambda,\infty)$ be a solution of (5.3). Then we get

$$\begin{aligned}\left|\zeta(\iota)-X_\psi^l(\iota,0)\zeta(0)-\int_0^\iota(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)Y_\psi^l(\iota,\varsigma)\hbar(\varsigma,\zeta(\varsigma),\Re\zeta(\varsigma))d\varsigma\right| \leq& \int_0^\iota(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)\varpi(\varsigma)\left\|Y_\psi^l(\iota,\varsigma)\right\|_\wp d\varsigma\\ \leq&\alpha\varpi(\iota), \quad \iota\in(0,\infty).\end{aligned}$$

Let $\chi \in C(\Lambda,\infty)$ the unique mild solution of the semilinear problem

$$\begin{cases} {}^C D_{0^+}^{l,\psi}\chi(\iota) = -\mathfrak{A}\chi(\iota)+\hbar(\iota,\chi(\iota),\Re\chi(\iota)), \iota\in(0,\infty),\\ \chi(0)=\zeta(0)\end{cases} \tag{6.1}$$

We have

$$\chi(\iota) = X_\psi^l(\iota,0)\zeta(0)+\int_0^\iota(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)Y_\psi^l(\iota,\varsigma)\hbar(\varsigma,\zeta(\varsigma),\Re\zeta(\varsigma))d\varsigma, \quad \iota\in(0,\infty).$$

It follows that

$$\begin{aligned}|\zeta(\iota)-\chi(\iota)| \leq& \left|\zeta(\iota)-\int_0^\iota(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)Y_\psi^l(\iota,\varsigma)\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))d\varsigma\right|\\ &+\int_0^\iota(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)\left\|Y_\psi^l(\iota,\varsigma)\right\|_\wp|\hbar(\varsigma,\zeta(\varsigma),\Re\zeta(\varsigma))-\hbar(\varsigma,\chi(\varsigma),\Re\chi(\varsigma))|d\varsigma\\ \leq&\alpha\varpi(\iota)+\frac{M_V}{\Gamma(l)}\int_0^\iota(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)[\ell_1(\iota)|\zeta(\varsigma)-\chi(\varsigma)|+\ell_2(\iota)|\Re\zeta(\varsigma)-\Re\chi(\varsigma)|]d\varsigma\\ \leq&\alpha\varpi(\iota)+\frac{M_V}{\Gamma(l)}\int_0^\iota(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)[\ell_1(\iota)|\zeta(\varsigma)-\chi(\varsigma)|+\ell_2(\iota)|\zeta(\varsigma)-\chi(\varsigma)|j^*]d\varsigma\\ \leq&\alpha\varpi(\iota)+\frac{M_V\ell(\iota)(1+j^*)}{\Gamma(l)}\int_0^\iota(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)|\zeta(\varsigma)-\chi(\varsigma)|d\varsigma,\end{aligned}$$

where $\ell(\iota)=\sup\limits_{\iota\in\Lambda}(\ell_1(\iota),\ell_2(\iota))$. By Corollary 2.11, we obtain

$$|\zeta(\iota)-\chi(\iota)|\leq\alpha\varpi(\iota)E_l(M_V\ell(\iota)(1+j^*)(\psi(\iota)-\psi(0))^l).$$

□

7 Example

The example that follows is examined in the final section to back up the theorem 4.1's result. Examine the following equation for a fractional partial differential:

$$\begin{cases} {}^C D_{0^+}^{\frac{1}{2},\iota}\chi(\iota,y) = \dfrac{\partial^2}{\partial y^2}\chi(\iota,y)+\dfrac{e^{-\iota}}{9+e^\iota}\cos\left(\chi(\iota,y)+\displaystyle\int_0^\iota\cos(\iota\varsigma)\chi(\varsigma,y)d\varsigma\right),\\ \chi(\iota,0)=\chi(\iota,\pi), \quad \iota\in\Lambda\\ \chi(0,y)=\chi_0(y), \quad y\in[0,\pi]\end{cases} \tag{7.1}$$

