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Fixed point theorems satisfying rational tower-type mapping in a complete metric spaces

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Abstract

In this paper, we define rational type Geraghty tower contraction mapping and prove the existence of such finite and infinite rational Geraghty tower theorem(s) in complete metric spaces. The results we establish in this paper extend, improve, generalise and unify some existing results in the literature.

Keywords: Fixed point, metric spaces, finite, infinite rational Geraghty tower contraction maps
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1 Introduction

In 1922, Polish mathematician Stefan Banach established a remarkable fixed point theorem known today as Banach Contraction Principle (BCP) which is one of the most important results of analysis and considered as the main source of metric fixed point theory. It is the most widely applied fixed point result in many branches of mathematics because it requires the structure of complete metric space with contractive condition on the map which is easy to test in this setting. In Banach's theorem, X is a complete metric space with metric d and $f : X \to X$ is required to be a contraction, that is there must exist $L < 1$ such that $d(f(x), f(y)) \leq Ld(x, y)$ for all $x, y \in X$. The conclusion is that f has a fixed point, in fact exactly one of them. In 1969, Meir and Keeler [18] obtained the following interesting fixed point result as also an extension of BCP. Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $\epsilon \leq d(x, y) < \epsilon + \delta(\epsilon)$ implies $d(Tx, Ty) < \epsilon$ for all $x, y \in X$. Then T has a unique fixed point.

In fact, there are vast amount of literatures dealing with extensions or generalizations of this remarkable theorem. This has encountered in so many extensions/generalizations to mention a few as recorded in [3, 10, 11, 13, 15, 16, 17, 18, 28, 29, 30, 31, 32, 33] etc and their references therein.

In this paper, it is almost impossible to cover all the known extensions or generalizations of the celebrated contraction principle due to S. Banach [2], which appeared in literature in 1922. However, an attempt is made to present some

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extensions of the BCP in which the conclusion is obtained under mild modified conditions and which play important role in the development of metric fixed point theory.

Somewhere in combinatorial set theory we do encounter tower of infinite ordinals, a positive result i.e; $\omega^{\omega^{\omega}} \to (\omega^{\omega^{\omega}}, 3)$ and in elementary calculus, integration of a tower function of the types $x \mapsto x^{x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}}$ and $x \mapsto x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$ poses great amount of difficulty and interesting in its own right. In metrical fixed point theories, especially, contraction mappings involving such tower to our knowledge has not been in print since the time of Banach. This is what we propose to do in this present paper. We shall define and prove Geraghty tower contraction mapping theorems in the setting of metric spaces.

Among these generalizations cited above, what actually cut our fancy is the one given by M. Geraghty [13] and we will try to get it's tower form and other related forms.

Decade ago, Amini-Harandi and Emami [1] characterized the result of Geraghty in the context of a partially ordered complete metric space with some application to ordinary differential equations. Gordji et al. [14] defined the notion of ψ-Geraghty type contraction and supposedly improved and extended the results of Amini-Harandi and Emami [1]. Cho, Bae and Karapinar [9] defined the concept of α-Geraghty contraction type maps in the setting of a metric space and proved the existence and uniqueness of a fixed point of such maps in the context of a complete metric space. Popescu [27] generalized the results obtained in Cho et al. [9] and gave other conditions to prove the existence and uniqueness of a fixed point of α-Geraghty contraction type maps in the context of a complete metric space. See also [4, 7, 8, 19, 22, 23, 24, 25] for other results in fixed point theory.

In this present paper, we will define finite and infinite rational Geraghty metric tower map and prove that fixed point theorems containing such tower contraction exists. Our results in this work includes in its full strength Banach contraction principle and Geraghty contraction mapping, [2, 3, 13, 21, 28]. We hope that the results of this paper will open another important aspect of contraction maps in literature of metrical fixed point theory.

2 Preliminaries

Definition 2.1. [26] Let (X, d) be a metric space, $\{x_k\}_{k\in\mathbb{N}}$ a sequence in X, and let $x \in X$. Then,

(a) The sequence $\{x_k\}_{k\in\mathbb{N}}$ is said to be convergent in (X, d) and converges to x_0, if for given $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_k, x_0) < \epsilon$ for all $k \geq n_0$ and this fact is represented by $\lim_{k\to\infty} x_k = x_0$ or $x_k \to x_0$ as $k \to \infty$.

(b) The sequence $\{x_k\}_{k\in\mathbb{N}}$ is said to be Cauchy sequence in (X, d) if for given $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_k, x_{k+p}) < \epsilon$ for all $k \geq n_0,\ p > 0$ or equivalently, $\lim_{k\to\infty} d(x_k, x_{k+p}) = 0$ for all $p > 0$.

(c) (X, d) is said to be a complete metric space if every Cauchy sequence in X converges to some $x \in X$.

Definition 2.2. [13] S is the class of functions $\alpha : \mathbb{R}^+ \to [0, 1)$ with

(i) $\mathbb{R}^+ = \{t \in \mathbb{R} | t > 0\}$,

(ii) $\alpha(t_n) \to 1 \implies t_n \to 0$ as $n \to \infty$.

Definition 2.3. [4] For each $n \in \mathbb{N}$, let $\mathcal{S}_n$ denote the class of n-tuples of functions $(\beta_1, \beta_2, \beta_3, \cdots, \beta_n)$, where for each $i \in \{1, 2, \cdots, n\}$, $\beta_i : \mathbb{R}_+ \cup \{\infty\} \to [0, 1)$ and the following implications holds: $\beta_i(t_k) := \beta_1(t_k) + \beta_2(t_k) + \cdots + \beta_n(t_k) \to 1$ implies $t_k \to 0$. It follows that, for each $m \in \{1, 2, \cdots, n\}$, if $(\beta_1, \beta_2, \beta_3, \cdots, \beta_m) \in \mathcal{S}_m$, then $\{\underbrace{\beta_1, \beta_2, \beta_3, \cdots, \beta_m, 0, 0, 0 \cdots, 0}_{n-m\ entries}\} \in \mathcal{S}_n$, where 0 is a zero function.

Remark 2.4. Note that, if $(\underbrace{\beta, \beta, \cdots, \beta}_{n\ entries}) \in \mathcal{S}_n$, then we also have the following: $\beta(t_k) \to \frac{1}{n}$ implies $t_k \to 0$.

We will in this paper take $\mathcal{F}_{Ger}$ as the class of all Geraghty functions.

Theorem 2.5. [13] Let X be a complete metric space. Let $f : X \to X$ with $d(f(x), f(y)) < d(x, y)$, for all $x, y \in X$. Let $x_0 \in X$ and set $f(x_{n-1}) = x_n$ for all $n > 0$. Then $x_n \to x^*$ in X, with x^* a unique fixed point of f, iff for any two sub-sequences x_{h_n} and x_{h_k} with $x_{h_n} \neq x_{h_k}$, we have that $\Omega_n \to 1$ only if $d_n \to 0$.

Remark 2.6. In Theorem 2.5, we take for any pair of sequences x_n and y_n with $x_n \neq y_n$, we write $d_n = d(x_n, y_n)$ and $\Omega_n = \frac{d(f(x_n),f(y_n))}{d(x_n,y_n)}$.

Theorem 2.7. [13] Let $f : X \to X$ be a contraction on a complete metric space. Let $x_0 \in X$ and set $f(x_{n-1}) = x_n$ for all $n > 0$. Then $x_n \to x^*$ in X, where x^* a unique fixed point of f in X, iff there exists an α in $\mathcal{F}_{Ger}$ such that for all n, m

$$d(f(x_n), f(x_m)) \leq \alpha(d(x_n, x_m))d(x_n, x_m). \tag{2.1}$$

Theorem 2.8. [13] Let (X, ρ) be a complete metric space and $T : X \to X$ such that there is an $\alpha \in \mathcal{F}_{Ger}$ satisfying

$$\rho(Tx, Ty) \leq \alpha(\rho(x, y))\rho(x, y), \tag{2.2}$$

for all $x, y \in X$. Then the sequence $\{x_k\}_{k\in\mathbb{N}}$ defined by $Tx_{k-1} = x_k$ converges to a unique fixed point of T in X for $k \geq 1$.

Definition 2.9. [21] Let (X, d) be a metric space and $T : X \to X$ such that there is $\{\alpha, \beta, \gamma, \delta, \epsilon\} \in \mathcal{F}_{Ger}$. Then T is called a metric tower map if

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y)^{A_\beta^{B_\gamma^{C_\delta^{D_\epsilon}}}}, \tag{2.3}$$

where, $A_\beta := \beta(d(x, y))d(x, Tx)$, $B_\gamma := \gamma(d(x, y))d(y, Ty)$, $C_\delta := \delta(d(x, y))d(x, Ty)$; $D_\epsilon := \epsilon(d(x, y))d(y, Tx)$, for all distinct $x, y \in X$

Remark 2.10. This type of tower map is called Geraghty tower of order 5. Also observe that we can possibly change the positions of $A_\beta, B_\gamma, C_\delta$ and D_ϵ. For example see Okeke and Francis [21].

Following Okeke et al. [25], and Okeke and Francis [21] we have the immediate constructions. For each $n \in \mathbb{N}$, let $\mathcal{F}_{nGer}$ be the class of n-tuples of functions $\{\mu_1, \mu_2, \cdots \mu_n\}$ and for each $i \in \{1, 2, 3, \cdots, n\}$, the map $\mu_i : \mathbb{R}_+ \cup \{\infty\} \to [0, 1)$, so that we have the following $\mu_i(t_k) := \mu_1(t_k)\mu_2(t_k)\cdots\mu_n(t_k) \to 1 \implies t_k \to 0$. Now for each $m \in \{1, 2, \cdots, n\}$, suppose that $\{\mu_1, \mu_2, \cdots \mu_m\} \in \mathcal{F}_{mGer}$, then $\{\underbrace{\mu_1, \mu_2, \cdots \mu_m, 0, 0, 0 \cdots, 0}_{n-m \ times}\} \in \mathcal{F}_{mGer}$, where 0 is a zero function. Again, if $\underbrace{\{\mu, \mu, \cdots \mu\}}_{n \ times} \in \mathcal{F}_{nGer}$, then $\mu_i(t_k) \to \frac{1}{n}$ implies $t_k \to 0$ for all i. If $\{\mu_1, \mu_2, \cdots \mu_n\} \in \mathcal{F}_{nGer}$, then $\pi(\mu_1, \mu_2, \cdots \mu_n) \in \mathcal{F}_{nGer}$ is a permutation of $(\mu_1, \mu_2, \cdots \mu_n)$. If $(\mu_1, \mu_2, \cdots \mu_n) \in \mathcal{F}_{nGer}$, then its subsequences i.e; $(\mu_{n_1}, \mu_{n_2}, \cdots \mu_{n_m}) \in \mathcal{F}_{mGer}$ for each $m \in \{1, 2, \cdots, n\}$. $\mu_{n_i} \neq \mu_{n_j}$ for all $i, j \in \{1, 2, \cdots, m\}$, where $\mu_{n_i} \in \{\mu_1, \mu_2, \cdots \mu_n\}$.

