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Lie symmetries, conservation laws, optimal system and power series solutions of (3+1)-dimensional fractional Zakharov-Kuznetsov equation

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Abstract

In this paper, the Lie symmetry analysis method is applied to the high dimensional fractional Zakharov-Kuznetsov equation. All Lie symmetries and the corresponding conserved vectors for the equation are obtained. The one-dimensional optimal system is utilized to reduce the aimed equation with Riemann-Liouville fractional derivative to a low dimensional fractional partial differential equation with Erdélyi-Kober fractional derivative. Then the power series solution of the reduced equation is given. Moreover, some other low-dimensional reduced fractional differential equations with Riemann-Liouville fractional derivatives are obtained and can be solved by different methods in the literatures herein.

Keywords: Lie symmetry analysis, fractional Zakharov-Kuznetsov equation, conservation laws, one-dimensional optimal system, power series solution
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1 Introduction

Nonelinear partial differential equations are increasingly used to model nonlinear physical phenomena. Among them, the following (2+1)-dimensional Zakharov-Kuznetsov (Z-K) equation is considered:

$$u_t + (u_{xx} + u_{yy} + u^p)_x = 0, \quad p = 2, 3, 4. \tag{1.1}$$

which, introduced by Zakharov and Kuznetsov, was firstly derived for describing weakly nonlinear ion-acoustic waves in strongly magnetized lossless plasma in two dimensions [44]. The Z-K equation governs the behavior of weakly nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [27, 28]. Various numerical and analytical methods have been subsequently applied to solve the Z-K and modified Z-K equations [1, 11, 19, 20, 22, 37]. Recently, fractional Z-K equation was introduced and studied in [2, 38]. In [38], Yang et al. studied the conservation laws of space-time fractional mZK equation for Bossby solitary waves with complete Coriolis force. In [2], Al-deiakeh et al. used Lie symmetry analysis method to study

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the time-fractional (2+1)-dimensional Zakharov-Kuznetsov (q, p, r) equation and obtained analytical solutions by the power series method.

In this paper, Eq. (1.1) are extended to the following (3+1)-dimensional time-fractional version:

$$D_t^\alpha u + (u_{xx} + u_{yy} + u_{zz} + u^p)_x = 0, \quad 0 < \alpha < 1, \quad p = 2, 3, 4. \tag{1.2}$$

As a generalization of the classical calculus, fractional calculus can be traced back to the letter written by L'Hôspital to Leibniz in 1695. Since then, it has gradually gained the attention of mathematicians. Especially in recent decades, it has developed rapidly and been successfully applied in many fields of science and technology [12, 21, 32, 35]. Therefore, it is very important to find the solution of fractional differential equation. So far, there have been some numerical and analytical methods, such as Adomian decomposition method [5], finite difference method [24], homotopy perturbation method [25], the sub-equation method [47], the variational iteration method [26], Lie symmetry analysis method [9], invariant subspace method [8] and so on. Among them, Lie symmetry analysis method has received an increasing attention.

Lie symmetry analysis method was founded by Norwegian mathematician Sophus Lie at the end of the nineteenth century and then further developed by some other mathematicians, such as Ovsiannikov [31], Olver [30], Ibragimov [14, 15, 16] and so on. As a modern method among many analytic techniques, Lie symmetry analysis has been extended to fractional differential equations (FDEs) by Gazizov et al. [9] in 2007. It was then effectively applied to various models of the (1+1)-dimensional FDEs [6, 7, 10, 29, 39, 40, 41, 42, 43, 45, 46] and the (2+1)-dimensional FDEs [3, 33, 34, 48] occurring in different areas of applied science.

This paper applies Lie symmetry analysis method to study the (3+1)-dimensional time-fractional Z-K equation. We aim to find all Lie symmetries admitted by the equation and construct the corresponding conserved vector for each symmetry by the new conservation theorem and the generalization of Noether operator. The one-dimensional optimal system obtained through Olver's method [30] is used to reduce the dimensionality of Eq. (1.2), where the reduced equation is then solved simultaneously to obtain the power series solution.

As we all know, there are many types of definitions for fractional derivative, such as Riemann-Liouville type, Caputo type, Weyl type and so on. This paper adopts Riemann-Liouville fractional derivative defined by

$$ {}_aD_t^\alpha f(t,x) = D_t^n \; {}_aI_t^{n-\alpha} f(t,x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(s,x)}{(t-s)^{\alpha-n+1}} \mathrm{d}s, & n-1 < \alpha < n, n \in \mathbb{N} \\ D_t^n f(t,x), & \alpha = n \in \mathbb{N} \end{cases} $$

for $t > a$. We denote the operator ${}_0D_t^\alpha$ as D_t^α throughout this paper.

This paper is organized as follows. In Section 2, Lie symmetry analysis of Eq. (1.2) is presented. In Section 3, the conserved vectors for all the symmetries admitted by Eq. (1.2) are constructed. In Section 4, the one-dimensional optimal system is derived. In Section 5, similarity reductions and exact solutions are obtained. The conclusion is given in the last section.

2 Lie symmetry analysis of Eq. (1.2)

Consider the (3+1)-dimensional fractional Z-K equation (1.2), which is assumed to be invariant under the one-parameter (ϵ) Lie group of continuous point transformations, i.e.

$$\begin{aligned} t^* &= t + \epsilon\tau(t,x,y,z,u) + o(\epsilon), \quad & x^* &= x + \epsilon\xi(t,x,y,z,u) + o(\epsilon), \\ y^* &= y + \epsilon\zeta(t,x,y,z,u) + o(\epsilon), \quad & z^* &= z + \epsilon\theta(t,x,y,z,u) + o(\epsilon), \\ u^* &= u + \epsilon\eta(t,x,y,z,u) + o(\epsilon), \quad & D_{t^*}^\alpha u^* &= D_t^\alpha u + \epsilon\eta^{\alpha,t} + o(\epsilon), \\ D_{x^*} u^* &= D_x u + \epsilon\eta^x + o(\epsilon), \quad & D_{y^*} u^* &= D_y u + \epsilon\eta^y + o(\epsilon), \\ D_{z^*} u^* &= D_z u + \epsilon\eta^z + o(\epsilon), \quad & D_{x^*}^2 u^* &= D_x^2 u + \epsilon\eta^{xx} + o(\epsilon), \\ D_{y^*}^2 u^* &= D_y^2 u + \epsilon\eta^{yy} + o(\epsilon), \quad & D_{z^*}^2 u^* &= D_z^2 u + \epsilon\eta^{zz} + o(\epsilon), \\ D_{x^*}^3 u^* &= D_x^3 u + \epsilon\eta^{xxx} + o(\epsilon), \quad & D_{y^*}^2 D_{x^*} u^* &= D_y^2 D_x u + \epsilon\eta^{xyy} + o(\epsilon), \end{aligned} \tag{2.1}$$

$$D_{z^*}^2 D_{x^*} u^* = D_z^2 D_x u + \epsilon \eta^{xzz} + o(\epsilon), \quad \cdots,$$

where τ, ξ, ζ, θ, η are infinitesimals and $\eta^{\alpha,t}$, η^x, η^y, η^z, η^{xx}, η^{yy}, η^{zz}, η^{xxx}, η^{xyy}, η^{xzz}, $\cdots$ are the corresponding prolongations of orders α, 1, 2, 3, $\cdots$, respectively. The corresponding group generator is defined by

$$X = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial y} + \theta\frac{\partial}{\partial z} + \eta\frac{\partial}{\partial u}. \tag{2.2}$$