where $l=\frac{1}{2}$, $\mathfrak{A}\chi=\dfrac{\partial^2}{\partial y^2}\chi(\iota,y)$, $\chi(\iota)=\chi(\iota,y)$,

$$\hbar(\iota,\chi(\iota),\Re\chi(\iota))=\frac{e^{-\iota}}{9+e^\iota}\cos\left(\chi(\iota,y)+\int_0^\iota\cos(\iota\varsigma)\chi(\varsigma,y)d\varsigma\right) \quad \text{and} \quad \psi(\iota)=\iota.$$

Let $\mathcal{X}$ be defined as $\mathcal{X} = L^2[0,\pi]$ and $\mathfrak{A}$ by $\mathfrak{A}\upsilon = -\upsilon^{''}$ on the domain

$$\mathcal{D}(\mathfrak{A}) = \{\upsilon(.) \in L^2[0,\pi],\ \upsilon,\ \upsilon^{'},\ \text{are absolutely continuous},\ \upsilon^{''} \in L^2[0,\pi],\ \upsilon(0) = \upsilon(\pi) = 0\}.$$

It is noticeable that $\mathfrak{A}$ has a discrete spectrum and the eigenvalues are $\{-n^2 :\ n \in \mathbb{N}$ with the corresponding normalized eigenvectors $e_n(y) = \sqrt{\frac{2}{\pi}} \sin ny$. Consequently,

$$\mathfrak{A}\upsilon = -\sum_{n=1}^{\infty} n^2 \langle \upsilon, e_n \rangle e_n, \quad \upsilon \in \mathcal{D}(\mathfrak{A}).$$

In addition, $\mathfrak{A}$ is the infinitesimal generator of a bounded analytic semigroup $(V(\iota))_{\iota \geq 0}$, where

$$V(\iota)\upsilon = \sum_{n=1}^{\infty} e^{-n^2 \iota} \langle \upsilon, e_n \rangle e_n, \quad \upsilon \in \mathcal{X}.$$

Surely, for all $\iota \geq 0$, $\|V(\iota)\| \leq e^{-\iota}$. Hence, we take $M_V = 1$, which implies that $\sup\limits_{\iota \in (0,\infty)} \|V(\iota)\| = 1$ and $(\mathcal{C}_1)$ are satisfied. For $l = \frac{1}{2}$, the operator $\mathfrak{A}^{\frac{1}{2}}$ is given by the following:

$$\mathfrak{A}^{\frac{1}{2}}\upsilon = -\sum_{n=1}^{\infty} n \langle \upsilon, e_n \rangle e_n, \quad \upsilon \in \mathcal{D}(\mathfrak{A}^{\frac{1}{2}}).$$

where $\mathcal{D}(\mathfrak{A}^{\frac{1}{2}}) = \{\upsilon \in \mathcal{X} :\ \sum\limits_{n=1}^{\infty} n \langle \upsilon, e_n \rangle e_n \in \mathcal{X}\}$ and $\left\|\mathfrak{A}^{-\frac{1}{2}}\right\| = 1$. Let $\mathcal{X}_{\frac{1}{2}} = \left(\mathcal{D}(\mathfrak{A}^{\frac{1}{2}}), \|\ \|_{\frac{1}{2}}\right)$, where $\|\chi\|_{\frac{1}{2}} = \left\|\mathfrak{A}^{\frac{1}{2}}\chi\right\|_{\mathcal{X}}$ for $\chi \in \mathcal{D}(\mathfrak{A}^{\frac{1}{2}})$. It's evident now that the purpose $\hbar : \Lambda \times [0,\pi] \times \mathbb{R} \to \mathbb{R}$ fulfils the following conditions:

1. for all $(\iota, \chi) \in \Lambda \times [0,\pi]$, $\hbar(\iota, \chi, .)$ is continuous.
2. for all $\varsigma \in \mathbb{R}$, $\hbar(., ., \varsigma)$ is measurable.
3. for all $\iota \in \Lambda$ and $\varsigma \in \mathbb{R}$, $\hbar(\iota, \varsigma)$ is differentiable and $\frac{\partial}{\partial \chi}\hbar(\iota, \chi, \varsigma) \in \mathcal{X}$.
4. $\hbar(0, ., .) = \hbar(\pi, ., .) = 0$.
5. there exists $C > 0$ such that for all $(\iota, \chi, \varsigma) \in \Lambda \times [0,\pi] \times \mathbb{R}$, $\left|\frac{\partial}{\partial \chi}\hbar(\iota, \chi, \varsigma)\right| \leq C$.

We now $\forall \varphi \in \mathcal{X}_{\frac{1}{2}}$ possess the following

$$\begin{aligned}
\langle \hbar(\iota, \varphi, \Re\varphi), e_n \rangle =& \int_0^{\pi} \left(\frac{e^{-\iota}}{9 + e^{\iota}} \cos\left(\varphi(\iota, y) + \int_0^{\iota} \cos(\iota\varsigma)\varphi(\varsigma, y) d\varsigma\right)\right) . \left(\sqrt{\frac{2}{\pi}} \sin ny\right) dy \\
=& \frac{1}{n} \int_0^{\pi} \frac{\partial}{\partial y} \left(\frac{e^{-\iota}}{9 + e^{\iota}} \cos\left(\varphi(\iota, y) + \int_0^{\iota} \cos(\iota\varsigma)\varphi(\varsigma, y) d\varsigma\right)\right) . \left(\sqrt{\frac{2}{\pi}} \cos ny\right) dy.
\end{aligned}$$

This suggests that $\hbar : \Lambda \times \mathcal{X}_{\frac{1}{2}} \times \mathcal{X}_{\frac{1}{2}} \to \mathcal{X}_{\frac{1}{2}}$. Moreover, $\forall o > 0$ by the Minkowski inequality, Our possessions include:

$$\begin{aligned}
&\sup_{\|\varphi\|_{\frac{1}{2}} \leq o} \|\hbar(\iota, \varphi, \Re\varphi)\|_{\frac{1}{2}} = \sup_{\|\varphi\|_{\frac{1}{2}} \leq o} \left\| \frac{\partial}{\partial y} \left(\frac{e^{-\iota}}{9 + e^{\iota}} \cos\left(\varphi(\iota, y) + \int_0^{\iota} \cos(\iota\varsigma)\varphi(\varsigma, y) d\varsigma\right)\right) \right\|_{\mathcal{X}} \\
&= \sup_{\|\varphi\|_{\frac{1}{2}} \leq o} \left(\int_0^{\pi} \left| \frac{\partial}{\partial y} \left(\frac{e^{-\iota}}{9 + e^{\iota}} \cos\left(\varphi(\varsigma, y) + \int_0^{\iota} \cos(\iota\varsigma)\varphi(\varsigma, y) d\varsigma\right)\right) \right|^2 dy \right)^{\frac{1}{2}} \\
&\leq \sup_{\|\varphi\|_{\frac{1}{2}} \leq o} \left(\int_0^{\pi} \left(\frac{e^{-\iota}}{9 + e^{\iota}} \sin\left(\varphi(\iota, y) + \int_0^{\iota} \cos(\iota\varsigma)\varphi(\varsigma, y) d\varsigma\right) \left(|\varphi(\iota, y)| + |\frac{\partial}{\partial y} \int_0^{\iota} \cos(\iota\varsigma)\varphi(\iota, y) d\varsigma| \right) \right)^2 dy \right)^{\frac{1}{2}} \\
&\leq \sup_{\|\varphi\|_{\frac{1}{2}} \leq o} \left(\frac{e^{-\iota}}{9 + e^{\iota}} \sin\left(\varphi(\iota, y) + \int_0^{\iota} \cos(\iota\varsigma)\varphi(\varsigma, y) d\varsigma\right) (\|\varphi\|_{\mathcal{X}} + j^*\nu \|\varphi\|_{\mathcal{X}}) \right) \\
&\leq (1 + j^*\nu) o \frac{e^{-\iota}}{9 + e^{\iota}} \sin\left(\varphi(\iota, y) + \int_0^{\iota} \cos(\iota\varsigma)\varphi(\varsigma, y) d\varsigma\right) \\
&\leq I_o(\iota).
\end{aligned}$$