3 Main Results

Theorem 3.1. Let (X, d) be a complete metric space and $T : X \to X$ such that there are $\{\alpha, \beta, \gamma, \delta, \epsilon\} \in \mathcal{F}_{Ger}$ satisfying;

$$d(Tx, Ty)^2 \leq \alpha(d(x, y))d(x, y)^{A_\beta^{B_\gamma^{C_\delta^{D_\epsilon}}}}, \tag{3.1}$$

where,

$$A_\beta := \frac{\beta(d(x, y))d(x, Tx)}{1 + \epsilon(d(x, y))d(y, Tx)},\ B_\gamma := \frac{\gamma(d(x, y))d(y, Ty)}{1 + \epsilon(d(x, y))d(y, Tx)},\ C_\delta := \frac{\delta(d(x, y))d(x, Ty)}{1 + \epsilon(d(x, y))d(y, Tx)},\ D_\epsilon := \frac{\epsilon(d(x, y))d(y, Tx)}{1 + \epsilon(d(x, y))d(y, Tx)}$$

for all distinct close $x, y \in X$ and $\ln^k(d(x, y)) \leq d(x, y)$, $d^k(x, y) \leq d(x, y)$ for all $k \in \mathbb{N}$. Then the sequence $\{x_k\}_{k\in\mathbb{N}}$ defined by $Tx_{k-1} = x_k$ converges to a unique fixed point of T in X.

Proof . Suppose, if possible, otherwise. Let $x_k = Tx_{k-1}$, then $x_{k+1} = Tx_k$ for all $k \in \mathbb{N}$. But $x_k \neq x_{k+1}$ and this implies that $d(x_k, x_{k+1}) > 0$, so that $d(x_k, x_{k+1}) = d(Tx_{k-1}, Tx_k)$. From inequality (3.1), take $x = x_{k-1}$ and $y = x_k$ for all $k \in \mathbb{N}$, thus

$$d(x_k, x_{k+1})^2 = d(Tx_{k-1}, Tx_k)^2$$

$$\leq \alpha(d(x_{k-1},x_k))d(x_{k-1},x_k)^{A_\beta^{B_\gamma^{C_\delta^{D_\epsilon}}}}, \tag{3.2}$$

where,

$$A_\beta := \frac{\beta(d(x_{k-1},x_k))d(x_{k-1},Tx_{k-1})}{1+\epsilon(d(x_{k-1},x_k))d(x_k,Tx_{k-1})}, \quad B_\gamma := \frac{\gamma(d(x_{k-1},x_k))d(x_k,Tx_k)}{1+\epsilon(d(x_{k-1},x_k))d(x_k,Tx_{k-1})},$$
$$C_\delta := \frac{\delta(d(x_{k-1},x_k))d(x_{k-1},Tx_k)}{1+\epsilon(d(x_{k-1},x_k))d(x_k,Tx_{k-1})}, \quad D_\epsilon := \frac{\epsilon(d(x_{k-1},x_k))d(x_k,Tx_{k-1})}{1+\epsilon(d(x_{k-1},x_k))d(x_k,Tx_{k-1})}.$$

So that,

$$A_\beta := \frac{\beta(d(x_{k-1},x_k))d(x_{k-1},x_k)}{1+\epsilon(d(x_{k-1},x_k))d(x_k,x_k)}, \quad B_\gamma := \frac{\gamma(d(x_{k-1},x_k))d(x_k,x_{k+1})}{1+\epsilon(d(x_{k-1},x_k))d(x_k,x_k)},$$
$$C_\delta := \frac{\delta(d(x_{k-1},x_k))d(x_{k-1},x_{k+1})}{1+\epsilon(d(x_{k-1},x_k))d(x_k,x_k)}, \quad D_\epsilon := \frac{\epsilon(d(x_{k-1},x_k))d(x_k,x_k)}{1+\epsilon(d(x_{k-1},x_k))d(x_k,x_k)}.$$

Thus, $A_\beta := \beta(d(x_{k-1},x_k))d(x_{k-1},x_k)$, $B_\gamma := \gamma(d(x_{k-1},x_k))d(x_k,x_{k+1})$, $C_\delta := \delta(d(x_{k-1},x_k))d(x_{k-1},x_{k+1})$; $D_\epsilon := \dfrac{\epsilon(d(x_{k-1},x_k))d(x_k,x_k)}{1+\epsilon(d(x_{k-1},x_k))d(x_k,x_k)} = 0$. Therefore, inequality (3.2) reduced to

$$\begin{aligned}
d(x_k,x_{k+1})^2 &\leq \alpha(d(x_{k-1},x_k))d(x_{k-1},x_k)^{A_\beta^{B_\gamma}} \\
&= \alpha(d(x_{k-1},x_k))\exp\Big\{\ln\Big(d(x_k,x_{k-1})\Big)^{A_\beta^{B_\gamma}}\Big\} \\
&= \alpha(d(x_{k-1},x_k))\exp\Big\{A_\beta^{B_\gamma}\ln\Big(d(x_k,x_{k-1})\Big)\Big\} \\
&\leq \alpha(d(x_{k-1},x_k))\Big\{A_\beta^{B_\gamma}\ln\Big(d(x_k,x_{k-1})\Big) + \frac{1}{2}A_\beta^{B_\gamma^2} \\
&\times \ln^2\Big(d(x_k,x_{k-1})\Big) + \cdots + \frac{1}{n!}A_\beta^{B_\gamma^n}\ln^n\Big(d(x_k,x_{k-1})\Big) + \cdots\Big\} \\
&= \alpha(d(x_{k-1},x_k))\Big\{\exp(B_\gamma\ln(A_\beta))\ln\Big(d(x_k,x_{k-1})\Big) \\
&+ \frac{1}{2}\exp(B_\gamma^2\ln(A_\beta))\ln^2\Big(d(x_k,x_{k-1})\Big) + \cdots + \frac{1}{n!}\exp(B_\gamma^n\ln(A_\beta))\ln^n\Big(d(x_k,x_{k-1})\Big) + \cdots\Big\} \\
&= \alpha(d(x_{k-1},x_k))\Big\{\Big(1 + B_\gamma\ln(A_\beta) + \frac{1}{2}B_\gamma^2\ln^2(A_\beta) + \cdots + \frac{1}{n!}B_\gamma^n\ln^n(A_\beta)\Big)\ln\Big(d(x_k,x_{k-1})\Big) \\
&+ \frac{1}{2}\Big(1 + B_\gamma^2\ln(A_\beta) + \frac{1}{2}B_\gamma^4\ln^2(A_\beta) + \cdots + \frac{1}{n!}B_\gamma^{2n}\ln^n(A_\beta)\Big)\ln^2\Big(d(x_k,x_{k-1})\Big) + \cdots \\
&+ \frac{1}{r!}\Big(1 + B_\gamma^r\ln(A_\beta) + \frac{1}{2}B_\gamma^{2r}\ln^2(A_\beta) + \cdots + \frac{1}{r!}B_\gamma^{nr}\ln^r(A_\beta)\Big)\ln^n\Big(d(x_k,x_{k-1})\Big) + \cdots\Big\} \\
&\leq \alpha(d(x_{k-1},x_k))\Big\{\Big(B_\gamma\ln(A_\beta) + \frac{1}{2}B_\gamma^2\ln^2(A_\beta) + \cdots + \frac{1}{n!}B_\gamma^n\ln^n(A_\beta)\Big)\ln\Big(d(x_k,x_{k-1})\Big) \\
&+ \frac{1}{2}\Big(B_\gamma^2\ln(A_\beta) + \frac{1}{2}B_\gamma^4\ln^2(A_\beta) + \cdots + \frac{1}{n!}B_\gamma^{2n}\ln^n(A_\beta)\Big)\ln^2\Big(d(x_k,x_{k-1})\Big) + \cdots \\
&+ \frac{1}{r!}\Big(B_\gamma^r\ln(A_\beta) + \frac{1}{2}B_\gamma^{2r}\ln^2(A_\beta) + \cdots + \frac{1}{r!}B_\gamma^{nr}\ln^r(A_\beta)\Big)\ln^n\Big(d(x_k,x_{k-1})\Big) + \cdots\Big\} \\
&\leq \alpha(d(x_{k-1},x_k))\Big\{\Big(B_\gamma\ln(A_\beta) + B_\gamma^2\ln^2(A_\beta) + \cdots + B_\gamma^n\ln^n(A_\beta)\Big)\ln\Big(d(x_k,x_{k-1})\Big) \\
&+ \Big(B_\gamma^2\ln(A_\beta) + B_\gamma^4\ln^2(A_\beta) + \cdots + B_\gamma^{2n}\ln^n(A_\beta)\Big)\ln^2\Big(d(x_k,x_{k-1})\Big) + \cdots \\
&+ \Big(B_\gamma^r\ln(A_\beta) + B_\gamma^{2r}\ln^2(A_\beta) + \cdots + B_\gamma^{nr}\ln^r(A_\beta)\Big)\ln^n\Big(d(x_k,x_{k-1})\Big) + \cdots\Big\}
\end{aligned}$$

$$= \alpha(d(x_{k-1}, x_k))\Big\{\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} B_{\gamma}^{nr}\ln^r(A_\beta)\ln^n\Big(d(x_k, x_{k-1})\Big)\Big\}$$

$$= \alpha(d(x_{k-1}, x_k))\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Big\{(\gamma(d(x_{k-1}, x_k))d(x_k, x_{k+1}))^{nr}$$

$$\times \ln^r(\beta(d(x_{k-1}, x_k))d(x_{k-1}, x_k))\ln^n\Big(d(x_k, x_{k-1})\Big)\Big\}$$

$$\le \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\alpha(d(x_{k-1}, x_k))\beta(d(x_{k-1}, x_k))\gamma^{nr}(d(x_{k-1}, x_k))d(x_k, x_{k-1})$$

$$= \Big(\sum_{r=0}^{\infty}\frac{\alpha(d(x_{k-1}, x_k))\beta(d(x_{k-1}, x_k))}{1-\gamma^r(d(x_{k-1}, x_k))}\Big)d(x_k, x_{k-1}), \tag{3.3}$$

$$\begin{aligned}
d(x_k, x_{k+1})^2 \le& \alpha(d(x_{k-1}, x_k))\sum_{j=0}^{\infty}\frac{\Big(A_\beta^{B_\gamma}\ln(d(x_k, x_{k-1}))\Big)^j}{j!}\\
=& \alpha(d(x_{k-1}, x_k))\sum_{j=0}^{\infty}\frac{A_\beta^{B_\gamma^j}\ln^j(d(x_k, x_{k-1}))}{j!}\\
\le& \alpha(d(x_{k-1}, x_k))\sum_{j=0}^{\infty}\frac{A_\beta^{B_\gamma^j}\ln(d(x_k, x_{k-1}))}{j!}\\
=& \alpha(d(x_{k-1}, x_k))\sum_{j=0}^{\infty}\frac{\exp\Big(B_\gamma^j\ln(A_\beta)\Big)\ln(d(x_k, x_{k-1}))}{j!}\\
=& \alpha(d(x_{k-1}, x_k))\sum_{j=0}^{\infty}\frac{d(x_k, x_{k-1})}{j!}\sum_{r=0}^{\infty}\frac{\Big(B_\gamma^j\ln(A_\beta)\Big)^r}{r!}\\
=& \alpha(d(x_{k-1}, x_k))\sum_{j=0}^{\infty}\frac{d(x_k, x_{k-1})}{j!}\sum_{r=0}^{\infty}\frac{B_\gamma^{jr}\ln^r(A_\beta)}{r!}\\
=& \alpha(d(x_{k-1}, x_k))\sum_{j=0}^{\infty}\frac{d(x_k, x_{k-1})}{j!}\sum_{r=0}^{\infty}\frac{\gamma^{jr}(d(x_{k-1}, x_k))d^{jr}(x_k, x_{k+1})\ln^r(\beta(d(x_{k-1}, x_k))d(x_{k-1}, x_k))}{r!}\\
=& \alpha(d(x_{k-1}, x_k))\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\frac{\gamma^{jr}(d(x_{k-1}, x_k))d^{jr}(x_k, x_{k+1})\ln^r(\beta(d(x_{k-1}, x_k))d(x_{k-1}, x_k))d(x_k, x_{k-1})}{j!r!}\\
\le& \alpha(d(x_{k-1}, x_k))\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\gamma^{jr}(d(x_{k-1}, x_k))d^{jr}(x_k, x_{k+1})\ln^r(\beta(d(x_{k-1}, x_k))d(x_{k-1}, x_k))d(x_k, x_{k-1})\\
\le& \alpha(d(x_{k-1}, x_k))\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\gamma^{jr}(d(x_{k-1}, x_k))d(x_k, x_{k+1})\beta(d(x_{k-1}, x_k))d(x_{k-1}, x_k)d(x_k, x_{k-1})\\
\le& \alpha(d(x_{k-1}, x_k))\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\gamma^{jr}(d(x_{k-1}, x_k))d(x_k, x_{k+1})\beta(d(x_{k-1}, x_k))d(x_k, x_{k-1})\\
=& \sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\alpha(d(x_{k-1}, x_k))\beta(d(x_{k-1}, x_k))\gamma^{jr}(d(x_{k-1}, x_k))d(x_k, x_{k+1})d(x_k, x_{k-1}).
\end{aligned}$$

