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Some convergence results using a new implicit iteration process in CAT(0) space

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Abstract

We prove the convergence of the newly defined iteration method to a common fixed point of a finite nonexpansive mappings family in CAT(0) space. A numerical example is given to check the convergence of the newly generalized iteration process. Many known results are extended and improved in this article.

Keywords: Implicit iteration process, Strong convergence, CAT(0) space
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1 Introduction

Let D be nonempty subset of a CAT(0) space E. Let S be a self-mapping of D. Then S is said to be nonexpansive if for all $x, y \in K$, the $d(Sx, Sy) \leq d(x, y)$ holds. Convergence theorems for nonexpansive mappings have been compiled by several authors. Xu and Ori [10] demonstrated weak convergence by introducing the implicit iteration process in a Hilbert space for a finite set of nonexpansive mappings $\{S_j : j \in I\}$ in 2001. In this article, we generalized the implicit iteration process which was given by Xu and Ori [10] as follows

$$
\begin{aligned}
&a_1 = s_1 a_0 + (1 - s_1) S_1 a_0, \\
&a_2 = s_2 a_1 + (1 - s_2) S_2 a_1, \\
&\vdots \\
&a_N = s_N a_{N-1} + (1 - s_N) S_N a_{N-1}, \\
&a_{N+1} = s_{N+1} a_N + (1 - s_{N+1}) S_1 a_N, \\
&\vdots
\end{aligned}
$$

Now, we transform this new iteration process in the context of CAT(0) space as

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$$a_1 = s_1 a_0 \bigoplus (1-s_1) S_1 a_0,$$
$$a_2 = s_2 a_1 \bigoplus (1-s_2) S_2 a_1,$$
$$\vdots$$
$$a_N = s_N a_{N-1} \bigoplus (1-s_N) S_N a_{N-1},$$
$$a_{N+1} = s_{N+1} a_N \bigoplus (1-s_{N+1}) S_1 a_N,$$
$$\vdots$$

More closely, we can transcribe the above table in the procedure

$$a_n = s_n a_{n-1} \bigoplus (1-s_n) S_n a_{n-1}, \quad \text{for all} \quad n \geq 1, \tag{1.1}$$

where $S_n = S_{n(modN)}$ (here the modN function adopts values in I). The aim of this paper is to study the implicit iteration process (1.1) for nonexpansive mappings in the setting of CAT(0) space. We shall develop the strong convergence of this new process to a common fixed point, needing some condition on the mappings defined above. The results obtained in this article simplify and cover the corresponding main findings of Xu and Ori [10], Chidume and Shahzad [5] and many others.

2 Preliminaries

For the sake of simplicity, we first recall a few definitions and conclusions. Let (E, d) be a metric space and $x, y \in E$ with $d(x, y) = l$. A geodesic path from x to y is a isometry $c : [0, l] \to X$ such that $c(0) = x$ and $c(l) = y$. The image of a geodesic path is called a geodesic segment. A metric space E is a (uniquely) geodesic space, if every two points of E are joined by only one geodesic segment. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic space E consists of three points x_1, x_2, x_3 of E and three geodesic segments joining each pair of vertices. A comparison triangle of a geodesic triangle $\Delta(x_1, x_2, x_3)$ is the triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean space $\mathbb{R}^2$ such that

$$d(x_i, x_j) = d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}),$$

for all $i, j = 1, 2, 3$. A geodesic space E is a CAT(0) space, if for each geodesic triangle $\Delta(x_1, x_2, x_3)$ in E and its comparison triangle $\overline{\Delta} := \Delta(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in $\mathbb{R}^2$, the CAT(0) inequality

$$d(x, y) = d_{\mathbb{R}^2}(\overline{x}, \overline{y})$$

is satisfied for all $x, y \in \Delta$ and $\overline{x}, \overline{y} \in \overline{\Delta}$. A thorough discussion of these spaces and their important role in various branches of mathematics are given [1, 4].

One approach is due to the famous mathematician Kirk [7, 8] who established a more general result to study the fixed point results in the setting of complete CAT(0) space. Among other things, he proved that every nonexpansive mapping defined on a bounded closed convex subset of a complete CAT(0) space has a fixed point. In this paper, we write $(1-t)x \bigoplus ty$ for the unique point z in the geodesic segment joining from x to y such that

$$d(z, x) = td(x, y), \qquad d(z, y) = (1-t)d(x, y).$$

We also, denote by $[x, y]$ the geodesic segment joining from x to y, i.e., $[x, y] = \{(1-t)x \bigoplus ty : t \in [0, 1]\}$. A subset of a CAT(0) space is convex if $[x, y] \subset C$ for all $x, y \in C$. For elementary facts about CAT(0) spaces, we refer the readers to [2, 3, 6]. We now give the definition of a mapping satisfy condition (B).