Consequently, $\hbar$ meets the requirements $(\mathcal{C}_2)$ in the following way:

$$\begin{aligned}
&\lim_{o\to+\infty}\inf\frac{1}{o}(\psi(\iota)-\psi(\varsigma))^{l-1}\psi'(\varsigma)I_o(\varsigma)d\varsigma\\
=&\lim_{o\to+\infty}\inf\frac{1}{o}\int_0^\iota(\iota-\varsigma)^{-\frac{1}{2}}(1+j^*\nu)o\frac{e^{-\iota}}{9+e^\iota}\sin\left(\varphi(\iota,y)+\int_0^\iota\cos(\iota\varsigma)\varphi(\varsigma,y)d\varsigma\right)d\varsigma\\
\leq&(1+j^*\nu)\left\|\frac{e^{-\iota}}{9+e^\iota}\sin\left(\varphi(\iota,y)+\int_0^\iota\cos(\iota\varsigma)\varphi(\varsigma,y)d\varsigma\right)\right\|_{L^\infty}\int_0^\iota(\iota-\varsigma)^{-\frac{1}{2}}\\
\leq&2\nu^{\frac{1}{2}}(1+j^*\nu)\left\|\left(\varphi(\iota,y)+\int_0^\iota\cos(\iota\varsigma)\varphi(\varsigma,y)d\varsigma\right)\right\|_{L^\infty}=\xi.
\end{aligned}$$

Thus, (7.1) has at least one mild solution.

Conclusion

We examined a semilinear ψ-fractional differential equation with starting conditions in this work, which involved the Volterra integral operator with an integral kernel. The Schauder fixed point theorem allowed us to determine the solution's existence. For the uniqueness of the solution, we utilized contraction principle. Additionally, we offered at least one $[0,\iota_{\max}]$ solution to (1.1). A few requirements are established for the HU stable, HU-R stable, generalized HU stable, and generalized HU-R stable of (1.1). We demonstrated the existence and uniqueness of (1.1) with an example.

Author contribution

All the authors have accepted responsibility for the entire content of this submitted manuscript and approved submission.

References

[1] M.S. Abdo, S.K. Panchal, and A.M. Saeed, *Fractional boundary value problem with ψ-Caputo fractional derivative*, Proc. Math. Sci. **129** (2019), no. 5, 65.

[2] M.R. Abdollahpour and M.T. Rassias, *Hyers-Ulam stability of hypergeometric differential equations*, Aequ. Math. **93** (2019), no. 4, 691–698.

[3] M.R. Abdollahpour, R. Aghayari, and M.T. Rassias, *Hyers-Ulam stability of associated Laguerre differential equations in a subclass of analytic functions*, J. Math. Anal. Appl. **437** (2016), 605–612.

[4] J. Aczel and J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, 1989.

[5] Y. Adjabi, F. Jarad, and T. Abdeljawad, *On generalized fractional operators and a Gronwall-type inequality with applications*, Filomat **31** (2017), no. 17, 5457–5473.

[6] R.P. Agarwal, H. Zhou, and Y. He, *Existence of fractional neutral functional differential equations*, Comput. Math. Appl. **59** (2010), no. 3, 1095–1100.

[7] R. Almeida, *A Caputo fractional derivative of a function with respect to another function*, Commun. Nonlinear Sci. Numer. Simul. **44** (2017), 460–481.