So the prolongation of the above group generator X has the form

$$Pr^{(\alpha,3)}X = X + \eta^{\alpha,t}\frac{\partial}{\partial u_t^\alpha} + \eta^x\frac{\partial}{\partial u_x} + \eta^{xxx}\frac{\partial}{\partial u_{xxx}} + \eta^{xyy}\frac{\partial}{\partial u_{xyy}} + \eta^{xzz}\frac{\partial}{\partial u_{xzz}}, \tag{2.3}$$

where

$$\begin{aligned}
\eta^x =& D_x\eta - u_t D_x\tau - u_x D_x\xi - u_y D_x\zeta - u_z D_x\theta,\\
\eta^y =& D_y\eta - u_t D_y\tau - u_x D_x\xi - u_y D_y\zeta - u_z D_y\theta,\\
\eta^z =& D_z\eta - u_t D_z\tau - u_x D_z\xi - u_y D_z\zeta - u_z D_z\theta,\\
\eta^{xx} =& D_x\eta^x - u_{xt} D_x\tau - u_{xx} D_x\xi - u_{xy} D_x\zeta - u_{xz} D_x\theta,\\
\eta^{yy} =& D_y\eta^y - u_{yt} D_y\tau - u_{xy} D_y\xi - u_{yy} D_y\zeta - u_{yz} D_y\theta,\\
\eta^{zz} =& D_z\eta^z - u_{zt} D_z\tau - u_{xz} D_z\xi - u_{yz} D_z\zeta - u_{zz} D_z\theta,\\
\eta^{xxx} =& D_x\eta^{xx} - u_{txx} D_x\tau - u_{xxx} D_x\xi - u_{yxx} D_x\zeta - u_{zxx} D_x\theta,\\
\eta^{xyy} =& D_x\eta^{yy} - u_{tyy} D_x\tau - u_{xyy} D_x\xi - u_{yyy} D_x\zeta - u_{zyy} D_x\theta,\\
\eta^{xzz} =& D_x\eta^{zz} - u_{tzz} D_x\tau - u_{xzz} D_x\xi - u_{yzz} D_x\zeta - u_{zzz} D_x\theta,
\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}
\eta^{\alpha,t} =& \frac{\partial^\alpha\eta}{\partial t^\alpha} + (\eta_u - \alpha D_t\tau)\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n\xi D_t^{\alpha-n}u_x - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n\zeta D_t^{\alpha-n}u_y - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n\theta D_t^{\alpha-n}u_z\\
&+ \sum_{n=1}^{\infty}\Big[\binom{\alpha}{n}\frac{\partial^n\eta_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}\tau\Big] D_t^{\alpha-n}u + \mu,
\end{aligned} \tag{2.5}$$

with

$$\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{t^{n-\alpha}(-u)^r}{k!\Gamma(n+1-\alpha)}\frac{\partial^m u^{k-r}}{\partial t^m}\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}.$$

Note that D_t, D_x, D_y and D_z are the total derivative with respect to t, x, y and z, respectively.

Remark 2.1. The infinitesimal transformations (2.1) should conserve the structure of the Riemann-Liouville fractional derivative operator, of which the lower limit in the integral is fixed. Therefore, the manifold $t = 0$ should be invariant with respect to such transformations. The invariance condition arrives at

$$\tau(t, x, y, z, u)|_{t=0} = 0. \tag{2.6}$$

Remark 2.2. From the expression of μ, if the infinitesimal η be linear with respect to the variable u, then $\mu = 0$, that is,

$$\frac{\partial^2\eta}{\partial u^2} = 0. \tag{2.7}$$

The one-parameter Lie symmetry transformations (2.1) are admitted by Eq. (1.2), if the following invariance criterion holds:

$$Pr^{(\alpha,3)}X\big(D_t^\alpha u + (u_{xx} + u_{yy} + u_{zz} + u^p)_x\big)|_{(1.2)} = 0, \tag{2.8}$$

which can be rewritten as

$$\big(\eta^{\alpha,t} + p(p-1)u^{p-2}u_x\eta + pu^{p-1}\eta^x + \eta^{xxx} + \eta^{xyy} + \eta^{xzz}\big)|_{(1.2)} = 0. \tag{2.9}$$

Putting $\eta^{\alpha,t}$, η^x, η^{xxx}, η^{xyy} and η^{xzz} into (2.9), a over-determined system of differential equations can be obtained by equating the coefficients of various derivatives of u to zero. Then, by solving the over-determined system, with the conditions (2.6) and (2.7), the following infinitesimals can be obtained:

$$\begin{aligned} &\tau = c_1 t, \quad \xi = \frac{\alpha}{3}c_1 x + c_3, \quad \zeta = \frac{\alpha}{3}c_1 y + c_2 z + c_4, \\ &\theta = \frac{\alpha}{3}c_1 z - c_2 y + c_5, \quad \eta = -\frac{2\alpha}{3(p-1)}c_1 u, \end{aligned} \tag{2.10}$$

where c_1, c_2, c_3, c_4 and c_5 are arbitrary constants. So Eq. (1.2) admitted the five-dimension Lie algebra L^5 spanned by

$$\begin{aligned} &X_1 = t\frac{\partial}{\partial t} + \frac{\alpha}{3}x\frac{\partial}{\partial x} + \frac{\alpha}{3}y\frac{\partial}{\partial y} + \frac{\alpha}{3}z\frac{\partial}{\partial z} - \frac{2\alpha}{3(p-1)}u\frac{\partial}{\partial u}, \\ &X_2 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad X_3 = \frac{\partial}{\partial x}, \quad X_4 = \frac{\partial}{\partial y}, \quad X_5 = \frac{\partial}{\partial z}. \end{aligned} \tag{2.11}$$

The corresponding one-parameter (ϵ) continuous transformation groups are defined by

$$h_i : \ (t, x, y, z, u) \to (t^*, x^*, y^*, z^*, u^*), \quad i = 1, 2, 3, 4, 5. \tag{2.12}$$

By solving the following Lie equations:

$$\begin{aligned} &\frac{\mathrm{d}(t^*, x^*, y^*, z^*, u^*)}{\mathrm{d}\epsilon} = (\tau, \xi, \zeta, \theta, \eta), \\ &(t^*, x^*, y^*, z^*, u^*)|_{\epsilon=0} = (t, x, y, z, u), \end{aligned} \tag{2.13}$$

we can get the following symmetry transformation groups corresponding to X_i $(i = 1, 2, 3, 4, 5)$:

$$\begin{aligned} &h_1 : \ (t, x, y, z, u) \to (e^{\epsilon}t, e^{\frac{\alpha}{3}\epsilon}x, e^{\frac{\alpha}{3}\epsilon}y, e^{\frac{\alpha}{3}\epsilon}z, e^{-\frac{2\alpha}{3(p-1)}\epsilon}u), \\ &h_2 : \ (t, x, y, z, u) \to (t, x, y + \epsilon z, z - \epsilon y, u), \\ &h_3 : \ (t, x, y, z, u) \to (t, x + \epsilon, y, z, u), \\ &h_4 : \ (t, x, y, z, u) \to (t, x, y + \epsilon, z, u), \\ &h_5 : \ (t, x, y, z, u) \to (t, x, y, z + \epsilon, u), \end{aligned} \tag{2.14}$$

where ϵ is any small real parameter.

3 Conservation laws of Eq. (1.2)

In this section, we will construct conservation laws for the obtained Lie symmetries (2.11) by using the generalization of the Noether operators and the new conservation theorem [18, 17]. The equation (1.2) are denoted as

$$F = D_t^{\alpha} u + (u_{xx} + u_{yy} + u_{zz} + u^p)_x = 0, \tag{3.1}$$

of which the formal Lagrangian is given by

$$\mathcal{L} = v(t, x, y, z)F = v(t, x, y, z)(D_t^{\alpha} u + (u_{xx} + u_{yy} + u_{zz} + u^p)_x), \tag{3.2}$$

where $v(t, x, y, z)$ is a new dependent variable. The Euler-Lagrange operator is

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (D_t^{\alpha})^* \frac{\partial}{\partial(D_t^{\alpha} u)} + \sum_{s=1}^{\infty}(-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1 \cdots i_s}}, \tag{3.3}$$

where $(D_t^{\alpha})^*$ is the adjoint operator of D_t^{α}. It is defined by the right-sided of Caputo fractional derivative, i.e.

$$(D_t^{\alpha})^* f(t, x) \equiv \ {}_t^c D_T^{\alpha} f(t, x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_t^T \frac{1}{(t-s)^{\alpha-n+1}} \frac{\partial^n}{\partial s^n} f(s, x)\mathrm{d}s, & n-1 < \alpha < n, n \in \mathbb{N} \\ D_t^n f(t, x), & \alpha = n \in \mathbb{N}. \end{cases}$$