Hence,

$$d(x_k, x_{k+1}) \le \sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\alpha(d(x_{k-1}, x_k))\beta(d(x_{k-1}, x_k))\gamma^{jr}(d(x_{k-1}, x_k))d(x_k, x_{k-1})$$

$$= \left(\sum_{j=0}^{\infty} \frac{\alpha(d(x_{k-1}, x_k))\beta(d(x_{k-1}, x_k))}{1 - \gamma^j(d(x_{k-1}, x_k))} \right) d(x_k, x_{k-1}). \tag{3.4}$$

Therefore, from inequality (3.3) or (3.4), we get;

$$d(x_k, x_{k+1}) \leq \sum_{j=0}^{\infty} \sum_{r=0}^{\infty} \alpha(d(x_{k-1}, x_k))\beta(d(x_{k-1}, x_k))\gamma^{jr}(d(x_{k-1}, x_k))d(x_k, x_{k-1}). \tag{3.5}$$

Therefore,

$$\begin{aligned} d(x_k, x_{k+1}) &\leq \sum_{j=0}^{\infty} \sum_{r=0}^{\infty} \alpha(d(x_{k-1}, x_k))\beta(d(x_{k-1}, x_k))\gamma^{jr}(d(x_{k-1}, x_k))d(x_k, x_{k-1}) \\ &\leq \sum_{j=0}^{\infty} \sum_{r=0}^{\infty} \alpha(d(x_{k-1}, x_k))\beta(d(x_{k-1}, x_k))\gamma^{jr}(d(x_{k-1}, x_k))d(x_{k-1}, x_{k-2}) \\ &\leq \sum_{j=0}^{\infty} \sum_{r=0}^{\infty} \alpha(d(x_{k-1}, x_k))\beta(d(x_{k-1}, x_k))\gamma^{jr}(d(x_{k-1}, x_k))d(x_{k-2}, x_{k-3}) \\ &\vdots \\ &\leq d(x_0, x_1). \end{aligned} \tag{3.6}$$

Thus, $\{d(x_k, x_{k+1})\}_{k \in \mathbb{N}}$ is a non-increasing sequence and bounded below for which the sequence converges to some real number $a \geq 0$ (say). If $a > 0$, then we see that

$$d(x_k, x_{k+1}) \leq \sum_{j=0}^{\infty} \sum_{r=0}^{\infty} \alpha(d(x_{k-1}, x_k))\beta(d(x_{k-1}, x_k))\gamma^{jr}(d(x_{k-1}, x_k))d(x_k, x_{k-1}). \tag{3.7}$$

Thus,

$$1 \leq \lim_{k \to \infty} \inf \left\{ \sum_{j=0}^{\infty} \sum_{r=0}^{\infty} \alpha(d(x_{k-1}, x_k))\beta(d(x_{k-1}, x_k))\gamma^{jr}(d(x_{k-1}, x_k)) \right\}. \tag{3.8}$$

This implies that $\lim_{k \to \infty} d(x_{k-1}, x_k) = 0$, which is a contradiction. Therefore,

$$\lim_{k \to \infty} d(x_k, x_{k+1}) = 0. \tag{3.9}$$

Now, we shall show that $\{x_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence in X. Suppose that there exists an $\epsilon^* > 0$ for which we define two sub-sequences $\{x_{k_j}\}_{j \in \mathbb{N}}$ and $\{x_{k_h}\}_{h \in \mathbb{N}}$ of $\{x_k\}_{k \in \mathbb{N}}$ such that for $k_j > k_h > k$, $x_{k_j} \geq x_{k_j - 1}$ and $x_{k_j} \neq x_{k_h}$ so that $d(x_{k_j}, x_{k_h}) > \epsilon^*$ and $d(x_{k_{j-1}}, x_{k_j}) < \eta$, $d(x_{k_{h-1}}, x_{k_h}) < \vartheta$. Now $d(x_{k_j}, x_{k_h})^2 = d(Tx_{k_{j-1}}, Tx_{k_{h-1}})^2$. Take $x = x_{k_{j-1}}$ and $y = x_{k_{h-1}}$, so inequality (3.1) becomes;

$$\begin{aligned} d(x_{k_j}, x_{k_h})^2 &= d(Tx_{k_{j-1}}, Tx_{k_{h-1}})^2 \\ &\leq \alpha(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_{h-1}})^{A_\beta^{B_\gamma^{C_\delta^{D_\epsilon}}}}, \end{aligned} \tag{3.10}$$

where,

$$A_\beta := \frac{\beta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, Tx_{k_{j-1}})}{1 + \epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, Tx_{k_{j-1}})}, \quad B_\gamma := \frac{\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, Tx_{k_{h-1}})}{1 + \epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, Tx_{k_{j-1}})},$$

$$C_\delta := \frac{\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, Tx_{k_{h-1}})}{1 + \epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, Tx_{k_{j-1}})}, \quad D_\epsilon := \frac{\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, Tx_{k_{j-1}})}{1 + \epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, Tx_{k_{j-1}})},$$

for $j_k > h_k > k$, hence,

$$A_\beta := \frac{\beta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_j})}{1+\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})}, \quad B_\gamma := \frac{\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})}{1+\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})},$$

$$C_\delta := \frac{\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})}{1+\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})}, \quad D_\epsilon := \frac{\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})}{1+\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})}.$$

So that,

$$\begin{aligned}
A_\beta &:= \frac{\beta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_j})}{1+\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})} \le \beta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_j}),\\
B_\gamma &:= \frac{\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})}{1+\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})} \le \gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h}),\\
C_\delta &:= \frac{\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})}{1+\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})} \le \delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}),\\
D_\epsilon &:= \frac{\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})}{1+\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})} \le \epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}),
\end{aligned}$$

for $k_j > k_h > k$. Thus, from inequality (3.10), we get;

