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Stability of orthogonally quintic functional equation on C^*-algebras with the type fixed point alternative

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Abstract

Using fixed point methods, we prove the stability of orthogonally quintic functional equation on C^*-algebras for the functional equation

$$Df(x,y) = f(3x+y) - 5f(2x+y) + f(2x-y) + 10f(x+y) - 5f(x-y) - 10f(y) - f(3x) + 3f(2x) + 27f(x).$$

Keywords: Fixed point, Hyers-Ulam stability, C^*-algebras, quintic functional equation
2020 MSC: 39B82, 46L05

1 Introduction

The stability problem of functional equations has originally been formulated by Ulam [26] in 1940: Under what condition does there exist a homomorphism near an approximate homomorphism? In following year, Hyers [18] answers the problem of Ulam under the assumption that the groups are Banach spaces. For more details about the result concerning such problems, we refer the reader to ([15, 16, 17, 25]). In 2003, Cădariu and Radu [7] applied the fixed point method to investigate the Jensen functional equation. The various problems of the stability of derivations and homomorphism have been studied during last few years (see also [8, 9, 11, 19, 27]).

The stability and hyperstability problems for various functional equations have been iintroduced by several author and they obtained many interesting results concerning the Hyers–Ulam stability (see for example [4, 6, 20, 23]). Moghimi and Najat investigated hyperstability and stability results for the Cauchy and Jensen functional equations on restricted domains [22]. El-Hady and Ogrekci studied stability problem of some fractional differential equations in the sense of Hyers-Ulam and Hyers-Ulam-Rassias based on some fixed point techniques [12]. Eshaghi and et al. introduced orthogonally sets and proved the real generalization of Banach fixed point theorem on this sets [13]. We start our work with the following definition, which can be consider the main definition of our paper [1, 3, 10, 14, 21].

Definition 1.1. Let $X \neq \emptyset$ and $\perp \subseteq X \times X$ be an binary relation. If $\perp$ satisfies the following condition

$$\exists x_0; (\forall y; y \perp x_0) \;\; or \;\; (\forall y; x_0 \perp y),$$

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it is called an orthogonally set (briefly O-set). We denote this O-set by $(X, \perp)$.

Definition 1.2. Let $(X, \perp)$ be an O-set. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called orthogonally sequence (briefly O-sequence) if

$$(\forall n; x_n \perp x_{n+1}) \;\; or \;\; (\forall n; x_{n+1} \perp x_n).$$

Definition 1.3. Let $(X, \perp, d)$ be an orthogonally metric space ($(X, \perp)$ is an O-set and (X, d) is a metric space). Then $f : X \to X$ is $\perp-$continuous in $a \in X$ if for each O-sequence $\{a_n\}_{n\in\mathbb{N}}$ in X if $a_n \to a$, then $f(a_n) \to f(a)$. Also f is $\perp-$continuous on X if f is $\perp-$continuous on each $a \in X$.

It is easy to see that every continuous mapping is $\perp-$continuous.

Definition 1.4. Let $(X, \perp, d)$ be an orthogonally metric space, then X is orthogonally complete (briefly O-complete) if every Cauchy O-sequence is convergent.

It is easy to see that every complete metric space is O-complete and the converse is not true.

Definition 1.5. Let $(X, \perp, d)$ be an orthogonally metric space and $0 < \lambda < 1$. A mapping $f : X \to X$ is said to be orthogonality contraction with Lipschitz constant λ if

$$d(fx, fy) \leq \lambda d(x, y) \quad if \; x \perp y.$$

Let H be a Hilbert space. Suppose that $f : H \to \mathbb{C}$ is a mapping satisfying

$$f(x) = \|x\|^2 \tag{1.1}$$

for all $x \in X$. It is natural that this equation is a quadratic functional equation. On the other hand by considering $x \perp y$ with $< x, y >= 0$ for $x, y \in H$, it is easy to see that the above function $f : H \to \mathbb{C}$ is an orthogonally additive functional equation, that is $f(x + y) = f(x) + f(y)$ if $x \perp y$. This means that orthogonality may change a functional equation. Recently, Bahraini et al. [2] proved a fixed point theorem in O-sets as follows:

Theorem 1.6. Let $(X, d, \perp)$ be an O-complete generalized metric space. Let $T : X \to X$ be a $\perp$-preserving, $\perp$-continuous and $\perp$-λ-contraction. Let $x_0 \in X$ satisfies for all $y \in X$, $x_0 \perp y$ or for all $y \in X$, $y \perp x_0$, and consider the "*O-sequence of successive approximations with initial element* x_0": x_0, $T(x_0)$, $T^2(x_0)$, ..., $T^n(x_0)$, ... Then, either $d(T^n(x_0), T^{n+1}(x_0)) = \infty$ for all $n \geq 0$, or there exists a positive integer n_0 such that $d(T^n(x_0), T^{n+1}(x_0)) < \infty$ for all $n > n_0$. If the second alternative holds, then

(i) the O-sequence of $\{T^n(x_0)\}$ is convergent to a fixed point x^* of T.