The adjoint equation to Eq. (1.2) is given by

$$F^* = \frac{\delta \mathcal{L}}{\delta u} = (D_t^\alpha)^* v - pu^{p-1} v_x - v_{xxx} - v_{xyy} - v_{xzz} = 0. \tag{3.4}$$

Next we will use the above adjoint equation and the new conservation theorem to construct conservation laws of Eq. (1.2). From the classical definition of the conservation laws, a vector $C = (C^t, C^x, C^y, C^z)$ is called a conserved vector for the governing equation if it satisfies the conservation equation $[D_t C^t + D_x C^x + D_y C^y + D_z C^z]_{F=0} = 0$. By using Noether theorem, the components of conserved vector can be obtained. Firstly, from the fundamental operator identity, i.e.

$$Pr^{(\alpha,3)} X + D_t \tau \cdot \mathcal{I} + D_x \xi \cdot \mathcal{I} + D_y \zeta \cdot \mathcal{I} + D_z \theta \cdot \mathcal{I} = W \cdot \frac{\delta}{\delta u} + D_t \mathcal{N}^t + D_x \mathcal{N}^x + D_y \mathcal{N}^y + D_z \mathcal{N}^z, \tag{3.5}$$

where $Pr^{(\alpha,3)} X$ is mentioned in (2.3), $\mathcal{I}$ is the identity operator and $W = \eta - \tau u_t - \xi u_x - \zeta u_y - \theta u_z$ is the characteristic for group generator X, we can get the Noether operators as follows:

$$\mathcal{N}^t = \tau \mathcal{I} + \sum_{k=0}^{n-1} (-1)^k D_t^{\alpha-1-k}(W) D_t^k \frac{\partial}{\partial(D_t^\alpha u)} - (-1)^n J(W, D_t^n \frac{\partial}{\partial(D_t^\alpha u)}), \tag{3.6}$$

$$\begin{aligned}\mathcal{N}^x =& \xi \mathcal{I} + W\Big(\frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xxx}} + D_y^2 \frac{\partial}{\partial u_{xyy}} + D_z^2 \frac{\partial}{\partial u_{xzz}}\Big) - D_x(W) D_x \frac{\partial}{\partial u_{xxx}} - D_y(W) D_y \frac{\partial}{\partial u_{xyy}} - D_z(W) D_z \frac{\partial}{\partial u_{xzz}} \\ &+ D_x^2(W) \frac{\partial}{\partial u_{xxx}} + D_y^2(W) \frac{\partial}{\partial u_{xyy}} + D_z^2(W) \frac{\partial}{\partial u_{xzz}},\end{aligned} \tag{3.7}$$

$$\mathcal{N}^y = \zeta \mathcal{I} + W D_x D_y \frac{\partial}{\partial u_{xyy}} - D_x(W) D_y \frac{\partial}{\partial u_{xyy}} + D_x D_y(W) \frac{\partial}{\partial u_{xyy}}, \tag{3.8}$$

$$\mathcal{N}^z = \theta \mathcal{I} + W D_x D_z \frac{\partial}{\partial u_{xzz}} - D_x(W) D_z \frac{\partial}{\partial u_{xzz}} + D_x D_z(W) \frac{\partial}{\partial u_{xzz}}, \tag{3.9}$$

where $n = [\alpha] + 1$ and J is given by

$$J(f, g) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \int_t^T \frac{f(\tau, x, y, z) g(\theta, x, y, z)}{(\theta - \tau)^{\alpha+1-n}} \mathrm{d}\theta \mathrm{d}\tau. \tag{3.10}$$

The components of the conserved vector are defined by

$$C^t = \mathcal{N}^t \mathcal{L}, \quad C^x = \mathcal{N}^x \mathcal{L}, \quad C^y = \mathcal{N}^y \mathcal{L}, \quad C^z = \mathcal{N}^z \mathcal{L}. \tag{3.11}$$

Case 1: $X_1 = t\frac{\partial}{\partial t} + \frac{\alpha}{3} x \frac{\partial}{\partial x} + \frac{\alpha}{3} y \frac{\partial}{\partial y} + \frac{\alpha}{3} z \frac{\partial}{\partial z} - \frac{2\alpha}{3(p-1)} u \frac{\partial}{\partial u}$

The characteristic of X_1 is

$$W = -\frac{2\alpha}{3(p-1)} u - t u_t - \frac{\alpha}{3} x u_x - \frac{\alpha}{3} y u_y - \frac{\alpha}{3} z u_z. \tag{3.12}$$

Therefore, for $0 < \alpha < 1$,

$$\begin{aligned}C^t =& v D_t^{\alpha-1}(W) + J(W, v_t) = v D_t^{\alpha-1}(-\frac{2\alpha}{3(p-1)} u - t u_t - \frac{\alpha}{3} x u_x - \frac{\alpha}{3} y u_y - \frac{\alpha}{3} z u_z) \\ &+ J(-\frac{2\alpha}{3(p-1)} u - t u_t - \frac{\alpha}{3} x u_x - \frac{\alpha}{3} y u_y - \frac{\alpha}{3} z u_z, v_t),\end{aligned} \tag{3.13}$$

$$\begin{aligned}C^x &= W(pu^{p-1}v + v_{xx} + v_{yy} + v_{zz}) - v_x D_x(W) - v_y D_y(W) - v_z D_z(W) + v \\ &\cdot (D_x^2(W) + D_y^2(W) + D_z^2(W)) = -(\frac{2\alpha}{3(p-1)}u + tu_t + \frac{\alpha}{3}(xu_x + yu_y + zu_z)) \\ &(pu^{p-1}v + v_{xx} + v_{yy} + v_{zz}) + \frac{(p+1)\alpha}{3(p-1)}(u_x v_x + u_y v_y + u_z v_z) + t(u_{xt}v_x + u_{yt}v_y \\ &+ u_{zt}v_z) + \frac{\alpha}{3}(x(u_{xx}v_x + u_{xy}v_y + u_{xz}v_z) + y(u_{xy}v_x + u_{yy}v_y + u_{yz}v_z) + z(u_{xz}v_x \\ &+ u_{yz}v_y + u_{zz}v_z)) + v(\frac{2p\alpha}{3(p-1)}(u_{xx} + u_{yy} + u_{zz}) + t(u_{xxt} + u_{yyt} + u_{zzt}) \\ &+ \frac{\alpha}{3}(x(u_{xxx} + u_{xyy} + u_{xzz}) + y(u_{xxy} + u_{yyy} + u_{yzz}) + z(u_{xxz} + u_{yyz} + u_{zzz}))),\end{aligned} \tag{3.14}$$

$$\begin{aligned}C^y =& v_{xy}W - v_y D_x(W) + vD_x D_y(W) = -v_{xy}(\frac{2\alpha}{3(p-1)}u + tu_t + \frac{\alpha}{3}xu_x + \frac{\alpha}{3}yu_y \\ &+ \frac{\alpha}{3}zu_z) + v_y(\frac{(p+1)\alpha}{3(p-1)}u_x + tu_{xt} + \frac{\alpha}{3}xu_{xx} + \frac{\alpha}{3}yu_{xy} + \frac{\alpha}{3}zu_{xz}) \\ &- v(\frac{2p\alpha}{3(p-1)}u_{xy} + tu_{xyt} + \frac{\alpha}{3}xu_{xxy} + \frac{\alpha}{3}yu_{xyy} + \frac{\alpha}{3}zu_{xyz}),\end{aligned} \tag{3.15}$$

$$\begin{aligned}C^z =& v_{xz}W - v_z D_x(W) + vD_x D_z(W) = -v_{xz}(\frac{2\alpha}{3(p-1)}u + tu_t + \frac{\alpha}{3}xu_x + \frac{\alpha}{3}yu_y \\ &+ \frac{\alpha}{3}zu_z) + v_z(\frac{(p+1)\alpha}{3(p-1)}u_x + tu_{xt} + \frac{\alpha}{3}xu_{xx} + \frac{\alpha}{3}yu_{xy} + \frac{\alpha}{3}zu_{xz}) \\ &- v(\frac{2p\alpha}{3(p-1)}u_{xz} + tu_{xzt} + \frac{\alpha}{3}xu_{xxz} + \frac{\alpha}{3}yu_{xyz} + \frac{\alpha}{3}zu_{xzz}).\end{aligned} \tag{3.16}$$