[8] R. Almeida, A.B. Malinowska, and M.T.T. Monteiro, *Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications*, Math. Meth. Appl. Sci. **41** (2018), 336–352.

[9] R. Almeida, M. Jleli, and B. Samet, *A numerical study of fractional relaxation– oscillation equations involving ψ-Caputo fractional derivative*, Rev. Real Acad. Cien. Exact. Fis. Natur. Ser. A. Mate. **113** (2019), no. 3, 1873–1891.

[10] S. Asawasamrit, A. Kijjathanakorn, S.K. Ntouya, and J. Tariboon, *Nonlocal boundary value problems for Hilfer fractional differential equations*, Bull. Korean Math. Soc. **55** (2018), 1639–1657.

[11] D.G. Bourgin, *Classes of transformations and bordering transformations*, Bull. Amer. Math. Soc. **57** (1951), 223–237.

[12] L. Cadariu, *Stabilitatea Ulam–Hyers–Bourgin Pentru Ecuatii Functionale*, Ed. Univ. Vest Timisoara, Timisara, 2007.

[13] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, 2002.

[14] S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg **62** (1992), 59–64.

[15] K. Deng, *Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions*, J. Math. Anal. Appl. **179** (1993), 630–637.

[16] K. Diethelm, *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics, Springer, New York, NY, USA, 2010.

[17] Z. Gajda, *On stability of additive mappings*, Int. J. Math. Math. Sci. **14** (1991), 431–434.

[18] L. Gaul, P. Klein, and S. Kemple, *Damping description involving fractional operators*, Mech. Syst. Signal Process **5** (1991), 81–88.

[19] K. Hilal and A. Kajouni, *Boundary value problems for hybrid differential equations with fractional order*, Adv. Differ. Equ. **2015** (2015), 1–19.

[20] D.H. Hyers, G. Isac, and T.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.

[21] D.H. Hyers and T.M. Rassias, *Approximate homomorphisms*, Aeque. Math. **44** (1992), 125–153.

[22] F. Jarad and T. Abdeljawad, *Generalized fractional derivatives and Laplace transform*, Discrete Contin. Dyn. Syst. Ser. S **13** (2019), no. 3, 709–722.

[23] O.K. Jaradat, A. Al-Omari, and S. Momani, *Existence of the mild solution for fractional semilinear initial value problems*, Nonlinear Anal. **69** (2008), 3153–3159.

[24] S.M. Jung, *Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, 2001.

[25] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer, 2011.

[26] S.-M. Jung, M.T. Rassias, and C. Mortici, *On a functional equation of trigonometric type*, Appl. Math. Comput. **252** (2015), 294–303.

[27] S.-M. Jung and M.T. Rassias, *A linear functional equation of third order associated to the Fibonacci numbers*, Abstr. Appl. Anal. **2014** (2014), Article ID 137468.

[28] S.-M. Jung, D. Popa, and M.T. Rassias, *On the stability of the linear functional equation in a single variable on complete metric groups*, J. Glob. Optim. **59** (2014), 165–171.

[29] P. Kannappan, *Functional Equations and Inequalities with Applications*, Springer, 2009.

[30] R.A. Khan and K. Shah, *Existence and uniqueness of solutions to fractional order multipoint boundary value problems*, Commun. Appl. Anal. **19** (2015), 515–526.

[31] V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities*, Academic Press, New York, 1969.

[32] Y.-H. Lee, S. Jung, and M.T. Rassias, *Uniqueness theorems on functional inequalities concerning cubic-quadratic-additive equation*, J. Math. Inequal. 12 (2018), no. 1, 43–61.

[33] Y. -H. Lee, S. Jung, and M.T. Rassias, *On an n-dimensional mixed type additive and quadratic functional equation*, Appl. Math. Comput. **228** (2014), 13–16.

[34] H. Liu and J.-Ch. Chang, *Existence for a class of partial differential equations with nonlocal conditions*, Nonlinear Anal.: Theory Meth. Appl. **70** (2009), no. 9, 3076–3083.