$$\begin{aligned}
d(x_{k_j}, x_{k_h})^2 &= d(Tx_{k_{j-1}}, Tx_{k_{h-1}})^2\\
&\le \alpha(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_{h-1}})^{A_\beta^{B_\gamma^{C_\delta^{D_\epsilon}}}}\\
&= \alpha(d(x_{k_{j-1}}, x_{k_{h-1}}))\exp\Big\{A_\beta^{B_\gamma^{C_\delta^{D_\epsilon}}}\ln(d(x_{k_{j-1}}, x_{k_{h-1}}))\Big\}\\
&= \alpha(d(x_{k_{j-1}}, x_{k_{h-1}}))\Big\{1 + A_\beta^{B_\gamma^{C_\delta^{D_\epsilon}}}\ln(d(x_{k_{j-1}}, x_{k_{h-1}})) + \frac{1}{2}A_\beta^{B_\gamma^{C_\delta^{D_\epsilon^2}}}\ln^2(d(x_{k_{j-1}}, x_{k_{h-1}})) + \cdots\\
&+ \frac{1}{n!}A_\beta^{B_\gamma^{C_\delta^{D_\epsilon^n}}}\ln^n(d(x_{k_{j-1}}, x_{k_{h-1}})) + \cdots\Big\}\\
&\le \alpha(d(x_{k_{j-1}}, x_{k_{h-1}}))\Big\{A_\beta^{B_\gamma^{C_\delta^{D_\epsilon}}}\ln(d(x_{k_{j-1}}, x_{k_{h-1}})) + \frac{1}{2}A_\beta^{B_\gamma^{C_\delta^{D_\epsilon^2}}}\ln^2(d(x_{k_{j-1}}, x_{k_{h-1}})) + \cdots\\
&+ \frac{1}{n!}A_\beta^{B_\gamma^{C_\delta^{D_\epsilon^n}}}\ln^n(d(x_{k_{j-1}}, x_{k_{h-1}})) + \cdots\Big\}\\
&= \alpha(d(x_{k_{j-1}}, x_{k_{h-1}}))\Big\{\exp\Big\{B_\gamma^{C_\delta^{D_\epsilon}}\ln(A_\beta)\Big\}\ln(d(x_{k_{j-1}}, x_{k_{h-1}}))\\
&+ \frac{1}{2}\exp\Big\{B_\gamma^{C_\delta^{D_\epsilon^2}}\ln(A_\beta)\Big\}\ln^2(d(x_{k_{j-1}}, x_{k_{h-1}})) + \cdots + \frac{1}{n!}\exp\Big\{B_\gamma^{C_\delta^{D_\epsilon^n}}\ln(A_\beta)\Big\}\ln^n(d(x_{k_{j-1}}, x_{k_{h-1}})) + \cdots\Big\}\\
&= \alpha(d(x_{k_{j-1}}, x_{k_{h-1}}))\Big\{\Big(1 + B_\gamma^{C_\delta^{D_\epsilon}}\ln(A_\beta) + \frac{1}{2}B_\gamma^{C_\delta^{D_\epsilon^2}}\ln^2(A_\beta) + \cdots + \frac{1}{m!}B_\gamma^{C_\delta^{D_\epsilon^m}}\ln^m(A_\beta)\Big)\ln(d(x_{k_{j-1}}, x_{k_{h-1}}))\\
&+ \frac{1}{2}\Big(1 + B_\gamma^{C_\delta^{D_\epsilon^2}}\ln(A_\beta) + \frac{1}{2}B_\gamma^{C_\delta^{D_\epsilon^4}}\ln^2(A_\beta) + \cdots + \frac{1}{m!}B_\gamma^{C_\delta^{D_\epsilon^{2m}}}\ln^m(A_\beta)\Big)\ln^2(d(x_{k_{j-1}}, x_{k_{h-1}})) + \cdots\\
&+ \frac{1}{n!}\Big(1 + B_\gamma^{C_\delta^{D_\epsilon^n}}\ln(A_\beta) + \frac{1}{2}B_\gamma^{C_\delta^{D_\epsilon^{4n}}}\ln^2(A_\beta) + \cdots + \frac{1}{m!}B_\gamma^{C_\delta^{D_\epsilon^{mn}}}\ln^m(A_\beta)\Big)\ln^n(d(x_{k_{j-1}}, x_{k_{h-1}})) + \cdots\Big\}\\
&\le \alpha(d(x_{k_{j-1}}, x_{k_{h-1}}))\Big\{\Big(B_\gamma^{C_\delta^{D_\epsilon}}\ln(A_\beta) + \frac{1}{2}B_\gamma^{C_\delta^{D_\epsilon^2}}\ln^2(A_\beta) + \cdots + \frac{1}{m!}B_\gamma^{C_\delta^{D_\epsilon^m}}\ln^k(A_\beta)\Big)\ln(d(x_{k_{j-1}}, x_{k_{h-1}}))\\
&+ \frac{1}{2}\Big(B_\gamma^{C_\delta^{D_\epsilon^2}}\ln(A_\beta) + \frac{1}{2}B_\gamma^{C_\delta^{D_\epsilon^4}}\ln^2(A_\beta) + \cdots + \frac{1}{m!}B_\gamma^{C_\delta^{D_\epsilon^{2m}}}\ln^k(A_\beta)\Big)\ln^2(d(x_{k_{j-1}}, x_{k_{h-1}}))\\
&+ \cdots + \frac{1}{n!}\Big(B_\gamma^{C_\delta^{D_\epsilon^n}}\ln(A_\beta) + \frac{1}{2}B_\gamma^{C_\delta^{D_\epsilon^{4n}}}\ln^2(A_\beta) + \cdots + \frac{1}{m!}B_\gamma^{C_\delta^{D_\epsilon^{mn}}}\ln^m(A_\beta)\Big)\ln^n(d(x_{k_{j-1}}, x_{k_{h-1}})) + \cdots\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \alpha(d(x_{k_{j-1}}, x_{k_{h-1}}))\bigg\{\bigg(\bigg(\exp\bigg(C_\delta^{D_\epsilon}\ln(B_\gamma)\bigg)\ln(A_\beta)\bigg) + \frac{1}{2}\bigg(\exp\bigg(C_\delta^{D_\epsilon^2}\ln(B_\gamma)\bigg)\bigg)\ln^2(A_\beta) + \cdots \\
&+ \frac{1}{m!}\bigg(\exp\bigg(C_\delta^{D_\epsilon^m}\ln(B_\gamma)\bigg)\bigg)\ln^m(A_\beta)\ln(d(x_{k_{j-1}}, x_{k_{h-1}})) + \frac{1}{2}\bigg(\bigg(\exp\bigg(C_\delta^{D_\epsilon^2}\ln(B_\gamma)\bigg)\bigg)\bigg)\ln(A_\beta) \\
&+ \frac{1}{2}\bigg(\exp\bigg(C_\delta^{D_\epsilon^4}\ln(B_\gamma)\bigg)\bigg)\ln^2(A_\beta) + \cdots + \frac{1}{m!}\bigg(\exp\bigg(C_\delta^{D_\epsilon^{2m}}\ln(B_\gamma)\bigg)\bigg)\ln^m(A_\beta)\ln^2(d(x_{k_{j-1}}, x_{k_{h-1}})) + \cdots \\
&+ \frac{1}{n!}\bigg(\bigg(\exp\bigg(C_\delta^{D_\epsilon}\ln(B_\gamma)\bigg)\bigg)\ln(A_\beta) + \frac{1}{2}\bigg(\exp\bigg(C_\delta^{D_\epsilon^4}\ln(B_\gamma)\bigg)\bigg)\ln^2(A_\beta) + \cdots \\
&+ \frac{1}{m!}\bigg(\exp\bigg(C_\delta^{D_\epsilon^{mn}}\ln(B_\gamma)\bigg)\bigg)\ln^m(A_\beta)\bigg)\ln^n(d(x_{k_{j-1}}, x_{k_{h-1}})) + \cdots\bigg\} \\
&= \alpha(d(x_{k_{j-1}}, x_{k_{h-1}}))\bigg\{\bigg(\bigg(1 + C_\delta^{D_\epsilon}\ln(B_\gamma) + \frac{1}{2}C_\delta^{D_\epsilon^2}\ln^2(B_\gamma) + \cdots + \frac{1}{l!}C_\delta^{D_\epsilon^l}\ln^l(B_\gamma)\bigg)\bigg)\ln(A_\beta) \\
&+ \frac{1}{2}\bigg(\bigg(1 + C_\delta^{D_\epsilon^2}\ln(B_\gamma) + \frac{1}{2}C_\delta^{D_\epsilon^4}\ln^2(B_\gamma) + \cdots + \frac{1}{l!}C_\delta^{D_\epsilon^{2l}}\ln^l(B_\gamma)\bigg)\bigg)\ln^2(A_\beta) + \cdots \\
&+ \frac{1}{m!}\bigg(\bigg(1 + C_\delta^{D_\epsilon^l}\ln(B_\gamma) + \frac{1}{2}C_\delta^{D_\epsilon^{2l}}\ln^2(B_\gamma) + \cdots + \frac{1}{m!}C_\delta^{D_\epsilon^{ml}}\ln^l(B_\gamma)\bigg)\bigg)\ln^l(A_\beta) \times \ln(d(x_{k_{j-1}}, x_{k_{h-1}})) \\
&+ \frac{1}{2}\bigg(\bigg(\bigg(1 + C_\delta^{D_\epsilon^2}\ln(B_\gamma) + \frac{1}{2}C_\delta^{D_\epsilon^4}\ln^2(B_\gamma) + \cdots + \frac{1}{m!}C_\delta^{D_\epsilon^{2l}}\ln^l(B_\gamma)\bigg)\ln(A_\beta) \\
&+ \frac{1}{2}\bigg(\bigg(1 + C_\delta^{D_\epsilon^4}\ln(B_\gamma) + \frac{1}{2}C_\delta^{D_\epsilon^6}\ln^2(B_\gamma) + \cdots + \frac{1}{m!}C_\delta^{D_\epsilon^{4l}}\ln^l(B_\gamma)\bigg)\ln^2(A_\beta) + \cdots \\
&+ \frac{1}{m!}\bigg(\bigg(1 + C_\delta^{D_\epsilon^{2l}}\ln(B_\gamma) + \frac{1}{2}C_\delta^{D_\epsilon^{4l}}\ln^2(B_\gamma) + \cdots + \frac{1}{l!}C_\delta^{D_\epsilon^{2ml}}\ln^l(B_\gamma)\bigg)\ln^l(A_\beta)\bigg)\bigg)\bigg) \times \ln^2(d(x_{k_{j-1}}, x_{k_{h-1}})) \\
&+ \cdots + \frac{1}{n!}\bigg(\bigg(\bigg(1 + C_\delta^{D_\epsilon^l}\ln(B_\gamma) + \frac{1}{2}C_\delta^{D_\epsilon^{2l}}\ln^2(B_\gamma) + \cdots + \frac{1}{l!}C_\delta^{D_\epsilon^{ml}}\ln^l(B_\gamma)\bigg)\ln(A_\beta) \\
&+ \frac{1}{2}\bigg(\bigg(1 + C_\delta^{D_\epsilon^4}\ln(B_\gamma) + \frac{1}{2}C_\delta^{D_\epsilon^6}\ln^2(B_\gamma) + \cdots + \frac{1}{l!}C_\delta^{D_\epsilon^{4l}}\ln^l(B_\gamma)\bigg)\ln^2(A_\beta) + \cdots \\
&+ \frac{1}{l!}\bigg(\bigg(1 + C_\delta^{D_\epsilon^{lm}}\ln(B_\gamma) + \frac{1}{2}C_\delta^{D_\epsilon^{2lm}}\ln^2(B_\gamma) + \cdots + \frac{1}{l!}C_\delta^{D_\epsilon^{nlm}}\ln^m(B_\gamma)\bigg)\ln^l(A_\beta)\bigg)\bigg)\bigg) \\
&\times \ln^n(d(x_{k_{j-1}}, x_{k_{h-1}})) + \cdots\bigg\} \\
&\le \alpha(d(x_{k_{j-1}}, x_{k_{h-1}}))\bigg\{\bigg(C_\delta^{D_\epsilon}\ln(B_\gamma) + \frac{1}{2}C_\delta^{D_\epsilon^2}\ln^2(B_\gamma) + \cdots + \frac{1}{l!}C_\delta^{D_\epsilon^l}\ln^l(B_\gamma)\bigg)\ln(A_\beta) \\
&+ \frac{1}{2}\bigg(\bigg(C_\delta^{D_\epsilon^2}\ln(B_\gamma) + \frac{1}{2}C_\delta^{D_\epsilon^4}\ln^2(B_\gamma) + \cdots + \frac{1}{l!}C_\delta^{D_\epsilon^{2l}}\ln^l(B_\gamma)\bigg)\bigg)\ln^2(A_\beta) + \cdots \\
&+ \frac{1}{m!}\bigg(\bigg(C_\delta^{D_\epsilon^l}\ln(B_\gamma) + \frac{1}{2}C_\delta^{D_\epsilon^{2l}}\ln^2(B_\gamma) + \cdots + \frac{1}{m!}C_\delta^{D_\epsilon^{ml}}\ln^l(B_\gamma)\bigg)\bigg)\ln^l(A_\beta) \times \ln(d(x_{k_{j-1}}, x_{k_{h-1}})) \\
&+ \frac{1}{2}\bigg(\bigg(C_\delta^{D_\epsilon^2}\ln(B_\gamma) + \frac{1}{2}C_\delta^{D_\epsilon^4}\ln^2(B_\gamma) + \cdots + \frac{1}{m!}C_\delta^{D_\epsilon^{2l}}\ln^l(B_\gamma)\bigg)\ln(A_\beta) \\
&+ \frac{1}{2}\bigg(\bigg(C_\delta^{D_\epsilon^4}\ln(B_\gamma) + \frac{1}{2}C_\delta^{D_\epsilon^6}\ln^2(B_\gamma) + \cdots + \frac{1}{m!}C_\delta^{D_\epsilon^{4l}}\ln^l(B_\gamma)\bigg)\ln^2(A_\beta) + \cdots \\
&+ \frac{1}{m!}\bigg(\bigg(C_\delta^{D_\epsilon^{2l}}\ln(B_\gamma) + \frac{1}{2}C_\delta^{D_\epsilon^{4l}}\ln^2(B_\gamma) + \cdots + \frac{1}{l!}C_\delta^{D_\epsilon^{2ml}}\ln^l(B_\gamma)\bigg)\ln^l(A_\beta)\bigg)\bigg)\bigg) \times \ln^2(d(x_{k_{j-1}}, x_{k_{h-1}})) \\
&+ \cdots + \frac{1}{n!}\bigg(\bigg(C_\delta^{D_\epsilon^l}\ln(B_\gamma) + \frac{1}{2}C_\delta^{D_\epsilon^{2l}}\ln^2(B_\gamma) + \cdots + \frac{1}{l!}C_\delta^{D_\epsilon^{ml}}\ln^l(B_\gamma)\bigg)\ln(A_\beta) \\
&+ \frac{1}{2}\bigg(\bigg(C_\delta^{D_\epsilon^4}\ln(B_\gamma) + \frac{1}{2}C_\delta^{D_\epsilon^6}\ln^2(B_\gamma) + \cdots + \frac{1}{l!}C_\delta^{D_\epsilon^{4l}}\ln^l(B_\gamma)\bigg)\ln^2(A_\beta) + \cdots \\
&+ \frac{1}{l!}\bigg(\bigg(C_\delta^{D_\epsilon^{lm}}\ln(B_\gamma) + \frac{1}{2}C_\delta^{D_\epsilon^{2lm}}\ln^2(B_\gamma) + \cdots + \frac{1}{l!}C_\delta^{D_\epsilon^{nlm}}\ln^m(B_\gamma)\bigg)\ln^l(A_\beta)\bigg)\bigg)\bigg)
\end{aligned}
$$