Case 2: $X_2 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$

The characteristic of X_2 is

$$W = -zu_y + yu_z. \tag{3.17}$$

Therefore, for $0 < \alpha < 1$,

$$C^t = vD_t^{\alpha-1}(-zu_y + yu_z) + J(-zu_y + yu_z, v_t), \tag{3.18}$$

$$\begin{aligned}C^x =& (-zu_y + yu_z)(pu^{p-1}v + v_{xx} + v_{yy} + v_{zz}) - v_x(-zu_{xy} + yu_{xz}) \\ &- v_y(u_z - zu_{yy} + yu_{yz}) - v_z(-u_y - zu_{yz} + yu_{zz}) + v(-zu_{xxy} \\ &+ yu_{xxz} + 2u_{yz} - zu_{yyy} + yu_{yyz} - 2u_{yz} - zu_{yzz} + yu_{zzz}),\end{aligned} \tag{3.19}$$

$$C^y = v_{xy}(-zu_y + yu_z) - v_y(-zu_{xy} + yu_{xz}) + v(u_{xz} - zu_{xyy} + yu_{xyz}), \tag{3.20}$$

$$C^z = v_{xz}(-zu_y + yu_z) - v_z(-zu_{xy} + yu_{xz}) + v(-u_{xy} - zu_{xyz} + yu_{xzz}). \tag{3.21}$$

Case 3: $X_3 = \frac{\partial}{\partial x}$

The characteristic of X_3 is

$$W = -u_x. \tag{3.22}$$

Therefore, for $0 < \alpha < 1$,

$$C^t = -vD_t^{\alpha-1}(u_x) - J(u_x, v_t), \tag{3.23}$$

$$C^x = -u_x(pu^{p-1}v + v_{xx} + v_{yy} + v_{zz}) + u_{xx}v_x + u_{xy}v_y + u_{xz}v_z - v(u_{xxx} + u_{xyy} + u_{xzz}), \tag{3.24}$$

$$C^y = -u_x v_{xy} + u_{xx}v_y - vu_{xxy}, \tag{3.25}$$

$$C^z = -u_x v_{xz} + u_{xx}v_z - vu_{xxz}. \tag{3.26}$$

Case 4: $X_4 = \frac{\partial}{\partial y}$
The characteristic of X_4 is

$$W = -u_y. \tag{3.27}$$

Therefore, for $0 < \alpha < 1$,

$$C^t = -vD_t^{\alpha-1}(u_y) - J(u_y, v_t), \tag{3.28}$$

$$C^x = -u_y(pu^{p-1}v + v_{xx} + v_{yy} + v_{zz}) + u_{xy}v_x + u_{yy}v_y + u_{yz}v_z - v(u_{xxy} + u_{yyy} + u_{yzz}), \tag{3.29}$$

$$C^y = -u_y v_{xy} + u_{xy}v_y - vu_{xyy}, \tag{3.30}$$

$$C^z = -u_y v_{xz} + u_{xy}v_z - vu_{xyz}. \tag{3.31}$$

Case 5: $X_5 = \frac{\partial}{\partial z}$
The characteristic of X_5 is

$$W = -u_z. \tag{3.32}$$

Therefore, for $0 < \alpha < 1$,

$$C^t = -vD_t^{\alpha-1}(u_z) - J(u_z, v_t), \tag{3.33}$$

$$C^x = -u_z(pu^{p-1}v + v_{xx} + v_{yy} + v_{zz}) + u_{xz}v_x + u_{yz}v_y + u_{zz}v_z - v(u_{xxz} + u_{yyz} + u_{zzz}), \tag{3.34}$$

$$C^y = -u_z v_{xy} + u_{xz}v_y - vu_{xyz}, \tag{3.35}$$

$$C^z = -u_z v_{xz} + u_{xz}v_z - vu_{xzz}. \tag{3.36}$$

4 One-dimensional optimal system of Eq. (1.2)

In this section, the one-dimensional optimal system of Eq. (1.2) is obtained by the method introduced in [30]. It is easy to check that the group generators in (2.11) are closed under the Lie bracket defined by

$$[X_i, X_j] = X_iX_j - X_jX_i, \quad (i, j = 1, 2, 3, 4, 5). \tag{4.1}$$

The commutation relationships of these group generators can be seen in Table 1.

Table 1: The Commutation Table of Lie Algebra.

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	0	$-\frac{\alpha}{3}X_3$	$-\frac{\alpha}{3}X_4$	$-\frac{\alpha}{3}X_5$
X_2	0	0	0	X_5	-X_4
X_3	$\frac{\alpha}{3}X_3$	0	0	0	0
X_4	$\frac{\alpha}{3}X_4$	-X_5	0	0	0
X_5	$\frac{\alpha}{3}X_5$	X_4	0	0	0

Then we consider the action of the adjoint operator which is given by the Lie series

$$Ad(exp(\varepsilon X_i))X_j = X_j - \varepsilon[X_i, X_j] + \frac{\varepsilon^2}{2}[X_i, [X_i, X_j]] - \cdots, \tag{4.2}$$

where ε is an arbitrary parameter. According to (4.2), we calculate the adjoint action of the group generators in (2.11) which is listed in Table 2.

Table 2: The Adjoint Representation of Lie Algebra.

Ad	X_1	X_2	X_3	X_4	X_5
X_1	X_1	X_2	$e^{\frac{\alpha}{3}\varepsilon}X_3$	$e^{\frac{\alpha}{3}\varepsilon}X_4$	$e^{\frac{\alpha}{3}\varepsilon}X_5$
X_2	X_1	X_2	X_3	$\cos\varepsilon\ X_4-\sin\varepsilon\ X_5$	$\cos\varepsilon\ X_5+\sin\varepsilon\ X_4$
X_3	$X_1-\frac{\alpha}{3}\varepsilon X_3$	X_2	X_3	X_4	X_5
X_4	$X_1-\frac{\alpha}{3}\varepsilon X_4$	$X_2+\varepsilon X_5$	X_3	X_4	X_5
X_5	$X_1-\frac{\alpha}{3}\varepsilon X_5$	$X_2-\varepsilon X_4$	X_3	X_4	X_5

Assuming $X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5$, from Table 2, we can get the following expression:

$$Ad(exp(\varepsilon_1 X_1))X = a_1X_1 + a_2X_2 + a_3e^{\frac{\alpha}{3}\varepsilon}X_3 + a_4e^{\frac{\alpha}{3}\varepsilon}X_4 + a_5e^{\frac{\alpha}{3}\varepsilon}X_5, \tag{4.3}$$

which can be written as

$$Ad(exp(\varepsilon_1 X_1))X = (a_1, a_2, a_3, a_4, a_5)A_1(X_1, X_2, X_3, X_4, X_5)^T, \tag{4.4}$$

where

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & e^{\frac{\alpha}{3}\varepsilon_1} & 0 & 0 \\ 0 & 0 & 0 & e^{\frac{\alpha}{3}\varepsilon_1} & 0 \\ 0 & 0 & 0 & 0 & e^{\frac{\alpha}{3}\varepsilon_1} \end{pmatrix}.$$

Similar to A_1, we get

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cos\varepsilon_2 & -\sin\varepsilon_2 \\ 0 & 0 & 0 & \sin\varepsilon_2 & \cos\varepsilon_2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 & -\frac{\alpha}{3}\varepsilon_3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 1 & 0 & 0 & -\frac{\alpha}{3}\varepsilon_4 & 0 \\ 0 & 1 & 0 & 0 & \varepsilon_4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{\alpha}{3}\varepsilon_5 \\ 0 & 1 & 0 & -\varepsilon_5 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then the general adjoint matrix is constructed by

$$A = A_1A_2A_3A_4A_5 = \begin{pmatrix} 1 & 0 & -\frac{\alpha}{3}\varepsilon_3 & -\frac{\alpha}{3}\varepsilon_4 & -\frac{\alpha}{3}\varepsilon_5 \\ 0 & 1 & 0 & -\varepsilon_5 & \varepsilon_4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & e^{\frac{\alpha}{3}\varepsilon_1}\cos\varepsilon_2 & -e^{\frac{\alpha}{3}\varepsilon_1}\sin\varepsilon_2 \\ 0 & 0 & 0 & e^{\frac{\alpha}{3}\varepsilon_1}\sin\varepsilon_2 & e^{\frac{\alpha}{3}\varepsilon_1}\cos\varepsilon_2 \end{pmatrix}.$$

To conveniently derive invariant functions, we use these matrixes $(A_1, A_2, A_3, A_4, A_5)$ to construct Table 3, where $Ad(exp(\varepsilon X_i))X$ is marked P_i $(i = 1, 2, \cdots, 5)$.