Definition 2.1. A family $\{S_j : j \in I\}$ of N self-mappings of D with $F = \bigcap_{j=1}^{N} F(S_j) \neq \phi$ is said to be satisfy condition (B) on D if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$, for all $r \in (0, \infty)$ such that, for all $a \in D$

$$\max_{1 \leq l \leq N} d(a, S_l a) \geq f(d(a, F)).$$

Lemma 2.2. [1] Let E be a CAT(0) space. Then

$$d((1-s)a \bigoplus sb, c) \leq (1-s)d(a, c) + sd(b, c)$$

for all $a, b, c \in E$ and $s \in [0, 1]$.

Lemma 2.3. [1] Let E be a CAT(0) space. Then

$$d((1-s)a \bigoplus sb, c)^2 \leq (1-s)d(a,c)^2 + sd(b,c)^2 - s(1-s)d(a,b)^2,$$

for all $a, b, c \in E$ and $s \in [0,1]$.

Lemma 2.4. [9] Let $\{p_n\}, \{q_n\}$ and $\{r_n\}$ be sequences of nonnegative numbers satisfying the inequality

$$p_{n+1} \leq (1+q_n)p_n + r_n \qquad ,$$

for all $n \geq 1$. If $\sum_{n=1}^{\infty} q_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$, then $\lim_{n\to\infty} p_n$ exists.

3 Main Results

Theorem 3.1. Let E be a complete CAT(0) space and D be a nonempty closed convex subset of E. Let $\{S_j : j \in I\}$ be a nonexpansive self-mappings of D with $F = \bigcap_{j=1}^{N} F(S_j) \neq \phi$. Presume that $\{S_j : j \in I\}$ pleases condition(B). Let $\{s_n\}_{n\geq 1} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. Assume for $a_0 \in D$, describe the sequence $\{a_n\}$ as (1.1). Then $\{a_n\}$ strongly converges to a common fixed point of the mappings $\{S_j : j \in I\}$.

Proof . We claim that $\lim_{n\to\infty} d(a_n, S_m a_n) = 0$, for all $m \in I$. Let $a^* \in F$ and let $lim_{n\to\infty} d(a_n, a^*) = q$. If $q = 0$, then the conclusion follows by continuity of S_m. We consider

$$\begin{aligned} d(a_n, a^*) =& d(s_n a_{n-1} \bigoplus (1-s_n) S_n a_{n-1}, a^*) \\ \leq& s_n d(a_{n-1}, a^*) + (1-s_n) d(S_n a_{n-1}, a^*) \\ =& s_n d(a_{n-1}, a^*) + (1-s_n) d(S_j a_{n-1}, a^*) \\ \leq& s_n d(a_{n-1}, a^*) + (1-s_n) d(a_{n-1}, a^*) \\ \leq& d(a_{n-1}, a^*). \end{aligned}$$

This gives

$$d(a_n, a^*) \leq d(a_{n-1}, a^*) \tag{3.1}$$

Further, it implies that $\lim_{n\to\infty} d(a_n, a^*)$ exists by Lemma 2.4 and thus $\{a_n\}$ is bounded. There exists $\widetilde{R} > 0$ such that $a_n \in B_{\widetilde{R}}(0)$ holds for all $n \geq 1$. Before proving the assertion, first we need to prove that $\lim_{n\to\infty} d(S_n a_{n-1}, a_{n-1}) = 0$. Therefore, we calculate as follows:

$$\begin{aligned} d(a_n, a^*)^2 =& d(s_n a_{n-1} \bigoplus (1-s_n) S_n a_{n-1}, a^*)^2 \\ \leq& s_n d(a_{n-1}, a^*)^2 + (1-s_n) d(S_n a_{n-1}, a^*)^2 - s_n(1-s_n) d(S_n a_{n-1}, a_{n-1})^2 \\ \leq& s_n d(a_{n-1}, a^*)^2 + (1-s_n) d(a_{n-1}, a^*)^2 - s_n(1-s_n) d(S_n a_{n-1}, a_{n-1})^2 \\ \leq& d(a_{n-1}, a^*)^2 - s_n(1-s_n) d(S_n a_{n-1}, a_{n-1})^2. \end{aligned}$$