$$
\begin{aligned}
&\times \ln^n(d(x_{k_{j-1}}, x_{k_{h-1}})) + \cdots \Big\} \\
&= \alpha(d(x_{k_{j-1}}, x_{k_{h-1}}))\Big\{\Big(\Big(\exp(D_\epsilon \ln(C_\delta))\ln(B_\gamma) + \frac{1}{2}\exp(D_\epsilon^2 \ln(C_\delta))\ln^2(B_\gamma) + \cdots \\
&+ \frac{1}{l!}\exp(D_\epsilon^l \ln(C_\delta))\ln^l(B_\gamma)\Big)\ln(A_\beta) \\
&+ \frac{1}{2}\Big(\Big(\exp(D_\epsilon^2 \ln(C_\delta))\ln(B_\gamma) + \frac{1}{2}\exp(D_\epsilon^4 \ln(C_\delta))\ln^2(B_\gamma) + \cdots + \frac{1}{l!}\exp(D_\epsilon^{2l} \ln(C_\delta))\ln^l(B_\gamma)\Big)\ln^2(A_\beta) \\
&+ \cdots + \frac{1}{m!}\Big(\Big(\exp(D_\epsilon^l \ln(C_\delta))\ln(B_\gamma) + \frac{1}{2}\exp(D_\epsilon^{2l} \ln(C_\delta))\ln^2(B_\gamma) + \cdots \\
&+ \frac{1}{l!}\exp(D_\epsilon^{ml} \ln(C_\delta))\ln^m(B_\gamma)\Big)\ln^l(A_\beta)\Big)\Big)\Big)\ln(d(x_{k_{j-1}}, x_{k_{h-1}})) \\
&+ \frac{1}{2}\Big(\Big(\Big(\exp(D_\epsilon^2 \ln(C_\delta))\ln(B_\gamma) + \frac{1}{2}\exp(D_\epsilon^4 \ln(C_\delta))\ln^2(B_\gamma) + \cdots + \frac{1}{l!}\exp(D_\epsilon^{2l} \ln(C_\delta))\ln^l(B_\gamma)\Big)\ln(A_\beta) \\
&+ \frac{1}{2}\Big(\Big(\exp(D_\epsilon^4 \ln(C_\delta))\ln(B_\gamma) + \frac{1}{2}\exp(D_\epsilon^6 \ln(C_\delta))\ln^2(B_\gamma) + \cdots + \frac{1}{l!}\exp(D_\epsilon^{4l} \ln(C_\delta))\ln^l(B_\gamma)\Big)\ln^2(A_\beta) \\
&+ \cdots + \frac{1}{l!}\Big(\Big(\exp(D_\epsilon^{2l} \ln(C_\delta))\ln(B_\gamma) + \frac{1}{2}\exp(D_\epsilon^{4l} \ln(C_\delta))\ln^2(B_\gamma) + \cdots \\
&+ \frac{1}{l!}\exp(D_\epsilon^{2ml} \ln(C_\delta))\ln^l(B_\gamma)\Big)\ln^m(A_\beta)\Big)\Big)\Big)\ln^2(d(x_{k_{j-1}}, x_{k_{h-1}})) + \cdots \\
&+ \frac{1}{n!}\Big(\Big(\Big(\exp(D_\epsilon^l \ln(C_\delta))\ln(B_\gamma) + \frac{1}{2}\exp(D_\epsilon^{2l} \ln(C_\delta))\ln^2(B_\gamma) + \cdots + \frac{1}{l!}\exp(D_\epsilon^{ml} \ln(C_\delta))\ln^l(B_\gamma)\Big)\ln(A_\beta) \\
&+ \frac{1}{2}\Big(\Big(\exp(D_\epsilon^4 \ln(C_\delta))\ln(B_\gamma) + \frac{1}{2}\exp(D_\epsilon^6 \ln(C_\delta))\ln^2(B_\gamma) + \cdots + \frac{1}{l!}\exp(D_\epsilon^{4l} \ln(C_\delta))\ln^l(B_\gamma)\Big)\ln^2(A_\beta) \\
&+ \cdots + \frac{1}{l!}\Big(\Big(\exp(D_\epsilon^{ml} \ln(C_\delta))\ln(B_\gamma) + \frac{1}{2}\exp(D_\epsilon^{2ml} \ln(C_\delta))\ln^2(B_\gamma) + \cdots \\
&+ \frac{1}{l!}\exp(D_\epsilon^{nml} \ln(C_\delta))\ln^l(B_\gamma)\Big)\ln^m(A_\beta)\Big)\Big)\Big)\ln^n(d(x_{k_{j-1}}, x_{k_{h-1}})) + \cdots \Big\} \\
&= \alpha(d(x_{k_{j-1}}, x_{k_{h-1}}))\Big\{\Big(\Big(\Big(1 + D_\epsilon \ln(C_\delta) + \frac{1}{2}D_\epsilon^2 \ln^2(C_\delta) + \cdots + \frac{1}{r!}D_\epsilon^r \ln^r(C_\delta)\Big)\ln(B_\gamma) \\
&+ \frac{1}{2}\Big(1 + D_\epsilon^2 \ln(C_\delta) + \frac{1}{2}D_\epsilon^4 \ln^2(C_\delta) + \cdots + \frac{1}{r!}D_\epsilon^{2r} \ln^r(C_\delta)\Big)\ln^2(B_\gamma) + \cdots \\
&+ \frac{1}{r!}\Big(1 + D_\epsilon^r \ln(C_\delta) + \frac{1}{2}D_\epsilon^{2r} \ln^2(C_\delta) + \cdots + \frac{1}{l!}D_\epsilon^{lr} \ln^r(C_\delta)\Big)\ln^l(B_\gamma)\Big)\ln(A_\beta) \\
&+ \frac{1}{2}\Big(\Big(\Big(1 + D_\epsilon^2 \ln(C_\delta) + \frac{1}{2}D_\epsilon^4 \ln^2(C_\delta) + \cdots + \frac{1}{r!}D_\epsilon^{2r} \ln^r(C_\delta)\Big)\ln(B_\gamma) \\
&+ \frac{1}{2}\Big(1 + D_\epsilon^4 \ln(C_\delta) + \frac{1}{2}D_\epsilon^6 \ln^2(C_\delta) + \cdots + \frac{1}{r!}D_\epsilon^r \ln^r(C_\delta)\Big)\ln^2(B_\gamma) + \cdots \\
&+ \frac{1}{r!}\Big(1 + D_\epsilon^{2r} \ln(C_\delta) + \frac{1}{2}D_\epsilon^{4r} \ln^2(C_\delta) + \cdots + \frac{1}{l!}D_\epsilon^{lr} \ln^r(C_\delta)\Big)\ln^l(B_\gamma)\Big)\ln^2(A_\beta) + \cdots \\
&+ \frac{1}{r!}\Big(\Big(\Big(1 + D_\epsilon^r \ln(C_\delta) + \frac{1}{2}D_\epsilon^r \ln^2(C_\delta) + \cdots + \frac{1}{l!}D_\epsilon^{lr} \ln^r(C_\delta)\Big)\ln(B_\gamma) \\
&+ \frac{1}{2}\Big(1 + D_\epsilon^{2r} \ln(C_\delta) + \frac{1}{2}D_\epsilon^{4r} \ln^2(C_\delta) + \cdots + \frac{1}{l!}D_\epsilon^{rl} \ln^r(C_\delta)\Big)\ln^2(B_\gamma) + \cdots \\
&+ \frac{1}{r!}\Big(1 + D_\epsilon^{rl} \ln(C_\delta) + \frac{1}{2}D_\epsilon^{2rl} \ln^2(C_\delta) + \cdots + \frac{1}{l!}D_\epsilon^{rlm} \ln^r(C_\delta)\Big)\ln^l(B_\gamma)\Big)\ln^m(A_\beta)\Big)\Big)\Big) \times \ln(d(x_{k_{j-1}}, x_{k_{h-1}})) \\
&+ \frac{1}{2}\Big(\Big(\Big(1 + D_\epsilon^2 \ln(C_\delta) + \frac{1}{2}D_\epsilon^4 \ln^2(C_\delta) + \cdots + \frac{1}{r!}D_\epsilon^r \ln^r(C_\delta)\Big)\ln(B_\gamma)
\end{aligned}
$$

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\begin{aligned}
&+\frac{1}{2}\left(1+D_\epsilon^4\ln(C_\delta)+\frac{1}{2}D_\epsilon^8\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{4r}\ln^r(C_\delta)\right)\ln^2(B_\gamma)+\cdots\\
&+\frac{1}{l!}\left(1+D_\epsilon^{2r}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{4r}\ln^2(C_\delta)+\cdots+\frac{1}{l!}D_\epsilon^{2rl}\ln^r(C_\delta)\right)\ln^l(B_\gamma)\Bigg)\ln(A_\beta)\\
&+\frac{1}{2}\Bigg(\Bigg(\left(1+D_\epsilon^4\ln(C_\delta)+\frac{1}{2}D_\epsilon^8\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{4r}\ln^r(C_\delta)\right)\ln(B_\gamma)\\
&+\frac{1}{2}\left(1+D_\epsilon^6\ln(C_\delta)+\frac{1}{2}D_\epsilon^{12}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{6r}\ln^r(C_\delta)\right)\ln^2(B_\gamma)+\cdots\\
&+\frac{1}{m!}\left(1+D_\epsilon^{4r}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{8r}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{4r}\ln^r(C_\delta)\right)\ln^l(B_\gamma)\Bigg)\ln^2(A_\beta)+\cdots\\
&+\frac{1}{m!}\Bigg(\Bigg(\left(1+D_\epsilon^{2r}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{4r}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{2mr}\ln^r(C_\delta)\right)\ln(B_\gamma)\\
&+\frac{1}{2}\left(1+D_\epsilon^{4r}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{8r}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{4mr}\ln^r(C_\delta)\right)\ln^2(B_\gamma)+\cdots\\
&+\frac{1}{m!}\left(1+D_\epsilon^{2mr}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{4mr}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{2mlr}\ln^r(C_\delta)\right)\ln^l(B_\gamma)\Bigg)\ln^m(A_\beta)\Bigg)\Bigg)\Bigg)\\
&\times\ln^2(d(x_{k_{j-1}},x_{k_{h-1}}))+\cdots\\
&+\frac{1}{n!}\Bigg(\Bigg(\left(1+D_\epsilon^{r}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{2r}\ln^2(C_\delta)+\cdots+\frac{1}{l!}D_\epsilon^{lr}\ln^r(C_\delta)\right)\ln(B_\gamma)\\
&+\frac{1}{2}\left(1+D_\epsilon^{2r}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{4r}\ln^2(C_\delta)+\cdots+\frac{1}{l!}D_\epsilon^{lr}\ln^r(C_\delta)\right)\ln^2(B_\gamma)+\cdots\\
&+\frac{1}{m!}\left(1+D_\epsilon^{rm}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{2mr}\ln^2(C_\delta)+\cdots+\frac{1}{m!}D_\epsilon^{rml}\ln^r(C_\delta)\right)\ln^l(B_\gamma)\Bigg)\ln(A_\beta)\\
&+\frac{1}{2}\Bigg(\Bigg(\left(1+D_\epsilon^4\ln(C_\delta)+\frac{1}{2}D_\epsilon^8\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{r}\ln^r(C_\delta)\right)\ln(B_\gamma)\\
&+\frac{1}{2}\left(1+D_\epsilon^6\ln(C_\delta)+\frac{1}{2}D_\epsilon^{12}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{6r}\ln^r(C_\delta)\right)\ln^2(B_\gamma)+\cdots\\
&+\frac{1}{r!}\left(1+D_\epsilon^{4r}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{8r}\ln^2(C_\delta)+\cdots+\frac{1}{l!}D_\epsilon^{4lr}\ln^r(C_\delta)\right)\ln^l(B_\gamma)\Bigg)\ln^2(A_\beta)+\cdots\\
&+\frac{1}{m!}\Bigg(\Bigg(\left(1+D_\epsilon^{mr}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{2mr}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{mlr}\ln^r(C_\delta)\right)\ln(B_\gamma)\\
&+\frac{1}{2}\left(1+D_\epsilon^{2mr}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{4mr}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{2mlr}\ln^r(C_\delta)\right)\ln^2(B_\gamma)+\cdots\\
&+\frac{1}{n!}\left(1+D_\epsilon^{lrn}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{2nlr}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{nmlr}\ln^r(C_\delta)\right)\ln^l(B_\gamma)\Bigg)\ln^m(A_\beta)\Bigg)\Bigg)\Bigg)\\
&\times\ln^n(d(x_{k_{j-1}},x_{k_{h-1}}))+\cdots\Bigg\}\\
&\le\alpha(d(x_{k_{j-1}},x_{k_{h-1}}))\Bigg\{\Bigg(\Bigg(\Bigg(\left(D_\epsilon\ln(C_\delta)+\frac{1}{2}D_\epsilon^2\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{r}\ln^r(C_\delta)\right)\ln(B_\gamma)\\
&+\frac{1}{2}\left(D_\epsilon^2\ln(C_\delta)+\frac{1}{2}D_\epsilon^4\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{2r}\ln^r(C_\delta)\right)\ln^2(B_\gamma)+\cdots\\
&+\frac{1}{r!}\left(D_\epsilon^{r}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{2r}\ln^2(C_\delta)+\cdots+\frac{1}{l!}D_\epsilon^{lr}\ln^r(C_\delta)\right)\ln^l(B_\gamma)\Bigg)\ln(A_\beta)\\
&+\frac{1}{2}\Bigg(\Bigg(\left(D_\epsilon^2\ln(C_\delta)+\frac{1}{2}D_\epsilon^4\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{2r}\ln^r(C_\delta)\right)\ln(B_\gamma)\\
&+\frac{1}{2}\left(D_\epsilon^4\ln(C_\delta)+\frac{1}{2}D_\epsilon^6\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{r}\ln^r(C_\delta)\right)\ln^2(B_\gamma)+\cdots\\
&+\frac{1}{r!}\left(D_\epsilon^{2r}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{4r}\ln^2(C_\delta)+\cdots+\frac{1}{l!}D_\epsilon^{lr}\ln^r(C_\delta)\right)\ln^l(B_\gamma)\Bigg)\ln^2(A_\beta)+\cdots
\end{aligned}
$$