Lemma 4.1. For vector $X = \sum_{i=1}^{5} a_iX_i$ with $a_i \in \mathbf{R}$, the invariant function of symmetry algebra L^5 is obtained as $\Xi = F(a_1, a_2)$, where F is an arbitrary function.

Proof . Consider $g = \exp(\varepsilon Y)$ with $Y = \sum_{i=1}^{5} b_iX_i$ is any element of Lie group G generated by L^5. A real function Ξ on the Lie algebra L^5 is called an invariant if it satisfies the following condition:

$$\Xi[Ad(g)X] = \Xi(X) \quad \text{for all} \quad X \in L^5. \tag{4.5}$$

Table 3: Table for Construction of Invariant Functions.

	Coeff. X_1	Coeff. X_2	Coeff. X_3	Coeff. X_4	Coeff. X_5
P_1	a_1	a_2	$e^{\frac{\alpha}{3}\varepsilon}a_3$	$e^{\frac{\alpha}{3}\varepsilon}a_4$	$e^{\frac{\alpha}{3}\varepsilon}a_5$
P_2	a_1	a_2	a_3	$\cos\varepsilon\ a_4+\sin\varepsilon\ a_5$	$\cos\varepsilon\ a_5-\sin\varepsilon\ a_4$
P_3	a_1	a_2	$a_3-\frac{\alpha}{3}\varepsilon a_1$	a_4	a_5
P_4	a_1	a_2	a_3	$a_4-\frac{\alpha}{3}\varepsilon a_1$	$a_5+\varepsilon a_2$
P_5	a_1	a_2	a_3	$a_4-\varepsilon a_2$	$a_5-\frac{\alpha}{3}\varepsilon a_1$

That is,

$$\begin{aligned} Ad(exp(\varepsilon Y))X =&e^{-\varepsilon Y}Xe^{\varepsilon Y}=X-\varepsilon[Y,X]+\frac{\varepsilon^2}{2}[Y,[Y,X]]-\cdots \\ =&(a_1X_1+a_2X_2+a_3X_3+a_4X_4+a_5X_5)-(\vartheta_1X_1 \\ &+\vartheta_2X_2+\vartheta_3X_3+\vartheta_4X_4+\vartheta_5X_5)\varepsilon+O(\varepsilon^2), \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} &\vartheta_1=0,\quad \vartheta_2=0,\quad \vartheta_3=\frac{\alpha}{3}(b_3a_1-b_1a_3),\quad \vartheta_4=\frac{\alpha}{3}(b_4a_1-b_1a_4)+(b_5a_2-b_2a_5), \\ &\vartheta_5=\frac{\alpha}{3}(b_5a_1-b_1a_5)+(b_2a_4-b_4a_2). \end{aligned} \tag{4.7}$$

Then the condition (4.5) arrives at

$$\Xi(a_1,a_2,a_3,a_4,a_5)=\Xi(a_1-\varepsilon\vartheta_1,a_2-\varepsilon\vartheta_2,a_3-\varepsilon\vartheta_3,a_4-\varepsilon\vartheta_4,a_5-\varepsilon\vartheta_5).$$

By differentiating the right hand side of the above equation with respect to ε and then setting ε=0, we collect the coefficients of b_i to obtain the following system of first-order linear PDEs with constant coefficients:

$$\begin{aligned} &\frac{\alpha}{3}a_3\frac{\partial\Xi}{\partial a_3}+\frac{\alpha}{3}a_4\frac{\partial\Xi}{\partial a_4}+\frac{\alpha}{3}a_5\frac{\partial\Xi}{\partial a_5}=0, \\ &a_5\frac{\partial\Xi}{\partial a_4}-a_4\frac{\partial\Xi}{\partial a_5}=0, \\ &-\frac{\alpha}{3}a_1\frac{\partial\Xi}{\partial a_3}=0, \\ &-\frac{\alpha}{3}a_1\frac{\partial\Xi}{\partial a_4}+a_2\frac{\partial\Xi}{\partial a_5}=0, \\ &-a_2\frac{\partial\Xi}{\partial a_4}+\frac{\alpha}{3}a_1\frac{\partial\Xi}{\partial a_5}=0. \end{aligned} \tag{4.8}$$

On solving all above equations, we obtain the general invariant functions of Lie algebra L^5 with the form

$$\Xi(a_1,a_2,a_3,a_4,a_5)=F(a_1,a_2),$$

where F is an arbitrary function. □

Lemma 4.2. The Killing form of Lie algebra L^5 is $K\langle X,X\rangle=\frac{\alpha^2}{3}a_1^2$, which is a invariant function.

Proof . The Killing form of the symmetry algebra is defined as

$$K\langle X,X\rangle=\text{Trace}(adX\cdot adX), \tag{4.9}$$

where

$$adX=\begin{pmatrix} 0 & 0 & \frac{\alpha}{3}a_3 & \frac{\alpha}{3}a_4 & \frac{\alpha}{3}a_5 \\ 0 & 0 & 0 & a_5 & -a_4 \\ 0 & 0 & -\frac{\alpha}{3}a_1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\alpha}{3}a_1 & a_2 \\ 0 & 0 & 0 & -a_2 & \frac{\alpha}{3}a_1 \end{pmatrix}.$$

Therefore, $K\langle X, X\rangle = \frac{\alpha^2}{3}a_1^2$. □

Theorem 4.3. Based on Lemmas 4.1-4.2, the optimal system for the one-dimensional subalgebras of Eq. (1.2) can be spanned by

$$X_1,\ X_2,\ X_3,\ X_4,\ X_5,\ bX_3+dX_4,\ bX_3+dX_5,\ bX_4+dX_5,\ bX_3+cX_4+dX_5, \tag{4.10}$$

where b, c and d are free parameter.

Proof . The detailed process of this proof is similar to the literatures [30, 4, 13, 36, 23]. □

5 Similarity reductions and exact solutions of Eq. (1.2)

In this section, Eq. (1.2) can be reduced to some different (2+1)-, (1+1)- and (0+1)- dimensional time fractional differential equations by the obtained one-dimensional optimal system. In what follows, we consider the following cases.