The above relations imply that

$$2\delta^3 d(S_n a_{n-1}, a_{n-1})^2 \leq d(a_{n-1}, a^*)^2 - d(a_n, a^*)^2.$$

Hence $\sum_{n=1}^{\infty} d(S_n a_{n-1}, a_{n-1}) < \infty$. This implies that $\lim_{n\to\infty} d(S_n a_{n-1}, a_{n-1}) = 0$. Since

$$\begin{aligned} d(a_n, a_{n-1}) =& d(s_n a_{n-1} \bigoplus (1-s_n) S_n a_{n-1}, a_{n-1}) \\ \leq& (1-s_n) d(S_n a_{n-1}, a_{n-1}), \end{aligned}$$

it tracks that $\lim_{n\to\infty} d(a_n, a_{n-1}) = 0$. We have

$$d(a_n, S_n a_{n-1}) \leq d(a_n, a_{n-1}) + d(S_n a_{n-1}, a_{n-1}).$$

So, $\lim_{n\to\infty} d(a_n, S_n a_{n-1}) = 0$. Now,

$$d(a_n, S_n a_n) \leq d(a_n, S_n a_{n-1}) + d(S_n a_{n-1}, S_n a_n) \leq d(a_n, S_n a_{n-1}) + d(a_{n-1}, a_n).$$

Therefore, $\lim_{n\to\infty} d(a_n, S_n a_n) = 0$. For all $m \in I$

$$\begin{aligned} d(a_n, S_{n+m} a_n) \leq& d(a_n, a_{n+m}) + d(a_{n+m}, S_{n+m} a_{n+m}) + d(S_{n+m} a_{n+m}, S_{n+m} a_n) \\ \leq& d(a_n, a_{n+m}) + d(a_{n+m}, S_{n+m} a_{n+m}) + d(a_{n+m}, a_n) \end{aligned}$$

set by taking the limit $n \to \infty$ yields

$$\lim_{n\to\infty} d(a_n, S_{n+m} a_n) = 0, \tag{3.2}$$

for all $m \in I$. Accordingly, we have $\lim_{n\to\infty} d(a_n, S_m a_n) = 0$ for all $m \in I$. The relation (3.1) indicates that $d(a_n, F) \leq d(a_{n-1}, F)$ and by Lemma 2.4, $\lim_{n\to\infty} d(a_n, F)$ exists. So, by (3.2) and since $\{S_j : j \in I\}$ pleases condition (B), we accomplish that $\lim_{n\to\infty} f(d(a_n, F)) \leq 0$ implying $\lim_{n\to\infty} f(d(a_n, F)) = 0$, i.e. $\lim_{n\to\infty}(d(a_n, F)) = 0$. So, if for any $\epsilon > 0$, there exists a natural number $\hat{n_1}$ such that $d(a_n, F) < \frac{\epsilon}{3}$, for all $n \geq \hat{n_1}$. So, we can find $u^* \in F$ such that $d(a_{\hat{n_1}}, u^*) < \frac{\epsilon}{2}$. For all $n \geq \hat{n_1}$ and $l \geq 1$, we have

$$\begin{aligned} d(a_{n+l}, a_n) \leq& d(a_{n+l}, u^*) + d(u^*, a_n) \\ \leq& d(a_{\hat{n_1}}, u^*) + d(a_{\hat{n_1}}, u^*) \\ <& \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This proves that $\{a_n\}$ is a cauchy sequence and converges as E is complete. Let $\lim_{n\to\infty} a_n = v^*$. Then $v^* \in D$. It rests to display that $v^* \in F$. If $\hat{\epsilon} > 0$, then there exists a natural number $\hat{n_2}$ such that $d(a_n, v^*) < \frac{\hat{\epsilon}}{4}$, for all $n \geq \hat{n_2}$. Because $\lim_{n\to\infty} d(a_n, F) = 0$, there exists a natural number $\hat{n_3} \geq \hat{n_2}$ such that for all $n \geq \hat{n_3}$, we have $d(a_n, F) < \frac{\hat{\epsilon}}{4}$ and in particular $d(a_{n_3}, F) < \frac{\hat{\epsilon}}{4}$. Therefore, there is $w^* \in F$ such that $d(a_{\hat{n_3}}, w^*) < \frac{\hat{\epsilon}}{4}$. For any $j \in I$ and $n \geq \hat{n_3}$, we obtain

$$\begin{aligned} d(S_j v^*, v^*) \leq& d(S_j v^*, w^*) + d(w^*, v^*) \\ \leq& 2d(w^*, v^*) \\ \leq& 2d(w^*, a_{\hat{n_2}}) + 2d(a_{\hat{n_2}}, v^*) \\ <& \hat{\epsilon}. \end{aligned}$$