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\begin{aligned}
&+\frac{1}{r!}\Bigg(\Bigg(\Bigg(D_\epsilon^{r}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{r}\ln^2(C_\delta)+\cdots+\frac{1}{l!}D_\epsilon^{lr}\ln^r(C_\delta)\Bigg)\ln(B_\gamma)\\
&+\frac{1}{2}\Bigg(D_\epsilon^{2r}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{4r}\ln^2(C_\delta)+\cdots+\frac{1}{l!}D_\epsilon^{rl}\ln^r(C_\delta)\Bigg)\ln^2(B_\gamma)+\cdots\\
&+\frac{1}{r!}\Bigg(D_\epsilon^{rl}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{2rl}\ln^2(C_\delta)+\cdots+\frac{1}{l!}D_\epsilon^{rlm}\ln^r(C_\delta)\Bigg)\ln^l(B_\gamma)\Bigg)\ln^m(A_\beta)\Bigg)\Bigg)\Bigg)\times\ln(d(x_{k_{j-1}},x_{k_{h-1}}))\\
&+\frac{1}{2}\Bigg(\Bigg(\Bigg(D_\epsilon^{2}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{4}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{r}\ln^r(C_\delta)\Bigg)\ln(B_\gamma)\\
&+\frac{1}{2}\Bigg(D_\epsilon^{4}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{8}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{4r}\ln^r(C_\delta)\Bigg)\ln^2(B_\gamma)+\cdots\\
&+\frac{1}{l!}\Bigg(D_\epsilon^{2r}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{4r}\ln^2(C_\delta)+\cdots+\frac{1}{l!}D_\epsilon^{2rl}\ln^r(C_\delta)\Bigg)\ln^l(B_\gamma)\Bigg)\ln(A_\beta)\\
&+\frac{1}{2}\Bigg(\Bigg(\Bigg(D_\epsilon^{4}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{8}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{4r}\ln^r(C_\delta)\Bigg)\ln(B_\gamma)\\
&+\frac{1}{2}\Bigg(D_\epsilon^{6}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{12}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{6r}\ln^r(C_\delta)\Bigg)\ln^2(B_\gamma)+\cdots\\
&+\frac{1}{m!}\Bigg(D_\epsilon^{4r}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{8r}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{4r}\ln^r(C_\delta)\Bigg)\ln^l(B_\gamma)\Bigg)\ln^2(A_\beta)+\cdots\\
&+\frac{1}{m!}\Bigg(\Bigg(\Bigg(D_\epsilon^{2r}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{4r}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{2mr}\ln^r(C_\delta)\Bigg)\ln(B_\gamma)\\
&+\frac{1}{2}\Bigg(D_\epsilon^{4r}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{8r}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{4mr}\ln^r(C_\delta)\Bigg)\ln^2(B_\gamma)+\cdots\\
&+\frac{1}{m!}\Bigg(D_\epsilon^{2mr}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{4mr}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{2mlr}\ln^r(C_\delta)\Bigg)\ln^l(B_\gamma)\Bigg)\ln^m(A_\beta)\Bigg)\Bigg)\Bigg)\\
&\times\ln^2(d(x_{k_{j-1}},x_{k_{h-1}}))+\cdots\\
&+\frac{1}{n!}\Bigg(\Bigg(\Bigg(D_\epsilon^{r}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{2r}\ln^2(C_\delta)+\cdots+\frac{1}{l!}D_\epsilon^{lr}\ln^r(C_\delta)\Bigg)\ln(B_\gamma)\\
&+\frac{1}{2}\Bigg(D_\epsilon^{2r}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{4r}\ln^2(C_\delta)+\cdots+\frac{1}{l!}D_\epsilon^{lr}\ln^r(C_\delta)\Bigg)\ln^2(B_\gamma)+\cdots\\
&+\frac{1}{m!}\Bigg(D_\epsilon^{rm}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{2mr}\ln^2(C_\delta)+\cdots+\frac{1}{m!}D_\epsilon^{rml}\ln^r(C_\delta)\Bigg)\ln^l(B_\gamma)\Bigg)\ln(A_\beta)\\
&+\frac{1}{2}\Bigg(\Bigg(\Bigg(D_\epsilon^{4}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{8}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{r}\ln^r(C_\delta)\Bigg)\ln(B_\gamma)\\
&+\frac{1}{2}\Bigg(D_\epsilon^{6}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{12}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{6r}\ln^r(C_\delta)\Bigg)\ln^2(B_\gamma)+\cdots\\
&+\frac{1}{r!}\Bigg(D_\epsilon^{4r}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{8r}\ln^2(C_\delta)+\cdots+\frac{1}{l!}D_\epsilon^{4lr}\ln^r(C_\delta)\Bigg)\ln^l(B_\gamma)\Bigg)\ln^2(A_\beta)+\cdots\\
&+\frac{1}{m!}\Bigg(\Bigg(\Bigg(D_\epsilon^{mr}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{2mr}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{mlr}\ln^r(C_\delta)\Bigg)\ln(B_\gamma)\\
&+\frac{1}{2}\Bigg(D_\epsilon^{2mr}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{4mr}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{2mlr}\ln^r(C_\delta)\Bigg)\ln^2(B_\gamma)+\cdots\\
&+\frac{1}{n!}\Bigg(D_\epsilon^{lrn}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{2nlr}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{nmlr}\ln^r(C_\delta)\Bigg)\ln^l(B_\gamma)\Bigg)\ln^m(A_\beta)\Bigg)\Bigg)\Bigg)\\
&\times\ln^n(d(x_{k_{j-1}},x_{k_{h-1}}))+\cdots\Bigg\}\\
&=\alpha(d(x_{k_{j-1}},x_{k_{h-1}}))\Bigg\{\Bigg(\Bigg(\Bigg(D_\epsilon\ln(C_\delta)+\frac{1}{2}D_\epsilon^{2}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{r}\ln^r(C_\delta)\Bigg)\ln(B_\gamma)\\
&+\frac{1}{2}\Bigg(D_\epsilon^{2}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{4}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{2r}\ln^r(C_\delta)\Bigg)\ln^2(B_\gamma)+\cdots
\end{aligned}
$$