Case 1: X_1

The characteristic equation corresponding to the group generator X_1 is

$$\frac{\mathrm{d}t}{t} = \frac{3\mathrm{d}x}{\alpha x} = \frac{3\mathrm{d}y}{\alpha y} = \frac{3\mathrm{d}z}{\alpha z} = \frac{-3(p-1)\mathrm{d}u}{2\alpha u}, \tag{5.1}$$

from which, we obtain the similarity variables $xt^{-\frac{\alpha}{3}}$, $yt^{-\frac{\alpha}{3}}$, $zt^{-\frac{\alpha}{3}}$ and $ut^{\frac{2\alpha}{3(p-1)}}$. So we get the invariant solution of Eq. (1.2) as follows:

$$u = t^{-\frac{2\alpha}{3(p-1)}} f(\omega_1, \omega_2, \omega_3),\quad \omega_1 = xt^{-\frac{\alpha}{3}},\quad \omega_2 = yt^{-\frac{\alpha}{3}},\quad \omega_3 = zt^{-\frac{\alpha}{3}}. \tag{5.2}$$

Theorem 5.1. The similarity transformation $u = t^{-\frac{2\alpha}{3(p-1)}} f(\omega_1, \omega_2, \omega_3)$ with the similarity variables $\omega_1 = xt^{-\frac{\alpha}{3}}$, $\omega_2 = yt^{-\frac{\alpha}{3}}$, $\omega_3 = zt^{-\frac{\alpha}{3}}$ reduce Eq. (1.2) to the (2+1)-dimensional time fractional partial differential equation given by

$$(\mathcal{P}^{1-\frac{(3p-1)\alpha}{3(p-1)},\alpha}_{\frac{3}{\alpha},\frac{3}{\alpha},\frac{3}{\alpha}} f)(\omega_1, \omega_2, \omega_3) + pf^{p-1}f_{\omega_1} + f_{\omega_1\omega_1\omega_1} + f_{\omega_1\omega_2\omega_2} + f_{\omega_1\omega_3\omega_3} = 0. \tag{5.3}$$

where $(\mathcal{P}^{\iota,\kappa}_{\delta_1,\delta_2})$ is the left-hand Erdélyi-Kober fractional differential operator defined by

$$\begin{aligned}(\mathcal{P}^{\iota,\kappa}_{\delta_1,\delta_2,\delta_3}\psi)(\omega_1, \omega_2, \omega_3) :=& \prod_{j=0}^{m-1}(\iota + j - \frac{1}{\delta_1}\omega_1\frac{\mathrm{d}}{\mathrm{d}\omega_1} - \frac{1}{\delta_2}\omega_2\frac{\mathrm{d}}{\mathrm{d}\omega_2} - \frac{1}{\delta_2}\omega_3\frac{\mathrm{d}}{\mathrm{d}\omega_3})\\ &\times (\mathcal{K}^{\iota+\kappa,m-\kappa}_{\delta_1,\delta_2,\delta_3}\psi)(\omega_1, \omega_2, \omega_3),\ m = \begin{cases}[\kappa]+1, & \text{if } \kappa \notin \mathbb{N},\\ \kappa, & \text{if } \kappa \in \mathbb{N},\end{cases}\end{aligned} \tag{5.4}$$

where

$$(\mathcal{K}^{\iota,\kappa}_{\delta_1,\delta_2,\delta_3}\psi)(\omega_1, \omega_2, \omega_3) := \begin{cases}\frac{1}{\Gamma(\kappa)}\int_1^\infty (s-1)^{\kappa-1}s^{-(\iota+\kappa)}\psi(\omega_1 s^{\frac{1}{\delta_1}}, \omega_2 s^{\frac{1}{\delta_2}}, \omega_3 s^{\frac{1}{\delta_3}})\mathrm{d}s, & \kappa > 0,\\ \psi(\omega_1, \omega_2, \omega_3), & \kappa = 0,\end{cases} \tag{5.5}$$

is the left-hand Erdélyi-Kober fractional integral operator.

Proof . For $0 < \alpha < 1$, the Riemann-Liouville time fractional derivative of $u(t, x, y, z)$ can be obtained as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^\alpha}{\partial t^\alpha}(t^{-\frac{2\alpha}{3(p-1)}} f(\omega_1, \omega_2, \omega_3)) = \frac{\partial}{\partial t}\Big[\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}s^{-\frac{2\alpha}{3(p-1)}} f(xs^{-\frac{\alpha}{3}}, ys^{-\frac{\alpha}{3}}, zs^{-\frac{\alpha}{3}})\mathrm{d}s\Big].$$

Assuming $r = \frac{t}{s}$, we have

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} =& \frac{\partial}{\partial t}\Big[\frac{t^{1-\frac{(3p-1)\alpha}{3(p-1)}}}{\Gamma(1-\alpha)}\int_1^\infty (r-1)^{-\alpha}r^{\frac{(3p-1)\alpha}{3(p-1)}-2} f(\omega_1 r^{\frac{\alpha}{3}}, \omega_2 r^{\frac{\alpha}{3}}, \omega_3 r^{\frac{\alpha}{3}})\mathrm{d}r\Big]\\ =& \frac{\partial}{\partial t}\Big[t^{1-\frac{(3p-1)\alpha}{3(p-1)}}(\mathcal{K}^{1-\frac{2\alpha}{3(p-1)},1-\alpha}_{\frac{3}{\alpha},\frac{3}{\alpha},\frac{3}{\alpha}} f)(\omega_1, \omega_2, \omega_3)\Big].\end{aligned}$$

Because of $\omega_1 = xt^{-\frac{\alpha}{3}}$, $\omega_2 = yt^{-\frac{\alpha}{3}}$ and $\omega_3 = zt^{-\frac{\alpha}{3}}$, the following relation holds:

$$t\frac{\partial}{\partial t}\psi(\omega_1,\omega_2,\omega_3) = -\frac{\alpha}{3}\omega_1\frac{\mathrm{d}}{\mathrm{d}\omega_1}\psi - \frac{\alpha}{3}\omega_2\frac{\mathrm{d}}{\mathrm{d}\omega_2}\psi - \frac{\alpha}{3}\omega_3\frac{\mathrm{d}}{\mathrm{d}\omega_3}\psi.$$

Hence, we arrive at

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} =& t^{-\frac{(3p-1)\alpha}{3(p-1)}}\Big[(1-\frac{(3p-1)\alpha}{3(p-1)} - \frac{\alpha}{3}\omega_1\frac{\mathrm{d}}{\mathrm{d}\omega_1} - \frac{\alpha}{3}\omega_2\frac{\mathrm{d}}{\mathrm{d}\omega_2} - \frac{\alpha}{3}\omega_3\frac{\mathrm{d}}{\mathrm{d}\omega_3})\\ &\times(\mathcal{K}^{1-\frac{2\alpha}{3(p-1)},1-\alpha}_{\frac{3}{\alpha},\frac{3}{\alpha},\frac{3}{\alpha}}f)(\omega_1,\omega_2,\omega_3)\Big] = t^{-\frac{(3p-1)\alpha}{3(p-1)}}(\mathcal{P}^{1-\frac{(3p-1)\alpha}{3(p-1)},\alpha}_{\frac{3}{\alpha},\frac{3}{\alpha},\frac{3}{\alpha}}f)(\omega_1,\omega_2,\omega_3).\end{aligned}$$

Meanwhile,

$$(u_{xx}+u_{yy}+u_{zz}+u^p)_x = t^{-\frac{(3p-1)\alpha}{3(p-1)}}(pf^{p-1}f_{\omega_1} + f_{\omega_1\omega_1\omega_1} + f_{\omega_1\omega_2\omega_2} + f_{\omega_1\omega_3\omega_3}).$$

This completes the proof. □

Next we use the power series method introduced in [41] to derive the power series solution of (5.3). Assuming

$$f(\omega_1,\omega_2,\omega_3) = f(\omega) = \sum_{k=0}^{\infty} a_k\omega^k, \quad \omega = C_1\omega_1 + C_2\omega_2 + C_3\omega_3, \tag{5.6}$$

where a_k are constants to be known later, then one can get

$$f'(\omega) = \sum_{k=0}^{\infty}(k+1)a_{k+1}\omega^k, \quad f'''(\omega) = \sum_{k=0}^{\infty}(k+3)(k+2)(k+1)a_{k+3}\omega^k. \tag{5.7}$$

From [41], we have

$$(\mathcal{P}^{1-\frac{(3p-1)\alpha}{3(p-1)},\alpha}_{\frac{3}{\alpha},\frac{3}{\alpha},\frac{3}{\alpha}}f)(\omega_1,\omega_2,\omega_3) = (\mathcal{P}^{1-\frac{(3p-1)\alpha}{3(p-1)},\alpha}_{\frac{3}{\alpha}}f)(\omega) = \sum_{k=0}^{\infty}\frac{\Gamma(1-\frac{(kp-k+2)\alpha}{3(p-1)})}{\Gamma(1-\frac{((k+3)p-k-1)\alpha}{3(p-1)})}a_k\omega^k. \tag{5.8}$$