This gives that $S_j v^* = v^*$. Hence $v^* \in F(S_j)$ for all $j \in I$ and so $v^* \in F = \bigcap_{j=1}^{N} F(S_j)$. This is the result. □

Example 3.2. Let E be a real line with the Euclidean norm and $D = [0, 1]$. For $a \in$ D, $j = 1, 2, ...$ We define mappings S_j on D as follows:

$$S_j a = \frac{a}{j} \qquad \text{for all} \quad j = 1, 2, ..., N.$$

Let the sequence $\{a_n\}$ be described by the iterative scheme for a fixed $j = 12$ as (1.1). Obviously, S_j is nonexpansive. Also, $\bigcap_{j=1}^{12} F(S_j) = \{0\}$. It can be observed that all the conventions of Theorem 3.1 are fulfilled. For any $a_0 \in D = [0, 1]$, we put $s_{kn} = 1 - \frac{1}{\sqrt{k(7n+9)}}$. Also, $S_1 = a_n, S_2 = \frac{a_n}{2}, S_3 = \frac{a_n}{3}$ and upto so on.

Note: From Table 1 and Fig. 1, we obtain that iteration (1.1) converges faster when we take initial value near to common fixed point of N nonexpansive mappings. Also, it is clear that $\{a_n\}$ converges to 0, where $\bigcap_{j=1}^{12} F(S_j) = \{0\}$.

4 Conclusion

The extension of the linear version of convergence results to nonlinear spaces has its own importance. Here we extend a linear version of convergence results to the common fixed point of a finite nonexpansive mappings family for an implicit iteration method in the setting of Banach space to nonlinear CAT(0) spaces. Also, we gave an example to illustrate the facts.

Table 1: **The values of the sequence $\{a_n\}$ with different initial values.**

n	$a_0 = 0.1$ iteration(1.1)a_n	$a_0 = 0.2$ iteration(1.1)a_n	$a_0 = 0.5$ iteration(1.1)a_n
1	0.1	0.2	0.5
2	0.0947871396485731	0.189574279297146	0.473935698242866
3	0.0909414319043572	0.181882863808714	0.454707159521786
4	0.0881381795338868	0.176276359067774	0.440690897669434
5	0.0860122066522949	0.17202441330459	0.430061033261474
6	0.0843394124862045	0.168678824972409	0.421697062431022
7	0.0832257937835021	0.166451587567004	0.41612896891751
8	0.0820967277725585	0.164193455545117	0.410483638862792
9	0.0811411532388761	0.162282306477752	0.405705766194381
10	0.0802282439411137	0.160456487882227	0.401141219705568
11	0.0795132659218234	0.159026531843647	0.397566329609117
.	.	.	.
.	.	.	.
.	.	.	.
25	0.075728677657591	0.151457355315182	0.378643388287955
26	0.0756233019882729	0.151246603976546	0.378116509941365
27	0.075490602875869	0.150981205751738	0.377453014379345
.	.	.	.
.	.	.	.
97	0.0706582481586677	0.141316496317335	0.353291240793338
98	0.0706445735662075	0.141289147132415	0.353222867831038
99	0.0706266186328212	0.141253237265642	0.353133093164106
.	.	.	.
.	.	.	.
130	0.069776372660387	0.139552745320774	0.348881863301934
131	0.0697604601421406	0.139520920284281	0.348802300710703
132	0.0697446000384408	0.139489200076882	0.348723000192204
.	.	.	.

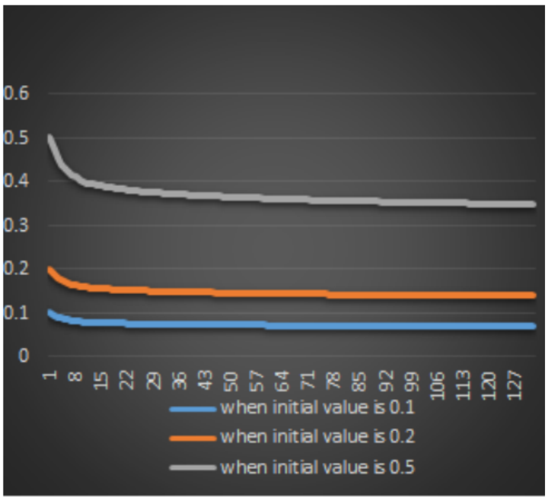

Figure 1: Represent the convergence of (1.1) iteration method to a common fixed point of N nonexpansive mappings.

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