(ii) x^* is the unique fixed point of T in $X^* = \{y \in X : d(T^n(x_0), y) < \infty\}$.

(iii) If $y \in X$, then

$$d(y, x^*) \leq \frac{1}{1-\lambda} d(y, T(y)).$$

In this paper, we apply the fixed point method to prove the stability problem for orthogonally $*$-quintic on C^*-algebras.

2 Main results

Throughout this section, assume that $(A, \|.\|_1, \perp_1)$ with $a \perp_1 b$ if $ab^* = b^*a = 0$ and $(B, \|.\|_2, \perp_2)$ with $a \perp_2 b$ if $ab^* = b^*a = 0$ be two C^*-algebras. For a given mapping $f : A \to B$, we define

$$Df(x, y) := f(3x + y) - 5f(2x + y) + f(2x - y) + 10f(x + y) - 5f(x - y) - 10f(y) - f(3x) + 3f(2x) + 27f(x)$$

for all $x, y \in A$ with $x \perp_1 y$ [6, 24]. We deal with the stability problem for the orthogonally $*$-quintic functional equation $Df(x, y) = 0$ in C^*-algebras.

Theorem 2.1. Let $f : A \to B$ be a mapping satisfying $f(0) = 0$ and for which there exist a function $\varphi : A^2 \to [0, \infty)$ such that

$$\|Df(x, y)\|_2 \le \varphi(x, y), \tag{2.1}$$

$$\|f(x^*) - f(x)^*\|_2 \le \varphi_m(x, x) \tag{2.2}$$

for all $x, y \in A$ with $x \perp_1 y$. If there exists a constant $0 < L < 1$ such that

$$\varphi(x, y) \le 32L\varphi\Big(\frac{x}{2}, \frac{y}{2}\Big) \tag{2.3}$$

for all $x, y \in A$ with $x \perp_1 y$, then there exists a unique orthogonally $*$-quintic mapping $Q : A \to B$ such that

$$\|f(x) - Q(x)\|_2 \le \frac{1}{1-L}\varphi_m(x, 0) \tag{2.4}$$

for all $x \in A$.

Proof . It follows that (2.3), we get

$$\lim_{n\to\infty} \frac{1}{32^n}\varphi(2^n x, 2^n y) = 0 \tag{2.5}$$

for all $x, y \in A$ with $x \perp_1 y$. Putting $y = 0$ in (2.1), we get

$$\|32f(x) - f(2x)\|_2 \le \varphi(x, 0)$$

for all $x \in A$. So

$$\|f(x) - \frac{1}{32}f(2x)\|_2 \le \frac{1}{32}\varphi(x, 0) < \varphi(x, 0) \tag{2.6}$$

for all $x \in A$. Consider the set

$$\Omega := \{g : \ g : X \to Y, \ g(x)\perp_2 \frac{1}{32}g(2x) \ \ or \ \ \frac{1}{32}g(2x)\perp_2 g(x), \ \ \forall x \in A\}.$$

For every $g, h \in \Omega$, define

$$d(g, h) = \inf\{K \in (0, \infty) : \ \ \|g(x) - h(x)\|_2 \le K\varphi(x, 0), \ \ \forall x \in A\}$$

Now, we put the $\perp$ relation orthogonal on Ω as follows: for all $g, h \in \Omega$

$$h\perp g \ \Leftrightarrow \ h(x) \perp_2 g(x) \ \ or \ \ g(x)\perp_2 h(x)$$

for all $x \in A$. It is easy to show that $(\Omega, d, \perp)$ is an O-complete generalized metric space. Now, we consider the mapping $T : \Omega \to \Omega$ defined by $Tg(x) = \frac{1}{32}g(2x)$ for all $x \in A$ and $g \in \Omega$. For all $g, h \in \Omega$ with $g \perp h$ and $x \in A$,

$$\begin{aligned}
d(g, h) < K \ &\Rightarrow \ \|g(x) - h(x)\|_2 \le K\varphi(x, 0) \\
&\Rightarrow \ \|\frac{1}{32}g(2x) - \frac{1}{32}h(2x)\|_2 \le \ \frac{K}{32} \ \varphi(2x, 0) \\
&\Rightarrow \ \|\frac{1}{32}g(2x) - \frac{1}{32}h(2x)\|_2 \le L \ K \ \varphi(x, 0) \\
&\Rightarrow \ d(Tg, Th) \le L \ K.
\end{aligned}$$