Substituting (5.6)-(5.8) into (5.3) arrives at the following equation:

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{\Gamma(1-\frac{(kp-k+2)\alpha}{3(p-1)})}{\Gamma(1-\frac{((k+3)p-k-1)\alpha}{3(p-1)})}a_k\omega^k + pC_1\sum_{k=0}^{\infty}(k+1)a_{k+1}\omega^k\big(\sum_{k=0}^{\infty}a_k\omega^k\big)^{p-1}\\ &+(C_1^3+C_1C_2^2+C_1C_3^2)\sum_{k=0}^{\infty}(k+3)(k+2)(k+1)a_{k+3}\omega^k = 0.\end{aligned} \tag{5.9}$$

In what follows, we equate the coefficients of different powers of ω to obtain the explicit expressions of a_k with $A(k) = (C_1^3+C_1C_2^2+C_1C_3^2)(k+3)(k+2)(k+1)$ and $B(k) = \frac{\Gamma(1-\frac{(kp-k+2)\alpha}{3(p-1)})}{\Gamma(1-\frac{((k+3)p-k-1)\alpha}{3(p-1)})}$. For $p=2$,

$$a_{k+3} = \frac{1}{A(k)}\Big[B(k)a_k + 2C_1\big(\sum_{i+j=k}(i+1)a_{i+1}a_j\big)\Big], \quad (k\geq 0). \tag{5.10}$$

For $p=3$,

$$a_{k+3} = \frac{1}{A(k)}\Big[B(k)a_k + 3C_1\big(\sum_{i+j+m=k}(i+1)a_{i+1}a_ja_m\big)\Big], \quad (k\geq 0). \tag{5.11}$$

For $p=4$,

$$a_{k+3} = \frac{1}{A(k)}\Big[B(k)a_k + 4C_1\big(\sum_{i+j+m+n=k}(i+1)a_{i+1}a_ja_ma_n\big)\Big], \quad (k\geq 0). \tag{5.12}$$

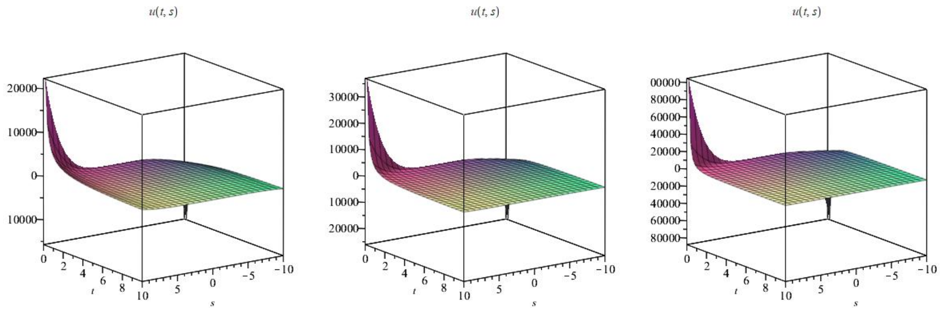
Thus we get the power series solution of Eq. (1.2) in the form

$$u(t,x,y,z) = t^{-\frac{2\alpha}{3(p-1)}} \sum_{k=0}^{\infty} a_k (C_1 x t^{-\frac{\alpha}{3}} + C_2 y t^{-\frac{\alpha}{3}} + C_3 z t^{-\frac{\alpha}{3}})^k, \tag{5.13}$$

where $a_0 = f(0)$, $a_1 = f'(0)$, $a_2 = \frac{f''(0)}{2}$, and a_{k+3} are defined as (5.10)-(5.12). Assuming $C_1 = C_2 = C_3 = 1$ and $s = x + y + z$ in (5.13), Tabs. 4-6 show some values of a_n and α when $p = 2, 3, 4$ in Eq. (1.2), respectively. Correspondingly, Figs. 1-3 illustrate the physical features for the power series solution (5.13) with different parameter values. As we can see from Figs. 1-3 that the power series solution of Eq. (1.2) is closely related to the order α of the fractional derivative and the index parameter p.

Table 4: Some of a_n with $a_0 = a_1 = a_2 = 1, p = 2$ for different fractional orders

	a_0	a_1	a_2	a_3	a_4	a_5
$\alpha = 0.4$	1	1	1	0.2481081295	0.08783865458	0.04233146436
$\alpha = 0.6$	1	1	1	0.2222222222	0.078040946266	0.03389013101
$\alpha = 0.8$	1	1	1	0.1962692286	0.06608613801	0.05057643726

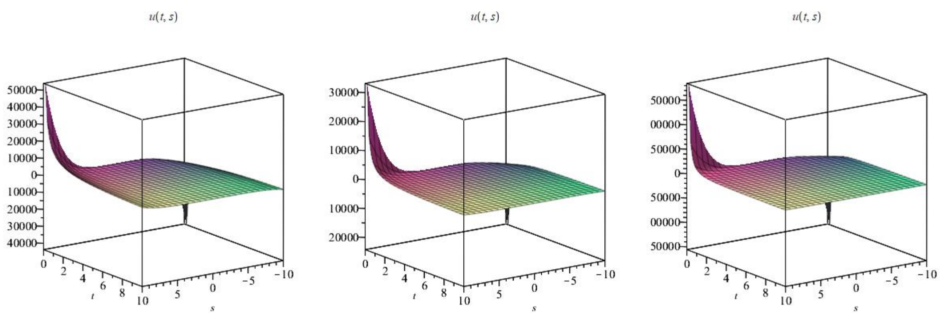


(a) α=0.4 (b) α=0.6 (c) α=0.8

Figure 1: Numerical simulation of the power series solution (5.13) with $p = 2$, $C_1 = C_2 = C_3 = 1$, $a_0 = a_1 = a_2 = 1$ and $s = x + y + z$.

Table 5: Some of a_n with $a_0 = a_1 = a_2 = 1, p = 3$ for different fractional orders

	a_0	a_1	a_2	a_3	a_4	a_5
$\alpha = 0.4$	1	1	1	0.3654490762	0.1731381435	0.1367412490
$\alpha = 0.6$	1	1	1	0.3474221223	0.1666666667	0.1319208180
$\alpha = 0.8$	1	1	1	0.3289015221	0.1601784183	0.1262128647



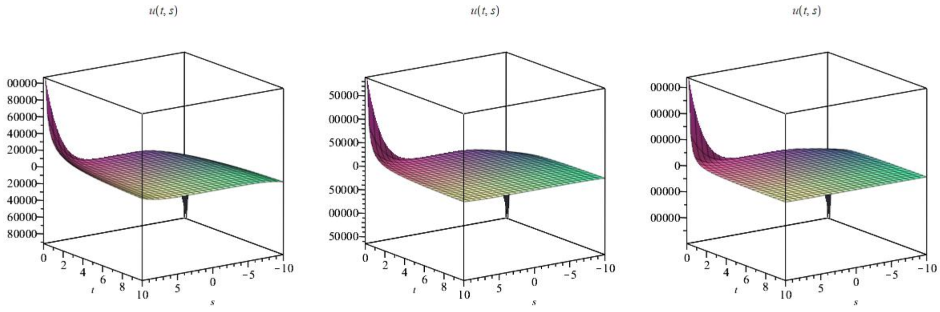
(a) α=0.4 (b) α=0.6 (c) α=0.8

Figure 2: Numerical simulation of the power series solution (5.13) with $p = 3$, $C_1 = C_2 = C_3 = 1$, $a_0 = a_1 = a_2 = 1$ and $s = x + y + z$.

Case 2: X_2

Table 6: Some of a_n with $a_0 = a_1 = a_2 = 1, p = 4$ for different fractional orders

	a_0	a_1	a_2	a_3	a_4	a_5
$\alpha = 0.4$	1	1	1	0.4783860185	0.2842548116	0.3006471284
$\alpha = 0.6$	1	1	1	0.4624405244	0.2788940982	0.2962146614
$\alpha = 0.8$	1	1	1	0.4458696571	0.2742986894	0.2915138317



(a) α=0.4 (b) α=0.6 (c) α=0.8

Figure 3: Numerical simulation of the power series solution (5.13) with $p = 4$, $C_1 = C_2 = C_3 = 1$, $a_0 = a_1 = a_2 = 1$ and $s = x + y + z$.