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\begin{aligned}
&+\frac{1}{r!}\left(D_\epsilon^r\ln(C_\delta)+\frac{1}{2}D_\epsilon^{2r}\ln^2(C_\delta)+\cdots+\frac{1}{l!}D_\epsilon^{lr}\ln^r(C_\delta)\right)\ln^l(B_\gamma)\Big)\ln(A_\beta)\\
&+\frac{1}{2}\Bigg(\Bigg(\bigg(D_\epsilon^2\ln(C_\delta)+\frac{1}{2}D_\epsilon^4\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{2r}\ln^r(C_\delta)\bigg)\ln(B_\gamma)\\
&+\frac{1}{2}\left(D_\epsilon^4\ln(C_\delta)+\frac{1}{2}D_\epsilon^6\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^r\ln^r(C_\delta)\right)\ln^2(B_\gamma)+\cdots\\
&+\frac{1}{r!}\left(D_\epsilon^{2r}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{4r}\ln^2(C_\delta)+\cdots+\frac{1}{l!}D_\epsilon^{lr}\ln^r(C_\delta)\right)\ln^l(B_\gamma)\Big)\ln^2(A_\beta)+\cdots\\
&+\frac{1}{r!}\Bigg(\Bigg(\bigg(D_\epsilon^r\ln(C_\delta)+\frac{1}{2}D_\epsilon^r\ln^2(C_\delta)+\cdots+\frac{1}{l!}D_\epsilon^{lr}\ln^r(C_\delta)\bigg)\ln(B_\gamma)\\
&+\frac{1}{2}\left(D_\epsilon^{2r}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{4r}\ln^2(C_\delta)+\cdots+\frac{1}{l!}D_\epsilon^{rl}\ln^r(C_\delta)\right)\ln^2(B_\gamma)+\cdots\\
&+\frac{1}{r!}\left(D_\epsilon^{rl}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{2rl}\ln^2(C_\delta)+\cdots+\frac{1}{l!}D_\epsilon^{rlm}\ln^r(C_\delta)\right)\ln^l(B_\gamma)\Big)\ln^m(A_\beta)\Big)\Big)\Big)\times\ln(d(x_{k_{j-1}},x_{k_{h-1}}))\\
&+\frac{1}{2}\Bigg(\Bigg(\bigg(D_\epsilon^2\ln(C_\delta)+\frac{1}{2}D_\epsilon^4\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^r\ln^r(C_\delta)\bigg)\ln(B_\gamma)\\
&+\frac{1}{2}\left(D_\epsilon^4\ln(C_\delta)+\frac{1}{2}D_\epsilon^8\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{4r}\ln^r(C_\delta)\right)\ln^2(B_\gamma)+\cdots\\
&+\frac{1}{l!}\left(D_\epsilon^{2r}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{4r}\ln^2(C_\delta)+\cdots+\frac{1}{l!}D_\epsilon^{2rl}\ln^r(C_\delta)\right)\ln^l(B_\gamma)\Big)\ln(A_\beta)\\
&+\frac{1}{2}\Bigg(\Bigg(\bigg(D_\epsilon^4\ln(C_\delta)+\frac{1}{2}D_\epsilon^8\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{4r}\ln^r(C_\delta)\bigg)\ln(B_\gamma)\\
&+\frac{1}{2}\left(D_\epsilon^6\ln(C_\delta)+\frac{1}{2}D_\epsilon^{12}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{6r}\ln^r(C_\delta)\right)\ln^2(B_\gamma)+\cdots\\
&+\frac{1}{m!}\left(D_\epsilon^{4r}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{8r}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{4r}\ln^r(C_\delta)\right)\ln^l(B_\gamma)\Big)\ln^2(A_\beta)+\cdots\\
&+\frac{1}{m!}\Bigg(\Bigg(\bigg(D_\epsilon^{2r}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{4r}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{2mr}\ln^r(C_\delta)\bigg)\ln(B_\gamma)\\
&+\frac{1}{2}\left(D_\epsilon^{4r}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{8r}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{4mr}\ln^r(C_\delta)\right)\ln^2(B_\gamma)+\cdots\\
&+\frac{1}{m!}\left(D_\epsilon^{2mr}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{4mr}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{2mlr}\ln^r(C_\delta)\right)\ln^l(B_\gamma)\Big)\ln^m(A_\beta)\Big)\Big)\Big)\\
&\times\ln^2(d(x_{k_{j-1}},x_{k_{h-1}}))+\cdots\\
&+\frac{1}{n!}\Bigg(\Bigg(\bigg(D_\epsilon^r\ln(C_\delta)+\frac{1}{2}D_\epsilon^{2r}\ln^2(C_\delta)+\cdots+\frac{1}{l!}D_\epsilon^{lr}\ln^r(C_\delta)\bigg)\ln(B_\gamma)\\
&+\frac{1}{2}\left(D_\epsilon^{2r}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{4r}\ln^2(C_\delta)+\cdots+\frac{1}{l!}D_\epsilon^{lr}\ln^r(C_\delta)\right)\ln^2(B_\gamma)+\cdots\\
&+\frac{1}{m!}\left(D_\epsilon^{rm}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{2mr}\ln^2(C_\delta)+\cdots+\frac{1}{m!}D_\epsilon^{rml}\ln^r(C_\delta)\right)\ln^l(B_\gamma)\Big)\ln(A_\beta)\\
&+\frac{1}{2}\Bigg(\Bigg(\bigg(D_\epsilon^4\ln(C_\delta)+\frac{1}{2}D_\epsilon^8\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^r\ln^r(C_\delta)\bigg)\ln(B_\gamma)\\
&+\frac{1}{2}\left(D_\epsilon^6\ln(C_\delta)+\frac{1}{2}D_\epsilon^{12}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{6r}\ln^r(C_\delta)\right)\ln^2(B_\gamma)+\cdots\\
&+\frac{1}{r!}\left(D_\epsilon^{4r}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{8r}\ln^2(C_\delta)+\cdots+\frac{1}{l!}D_\epsilon^{4lr}\ln^r(C_\delta)\right)\ln^l(B_\gamma)\Big)\ln^2(A_\beta)+\cdots\\
&+\frac{1}{m!}\Bigg(\Bigg(\bigg(D_\epsilon^{mr}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{2mr}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{mlr}\ln^r(C_\delta)\bigg)\ln(B_\gamma)\\
&+\frac{1}{2}\left(D_\epsilon^{2mr}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{4mr}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{2mlr}\ln^r(C_\delta)\right)\ln^2(B_\gamma)+\cdots
\end{aligned}
$$

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\begin{aligned}
&+\frac{1}{n!}\bigg(D_\epsilon^{lrn}\ln(C_\delta)+\frac{1}{2}D_\epsilon^{2nlr}\ln^2(C_\delta)+\cdots+\frac{1}{r!}D_\epsilon^{nmlr}\ln^r(C_\delta)\bigg)\ln^l(B_\gamma)\bigg)\ln^m(A_\beta)\bigg)\bigg)\bigg)\\
&\times\ln^n(d(x_{k_{j-1}},x_{k_{h-1}}))+\cdots\bigg\}\\
&\le\alpha(d(x_{k_{j-1}},x_{k_{h-1}}))\bigg\{\bigg(\bigg(\bigg(\bigg(D_\epsilon\ln(C_\delta)+D_\epsilon^2\ln^2(C_\delta)+\cdots+D_\epsilon^r\ln^r(C_\delta)\bigg)\ln(B_\gamma)\\
&+\bigg(D_\epsilon^2\ln(C_\delta)+D_\epsilon^4\ln^2(C_\delta)+\cdots+D_\epsilon^{2r}\ln^r(C_\delta)\bigg)\ln^2(B_\gamma)+\cdots\\
&+\bigg(D_\epsilon^r\ln(C_\delta)+D_\epsilon^{2r}\ln^2(C_\delta)+\cdots+D_\epsilon^{lr}\ln^r(C_\delta)\bigg)\ln^l(B_\gamma)\bigg)\ln(A_\beta)\\
&+\bigg(\bigg(\bigg(D_\epsilon^2\ln(C_\delta)+D_\epsilon^4\ln^2(C_\delta)+\cdots+D_\epsilon^{2r}\ln^r(C_\delta)\bigg)\ln(B_\gamma)\\
&+\bigg(D_\epsilon^4\ln(C_\delta)+D_\epsilon^6\ln^2(C_\delta)+\cdots+D_\epsilon^r\ln^r(C_\delta)\bigg)\ln^2(B_\gamma)+\cdots\\
&+\bigg(D_\epsilon^{2r}\ln(C_\delta)+D_\epsilon^{4r}\ln^2(C_\delta)+\cdots+D_\epsilon^{lr}\ln^r(C_\delta)\bigg)\ln^l(B_\gamma)\bigg)\ln^2(A_\beta)+\cdots\\
&+\bigg(\bigg(\bigg(D_\epsilon^r\ln(C_\delta)+D_\epsilon^{4r}\ln^2(C_\delta)+\cdots+D_\epsilon^{lr}\ln^r(C_\delta)\bigg)\ln(B_\gamma)\\
&+\bigg(D_\epsilon^{2r}\ln(C_\delta)+D_\epsilon^{4r}\ln^2(C_\delta)+\cdots+D_\epsilon^{rl}\ln^r(C_\delta)\bigg)\ln^2(B_\gamma)+\cdots\\
&+\bigg(D_\epsilon^{rl}\ln(C_\delta)+D_\epsilon^{2rl}\ln^2(C_\delta)+\cdots+D_\epsilon^{rlm}\ln^r(C_\delta)\bigg)\ln^l(B_\gamma)\bigg)\ln^m(A_\beta)\bigg)\bigg)\bigg)\times\ln(d(x_{k_{j-1}},x_{k_{h-1}}))\\
&+\bigg(\bigg(\bigg(D_\epsilon^2\ln(C_\delta)+D_\epsilon^4\ln^2(C_\delta)+\cdots+D_\epsilon^r\ln^r(C_\delta)\bigg)\ln(B_\gamma)\\
&+\bigg(D_\epsilon^4\ln(C_\delta)+D_\epsilon^8\ln^2(C_\delta)+\cdots+D_\epsilon^{4r}\ln^r(C_\delta)\bigg)\ln^2(B_\gamma)+\cdots\\
&+\bigg(D_\epsilon^{2r}\ln(C_\delta)+D_\epsilon^{4r}\ln^2(C_\delta)+\cdots+D_\epsilon^{2rl}\ln^r(C_\delta)\bigg)\ln^l(B_\gamma)\bigg)\ln(A_\beta)\\
&+\bigg(\bigg(\bigg(D_\epsilon^4\ln(C_\delta)+D_\epsilon^8\ln^2(C_\delta)+\cdots+D_\epsilon^{4r}\ln^r(C_\delta)\bigg)\ln(B_\gamma)\\
&+\bigg(D_\epsilon^6\ln(C_\delta)+D_\epsilon^{12}\ln^2(C_\delta)+\cdots+D_\epsilon^{6r}\ln^r(C_\delta)\bigg)\ln^2(B_\gamma)+\cdots\\
&+\bigg(D_\epsilon^{4r}\ln(C_\delta)+D_\epsilon^{8r}\ln^2(C_\delta)+\cdots+D_\epsilon^{4r}\ln^r(C_\delta)\bigg)\ln^l(B_\gamma)\bigg)\ln^2(A_\beta)+\cdots\\
&+\bigg(\bigg(\bigg(D_\epsilon^{2r}\ln(C_\delta)+D_\epsilon^{4r}\ln^2(C_\delta)+\cdots+D_\epsilon^{2mr}\ln^r(C_\delta)\bigg)\ln(B_\gamma)\\
&+\bigg(D_\epsilon^{4r}\ln(C_\delta)+D_\epsilon^{8r}\ln^2(C_\delta)+\cdots+D_\epsilon^{4mr}\ln^r(C_\delta)\bigg)\ln^2(B_\gamma)+\cdots\\
&+\bigg(D_\epsilon^{2mr}\ln(C_\delta)+D_\epsilon^{4mr}\ln^2(C_\delta)+\cdots+D_\epsilon^{2mlr}\ln^r(C_\delta)\bigg)\ln^l(B_\gamma)\bigg)\ln^m(A_\beta)\bigg)\bigg)\bigg)\\
&\times\ln^2(d(x_{k_{j-1}},x_{k_{h-1}}))+\cdots\\
&+\bigg(\bigg(\bigg(D_\epsilon^r\ln(C_\delta)+D_\epsilon^{2r}\ln^2(C_\delta)+\cdots+D_\epsilon^{lr}\ln^r(C_\delta)\bigg)\ln(B_\gamma)\\
&+\bigg(D_\epsilon^{2r}\ln(C_\delta)+D_\epsilon^{4r}\ln^2(C_\delta)+\cdots+D_\epsilon^{lr}\ln^r(C_\delta)\bigg)\ln^2(B_\gamma)+\cdots\\
&+\bigg(D_\epsilon^{rm}\ln(C_\delta)+D_\epsilon^{2mr}\ln^2(C_\delta)+\cdots+D_\epsilon^{rml}\ln^r(C_\delta)\bigg)\ln^l(B_\gamma)\bigg)\ln(A_\beta)\\
&+\bigg(\bigg(\bigg(D_\epsilon^4\ln(C_\delta)+D_\epsilon^8\ln^2(C_\delta)+\cdots+D_\epsilon^r\ln^r(C_\delta)\bigg)\ln(B_\gamma)
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg(D_\epsilon^{6}\ln(C_\delta)+D_\epsilon^{12}\ln^2(C_\delta)+\cdots+D_\epsilon^{6r}\ln^r(C_\delta)\Bigg)\ln^2(B_\gamma)+\cdots\\
&+\Bigg(D_\epsilon^{4r}\ln(C_\delta)+D_\epsilon^{8r}\ln^2(C_\delta)+\cdots+D_\epsilon^{4lr}\ln^r(C_\delta)\Bigg)\ln^l(B_\gamma)\Bigg)\ln^2(A_\beta)+\cdots\\
&+\Bigg(\Bigg(\Bigg(D_\epsilon^{mr}\ln(C_\delta)+D_\epsilon^{2mr}\ln^2(C_\delta)+\cdots+D_\epsilon^{mlr}\ln^r(C_\delta)\Bigg)\ln(B_\gamma)\\
&+\Bigg(D_\epsilon^{2mr}\ln(C_\delta)+D_\epsilon^{4mr}\ln^2(C_\delta)+\cdots+D_\epsilon^{2mlr}\ln^r(C_\delta)\Bigg)\ln^2(B_\gamma)+\cdots\\
&+\Bigg(D_\epsilon^{lrn}\ln(C_\delta)+D_\epsilon^{2nlr}\ln^2(C_\delta)+\cdots+D_\epsilon^{nmlr}\ln^r(C_\delta)\Bigg)\ln^l(B_\gamma)\Bigg)\ln^m(A_\beta)\Bigg)\Bigg)\Bigg)\\
&\times\ln^n(d(x_{k_{j-1}},x_{k_{h-1}}))+\cdots\Bigg\}\\
&=\alpha(d(x_{k_{j-1}},x_{k_{h-1}}))\Bigg\{\Bigg(\Bigg(\Bigg(\Bigg(\epsilon(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j})\ln(\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h}))\\
&+(\epsilon(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j}))^2\ln^2(\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h}))+\cdots\\
&+(\epsilon(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j}))^r\ln^r(\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h}))\Bigg)\\
&\times\ln(\gamma(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_h}))+\Bigg((\epsilon(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j})))^2\\
&\times\ln(\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h}))+(\epsilon(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j})))^4\\
&\times\ln^2(\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h}))+\cdots+(\epsilon(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j})))^{2r}\\
&\times\ln^r(\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h}))\ln^2(\gamma(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_h}))+\cdots\\
&+\Bigg((\epsilon(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j})))^r\ln(\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h}))\\
&+(\epsilon(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j}))^{2r}\ln^2(\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h}))+\cdots\\
&+(\epsilon(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j}))^{rl}\ln^r(\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h}))\\
&\times\ln^l(\gamma(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_h}))\ln(\beta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_j}))\\
&+\Bigg(\Bigg(\Bigg((\epsilon(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j}))^2\ln(\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h}))\\
&+(\epsilon(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j}))^4\ln^2(\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h}))+\cdots\\
&+(\epsilon(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j}))^{2r}\\
&\times\ln^r(\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h}))\Bigg)\ln(\gamma(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_h}))\\
&+\Bigg((\epsilon(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j}))^4\ln(\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h}))\\
&+(\epsilon(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j}))^6\ln^2(\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h}))+\cdots\\
&+(\epsilon(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j}))^r\ln^r(\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h}))\Bigg)\\
&\times\ln^2(\gamma(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_h}))+\cdots\\
&+\Bigg((\epsilon(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j}))^{2r}\ln(\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h}))\\
&+(\epsilon(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j}))^{4r}\ln^2(\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h}))+\cdots\\
&+(\epsilon(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j}))^{lr}\ln^r(\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h}))\Bigg)\\
&\times\ln^l(\gamma(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_h}))\Bigg)\ln^2(\beta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_j}))+\cdots
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg(\Bigg(\Bigg((\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{r}\ln(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\\
&+(\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{r}\ln^{2}(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))+\cdots\\
&+(\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{lr}\ln^{r}(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\Bigg)\\
&\times\ln(\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h}))\\
&+\Bigg((\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{2r}\ln(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\\
&+(\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{4r}\ln^{2}(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))+\cdots\\
&+(\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{lr}\ln^{r}(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\Bigg)\\
&\times\ln^{2}(\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h}))+\cdots\\
&+\Bigg((\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{rl}\ln(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\\
&+(\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{2rl}\ln^{2}(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))+\cdots\\
&+(\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{rlm}\ln^{r}(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\Bigg)\\
&\times\ln^{l}(\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h}))\Bigg)\ln^{m}(\beta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_j}))\Bigg)\Bigg)\\
&\times\ln(d(x_{k_{j-1}}, x_{k_{h-1}}))\\
&+\Bigg(\Bigg(\Bigg((\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{2}\ln(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\\
&+(\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{4}\ln^{2}(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))+\cdots\\
&+(\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{r}\ln^{r}(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\Bigg)\\
&\times\ln(\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h}))\\
&+\Bigg((\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{4}\ln(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\\
&+(\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{8}\ln^{2}(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))+\cdots\\
&+(\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{4r}\ln^{r}(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\Bigg)\\
&\times\ln^{2}(\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h}))+\cdots\\
&+\Bigg((\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{2r}\ln(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\\
&+(\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{4r}\ln^{2}(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))+\cdots\\
&+(\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{2rl}\ln^{r}(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\Bigg)\\
&\times\ln^{l}(\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h}))\Bigg)\ln(\beta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_j}))\\
&+\Bigg(\Bigg(\Bigg((\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{4}\ln(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\\
&+(\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{8}\ln^{2}(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))+\cdots\\
&+(\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{4r}\ln^{r}(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\Bigg)\\
&\times\ln(\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h}))\\
&+\Bigg((\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{6}\ln(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))
\end{aligned}
$$