[35] Z.H. Liu and J.H. Sun, *Nonlinear boundary value problems of fractional differential systems*, Comp. Math. Appl. **64** (2012), 463–475.

[36] H. Lmou, K. Hilal, and A. Kajouni, *A new result for ψ-Hilfer fractional Pantograph- Type Langevin equation and inclusions*, J. Math. **2022** (2022).

[37] F. Mainardi, *Fractional diffusive waves in viscoelastic solids*, J.L. Wegner, F.R. Norwood (Eds.), Nonlinear Waves in Solids, Fairfield, 1995.

[38] F. Mainardi, P. Paraddisi, and R. Gorenflo, *Probability distributions generated by fractional diffusion equations*, Kertesz, J., Kondor, I. (eds.) Econophysics: An Emerging Science. Kluwer Academic, Dordrecht, 2000.

[39] R. Metzler and J. Klafter, *Boundary value problems for fractional diffusion equations*, Phys. A **278** (2000), 107–125.

[40] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, NY, USA, 1993.

[41] C. Mortici, S. Jung, and M.T. Rassias, *On the stability of a functional equation associated with the Fibonacci numbers*, Abstr. Appl. Anal. **2014** (2014), Article ID 546046.

[42] C. Park and M.T. Rassias, *Additive functional equations and partial multipliers in C^*-algebras*, Rev. Real Acad. Cien. Exactas, Ser. A. Mate. **113** (2019), no. 3, 2261–2275.

[43] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[44] Th. M. Rassias, *Functional Equations and Inequalities*, Kluwer Academic Publishers, 2000.

[45] P.K. Sahoo and P. Kannappan, *Introduction to Functional Equations*, CRC Press, 2011.

[46] B. Samet and H. Aydi, *Lyapunov-type inequalities for an anti-periodic fractional boundary value problem involving ψ-Caputo fractional derivative*, J. Inequal. Appl. **2018** (2018), no. 1, 1–11.

[47] S.G. Samko, A.A. Kilbas, and O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon Breach, Yverdon, 1993.

[48] J. Sousa and E.C. de Oliveira, *A Gronwall inequality and the Cauchy-type problem by means of ψ-Hilfer operator*, Differ. Equ. Appl. **11** (2019), 87–106.

[49] A. Suechoei and P.S. Ngiamsunthorn, *Existence uniqueness and stability of mild solutions for semilinear ψ-Caputo fractional evolution equations*, Adv. Differ. Equ. **2020** (2020), 114.

[50] W. Smajdor, *Note on a Jensen type functional equation*, Publ. Math. Debrecen **63** (2003), 703–714.

[51] T. Trif, *On the stability of a functional equation deriving from an inequality of Popoviciu for convex functions*, J. Math. Anal. Appl. **272** (2002), 604–616.

[52] J. Wang, *Some further generalization of the Ulam-Hyers-Rassias stability of functional equations*, J. Math. Anal. Appl. **263** (2001), 406–423.

[53] J. Wang and Y. Zhou, *Mittag-Leffler–Ulam stabilities of fractional evolution equations*, Appl. Math. Lett. **25** (2012), no. 4, 723–728.

[54] J. Wu, X. Zhang, L. Liu, Y. Wu, and Y. Cui, *The convergence analysis and error estimation for unique solution of a p-Laplacian fractional differential equation with singular decreasing nonlinearity*, Bound. Value Probl. **2018** (2018), 1–15.

[55] H. Yang, *Existence of mild solutions for a class of fractional evolution equations with compact analytic semigroup*, Abstr. Appl. Anal. **2012** (2012).

[56] Y. Zhou and F. Jiao, *Existence of mild solutions for fractional neutral evolution equations*, Comp. Math. Appl. **59** (2010), 1063–1077.

[57] S. Zorlu and A. Gudaimat, *Approximate controllability of fractional evolution equations with ψ-Caputo derivative*, Symmetry **15** (2023), 1050.