Hence we see that

$$d(Tg, Th) \le L \ d(g, h)$$

for all $g, h \in \Omega$, that is, T is a strictly self-mapping of Ω with the Lipschitz constant L. Now, we show that T is a $\perp$-continuous. To this end, let $\{g_n\}_{n\in\mathbb{N}}$ be an O-sequence with $g_n \perp g_{n+1}$ or $g_{n+1} \perp g_n$ in $(\Omega, d, \perp)$ which convergent to $g \in \Omega$ and let $\epsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ and $K \in R^+$ with $K < \epsilon$ such that

$$\|g_n(x) - g(x)\|_2 \le K\varphi(x, 0,)$$

for all $x \in A$ and $n \ge N$ and so

$$\|\frac{1}{32}g_n(2x) - \frac{1}{32}g(2x)\|_2 \le \ \frac{K}{32} \ \varphi(2x, 0)$$

for all $x \in A$ and $n \geq N$. By inequality (2.3) and the define of T, we get

$$\|T(g_n)(x) - T(g)(x)\|_2 \leq LK\varphi(x, 0)$$

for all $x \in A$ and $n \geq N$. Hence

$$d(T(g_n), T(g)) \leq LK < \epsilon$$

for all $n \geq N$. It follows that T is $\perp$-continuous. By definition Ω, we have $f \perp_2 T(f)$ or $T(f) \perp_2 f$, by applying the inequality (2.6), we see that $d(T(f), f) \leq 1$. It follows from Theorem 1.6 that T has a unique fixed point $Q : X \to Y$ in the set $\Lambda : \{g \in \Omega : d(g, h) < \infty\}$, where Q is defined by

$$Q(x) = \lim_{n\to\infty} T^n g(x) = \lim_{n\to\infty} \frac{1}{32^n} g(2^n x) \tag{2.7}$$

for all $x \in A$. By Theorem 1.6,

$$d(Q, f) \leq \frac{1}{1 - L}.$$

It follows from (2.1), (2.5) and (2.7) that

$$\begin{aligned} \|DQ(x, y)\|_2 &= \lim_{n\to\infty} \frac{1}{32^n} \|Df(2^n x, 2^n y)\|_2 \\ &\leq \lim_{n\to\infty} \frac{1}{32^n} \varphi(2^n x, 2^n y) = 0 \end{aligned}$$

for all $x, y \in A$ with $x \perp_1 y$. So

$$Df(x, y) = 0$$

for all $x, y \in A$ with $x \perp_1 y$. It follows from (2.2), (2.5) and (2.7) that

$$\|Q(x^*) - Q(x)^*\|_2 = \lim_{n\to\infty} \frac{1}{32^n} \Big\| f(2^n x^*) - f(2^n x)^* \|_2 = 0$$

for all $x \in A$. Hence $Q : A \to B$ is an orthogonally $*$-quintic mapping. □

Corollary 2.2. Let θ be a positive real numbers and p a real numbers with $0 < p < 5$. Let $f : A \to B$ be a mapping

$$\|Df(x, y)\|_2 \leq \theta(\|x\|_1^p + \|y\|_1^p),$$

$$\|f(x^*) - f(x)^*\|_2 \leq 2\theta \|x\|_1^p$$

for all $x, y \in A$ with $x \perp_1 y$. Then there exists a unique orthogonally $*$-quintic mapping $Q : A \to B$ such that

$$\|f(x) - Q(x)\|_2 \leq \frac{2^p \theta}{32 - 2^p} \|x\|_1^p$$

for all $x \in A$.

Proof . Set $\varphi(x, y) = \epsilon(\|x\|_1^p + \|y\|_1^p)$ for all $x, y \in A$ with $x \perp_1 y$ and let $L = 2^{p-5}$ in Theorem 2.1. Then we get the desired result. □

Theorem 2.3. Let $f : A \to B$ be a mapping satisfying (2.1), (2.2) and $f(0) = 0$ and for which there exist function $\varphi : A^2 \to [0, \infty)$ such that

$$\varphi(x, y) \leq \frac{L}{32} \varphi(2x, 2y) \tag{2.8}$$

for all $x, y \in A$ with $x \perp_1 y$. Then there exists a unique orthogonally $*$-quintic mapping $Q : A \to B$ such that

$$\|f(x) - Q(x)\|_2 \leq \frac{1}{32(1 - L)} \varphi(x, 0) \tag{2.9}$$

for all $x \in A$.

Proof . The proof is similar to the proof Theorem 2.1. □

Corollary 2.4. Let θ be a positive real numbers and p a real numbers with $p > 5$. Let $f : A \to B$ be a mapping

$$\|Df(x,y)\|_2 \le \theta(\|x\|_1^p + \|y\|_1^p),$$

$$\|f(x^*) - f(x)^*\|_2 \le 2\theta\|x\|_1^p$$

for all $x, y \in A$ with $x \perp_1 y$. Then there exists a unique orthogonally $*$-quintic mapping $Q : A \to B$ such that

$$\|f(x) - Q(x)\|_2 \le \frac{\theta}{2^p - 32}\|x\|_1^p$$

for all $x \in A$.

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