The characteristic equation corresponding to the group generator X_2 is

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}y}{z} = \frac{\mathrm{d}z}{-y} = \frac{\mathrm{d}u}{0}, \tag{5.14}$$

from which, we obtain the similarity variables t, x, $y^2 + z^2$ and u. So we get the invariant solution of Eq. (1.2) as follows:

$$u = f(\omega_1, \omega_2, \omega_3), \quad \omega_1 = t, \quad \omega_2 = x, \quad \omega_3 = y^2 + z^2. \tag{5.15}$$

Substituting (5.15) into Eq. (1.2), we have the following reduced equation:

$$D^\alpha_{\omega_1} f + pf^{p-1} f_{\omega_2} + f_{\omega_2\omega_2\omega_2} + 4\omega_3 f_{\omega_2\omega_3\omega_3} = 0. \tag{5.16}$$

Case 3: X_3

The characteristic equation corresponding to the group generator X_3 is

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}z}{0} = \frac{\mathrm{d}u}{0}, \tag{5.17}$$

from which, we obtain the similarity variables t, y, z and u. So we get the invariant solution of Eq. (1.2) as follows:

$$u = f(\omega_1, \omega_2, \omega_3), \quad \omega_1 = t, \quad \omega_2 = y, \quad \omega_3 = z. \tag{5.18}$$

Substituting (5.18) into Eq. (1.2), we have the following reduced equation:

$$D^\alpha_{\omega_1} f = 0, \tag{5.19}$$

from which, we can easily get $f(\omega_1, \omega_2, \omega_3) = \omega_1^{\alpha-1} g(\omega_2, \omega_3)$, that is, $u = t^{\alpha-1} g(y, z)$ with g an arbitrary function.

Case 4: X_4

The characteristic equation corresponding to the group generator X_4 is

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}y}{1} = \frac{\mathrm{d}z}{0} = \frac{\mathrm{d}u}{0}, \tag{5.20}$$

from which, we obtain the similarity variables t, x, z and u. So we get the invariant solution of Eq. (1.2) as follows:

$$u = f(\omega_1, \omega_2, \omega_3), \quad \omega_1 = t, \quad \omega_2 = x, \quad \omega_3 = z. \tag{5.21}$$

Substituting (5.21) into Eq. (1.2), we have the following reduced equation:

$$D_{\omega_1}^{\alpha} f + p f^{p-1} f_{\omega_2} + f_{\omega_2\omega_2\omega_2} + f_{\omega_2\omega_3\omega_3} = 0. \tag{5.22}$$

Case 5: X_5

The characteristic equation corresponding to the group generator X_5 is

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}z}{1} = \frac{\mathrm{d}u}{0}, \tag{5.23}$$

from which, we obtain the similarity variables t, x, y and u. So we get the invariant solution of Eq. (1.2) as follows:

$$u = f(\omega_1, \omega_2, \omega_3), \quad \omega_1 = t, \quad \omega_2 = x, \quad \omega_3 = y. \tag{5.24}$$

Substituting (5.24) into Eq. (1.2), we have the following reduced equation:

$$D_{\omega_1}^{\alpha} f + p f^{p-1} f_{\omega_2} + f_{\omega_2\omega_2\omega_2} + f_{\omega_2\omega_3\omega_3} = 0. \tag{5.25}$$

Case 6: $bX_3 + dX_4$

The characteristic equation corresponding to the group generator $bX_3 + dX_4$ is

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{b} = \frac{\mathrm{d}y}{d} = \frac{\mathrm{d}z}{0} = \frac{\mathrm{d}u}{0}, \tag{5.26}$$

from which, we obtain the similarity variables t, $dx - by$, z and u. So we get the invariant solution of Eq. (1.2) as follows:

$$u = f(\omega_1, \omega_2, \omega_3), \quad \omega_1 = t, \quad \omega_2 = dx - by, \quad \omega_3 = z. \tag{5.27}$$

Substituting (5.27) into Eq. (1.2), we have the following reduced equations:

$$D_{\omega_1}^{\alpha} f + pdf^{p-1} f_{\omega_2} + (d^3 + db^2) f_{\omega_2\omega_2\omega_2} + f_{\omega_2\omega_3\omega_3} = 0. \tag{5.28}$$

Case 7: $bX_3 + dX_5$

The characteristic equation corresponding to the group generator $bX_3 + dX_5$ is

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{b} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}z}{d} = \frac{\mathrm{d}u}{0}, \tag{5.29}$$

from which, we obtain the similarity variables t, $dx - bz$, y and u. So we get the invariant solution of Eq. (1.2) as follows:

$$u = f(\omega_1, \omega_2, \omega_3), \quad \omega_1 = t, \quad \omega_2 = dx - bz, \quad \omega_3 = y. \tag{5.30}$$

Substituting (5.30) into Eq. (1.2), we have the following reduced equations:

$$D_{\omega_1}^{\alpha} f + pdf^{p-1} f_{\omega_2} + (d^3 + db^2) f_{\omega_2\omega_2\omega_2} + f_{\omega_2\omega_3\omega_3} = 0. \tag{5.31}$$

Case 8: $bX_4 + dX_5$

The characteristic equation corresponding to the group generator $bX_4 + dX_5$ is

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}y}{b} = \frac{\mathrm{d}z}{d} = \frac{\mathrm{d}u}{0}, \tag{5.32}$$

from which, we obtain the similarity variables t, $dy - bz$, x and u. So we get the invariant solution of Eq. (1.2) as follows:

$$u = f(\omega_1, \omega_2, \omega_3), \quad \omega_1 = t, \quad \omega_2 = dy - bz, \quad \omega_3 = x. \tag{5.33}$$

Substituting (5.33) into Eq. (1.2), we have the following reduced equations:

$$D_{\omega_1}^{\alpha} f + p f^{p-1} f_{\omega_3} + (d^2 + b^2) f_{\omega_2\omega_2\omega_3} + f_{\omega_3\omega_3\omega_3} = 0. \tag{5.34}$$

Case 9: $bX_3 + cX_4 + dX_5$

The characteristic equation corresponding to the group generator $bX_3 + cX_4 + dX_5$ is

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{b} = \frac{\mathrm{d}y}{c} = \frac{\mathrm{d}z}{d} = \frac{\mathrm{d}u}{0}, \tag{5.35}$$

from which, we obtain the similarity variables t, $\frac{1}{b}x + \frac{1}{c}y - \frac{2}{d}z$ and u. So we get the invariant solution of Eq. (1.2) as follows:

$$u = f(\omega_1, \omega_2), \quad \omega_1 = t, \quad \omega_2 = \frac{1}{b}x + \frac{1}{c}y - \frac{2}{d}z. \tag{5.36}$$

Substituting (5.36) into Eq. (1.2), we have the following reduced equations:

$$D^{\alpha}_{\omega_1} f + \frac{p}{b} f^{p-1} f_{\omega_2} + \frac{1}{b}(\frac{1}{b^2} + \frac{1}{c^2} + \frac{4}{d^2}) f_{\omega_2\omega_2\omega_2} = 0. \tag{5.37}$$

Note that Eqs. (5.16), (5.19), (5.22), (5.25), (5.28) , (5.31), (5.34) and (5.37) are the (2+1)-, (1+1)- and (0+1)-dimensional fractional Z-K equations with Riemann-Liouville fractional derivative, respectively, which have been studied in [38, 2, 34] and references therein.

6 Conclusion

This paper extends the (1+1)-dimensional and (2+1)-dimensional fractional partial differential equations to the (3+1)-dimensional fractional partial differential equations. Lie symmetry analysis method successfully reduced the high dimensional fractional partial differential equation to a low dimensional equation and obtained the power series solution of the reduced equation. This indicates that Lie symmetry analysis method can be effectively applied to some higher dimensional fractional partial differential equations in physical science and engineering.

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