$$
\begin{aligned}
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{12} \ln^2(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})) + \cdots \\
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{6r} \ln^r(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\Big) \\
&\times \ln^2(\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})) + \cdots \\
&+ \Big((\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{4r} \ln(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})) \\
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{8r} \ln^2(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})) + \cdots \\
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{4r} \ln^r(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\Big) \\
&\times \ln^l(\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h}))\Big) \ln^2(\beta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_j})) + \cdots \\
&+ \Bigg(\Bigg(\Bigg((\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{2r} \ln(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})) \\
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{4r} \ln^2(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})) + \cdots \\
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{2mr} \ln^r(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\Big) \\
&\times \ln(\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})) \\
&+ \Big((\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{4r} \ln(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})) \\
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{8r} \ln^2(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})) + \cdots \\
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{4mr} \ln^r(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\Big) \\
&\times \ln^2(\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})) + \cdots \\
&+ \Big((\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{2mr} \ln(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})) \\
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{4mr} \ln^2(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})) + \cdots \\
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{2rml} \ln^r(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\Big) \\
&\times \ln^l(\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h}))\Big) \ln^m(\beta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_j}))\Bigg)\Bigg)\Bigg) \\
&\times \ln^2(d(x_{k_{j-1}}, x_{k_{h-1}})) + \cdots \\
&+ \Bigg(\Bigg(\Bigg(\Bigg((\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{r} \ln(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})) \\
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{2r} \ln^2(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})) + \cdots \\
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{2mr} \ln^r(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\Big) \\
&\times \ln(\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})) \\
&+ \Big((\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{2r} \ln(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})) \\
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{4r} \ln^2(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})) + \cdots \\
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{lr} \ln^r(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\Big) \\
&\times \ln^2(\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})) + \cdots \\
&+ \Big((\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{rm} \ln(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})) \\
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{2mr} \ln^2(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})) + \cdots
\end{aligned}
$$

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\begin{aligned}
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{rml} \ln^r(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\Big) \\
&\times \ln^l(\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h}))\Big) \ln(\beta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_j})) \\
&+ \Bigg(\Bigg(\Bigg((\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^4 \ln(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})) \\
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^8 \ln^2(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})) + \cdots \\
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^r \ln^r(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\Big) \\
&\times \ln(\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})) \\
&+ \Big((\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^6 \ln(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})) \\
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{12} \ln^2(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})) + \cdots \\
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{6r} \ln^r(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\Big) \\
&\times \ln^2(\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})) + \cdots \\
&+ \Big((\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{4r} \ln(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})) \\
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{8r} \ln^2(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})) + \cdots \\
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{4lr} \ln^r(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\Big) \\
&\times \ln^l(\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h}))\Big) \ln^2(\beta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_j})) + \cdots \\
&+ \Bigg(\Bigg(\Bigg(\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{mr} \ln(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})) \\
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{2mr} \ln^2(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})) + \cdots \\
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{mrl} \ln^r(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\Big) \\
&\times \ln(\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})) \\
&+ \Big((\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{2mr} \ln(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})) \\
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{4mr} \ln^2(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})) + \cdots \\
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{2mrl} \ln^r(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\Big) \\
&\times \ln^2(\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})) + \cdots \\
&+ \Big((\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{lrn} \ln(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})) \\
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{2lrn} \ln^2(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})) + \cdots \\
&+ (\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j}))^{rnlm} \ln^r(\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}))\Big) \\
&\times \ln^l(\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h}))\Big) \ln^m(\beta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_j}))\Big)\Big)\Big) \\
&\times \ln^n(d(x_{k_{j-1}}, x_{k_{h-1}})) + \cdots \Big\} \\
&\leq \alpha(d(x_{k_{j-1}}, x_{k_{h-1}}))\Big\{\Bigg(\Bigg(\Bigg(\Bigg(\epsilon(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}) \\
&+ \epsilon^2(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}) + \cdots
\end{aligned}
$$

$$
\begin{aligned}
&+\epsilon^r(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j})\delta^r(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h})\Big)\\
&\times\gamma(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_h})+\Big(\epsilon^2(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j}))\\
&\times\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h})+\epsilon^4(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j}))\\
&\times\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h})+\cdots+\epsilon^{2r}(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j}))\\
&\times\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h})\gamma(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_h})+\cdots\\
&+\Big(\epsilon^r(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j})\Big)\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h})\\
&+\epsilon^{2r}(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j})\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h})+\cdots\\
&+\epsilon^{rl}(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j})\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h})\\
&\times\gamma(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_h})\beta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_j})\\
&+\Bigg(\Bigg(\Bigg(\epsilon^2(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j})\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h})\\
&+\epsilon^4(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j})\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h})+\cdots\\
&+\epsilon^{2r}(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j})\\
&\times\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h})\Bigg)\gamma(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_h})\\
&+\Big(\epsilon^4(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j})\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h})\\
&+\epsilon^6(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j})\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h})+\cdots\\
&+\epsilon^r(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j})\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h})\Big)\\
&\times\gamma(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_h})+\cdots\\
&+\Big(\epsilon^{2r}(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j})\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h})\\
&+\epsilon^{4r}(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j})\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h})+\cdots\\
&+\epsilon^{lr}(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j})\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h})\Big)\\
&\times\gamma(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_h})\Bigg)\beta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_j})+\cdots\\
&+\Bigg(\Bigg(\Bigg(\epsilon^r(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j})\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h})\\
&+\epsilon^r(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j})\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h})+\cdots\\
&+\epsilon^{lr}(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j})\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h})\Big)\\
&\times\gamma(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_h})\\
&+\Big(\epsilon^{2r}(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j})\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h})\\
&+\epsilon^{4r}(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j})\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h})+\cdots\\
&+\epsilon^{lr}(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j})\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h})\Big)\\
&\times\gamma(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_h})+\cdots\\
&+\Big(\epsilon^{lr}(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j})\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h})\\
&+\epsilon^{2rl}(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{h-1}},x_{k_j})\delta(d(x_{k_{j-1}},x_{k_{h-1}}))d(x_{k_{j-1}},x_{k_h})+\cdots
\end{aligned}
$$

$$+\,\epsilon^{rlm}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})\Bigg)$$

$$\times\,\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})\Bigg)\beta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_j})\Bigg)\Bigg) \times d(x_{k_{j-1}}, x_{k_{h-1}})$$

$$+\,\Bigg(\Bigg(\Bigg(\epsilon^{2}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})$$

$$+\,\epsilon^{4}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}) + \cdots$$

$$+\,\epsilon^{r}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})\Bigg) \times \gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})$$

$$+\,\Bigg(\epsilon^{4}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})$$

$$+\,\epsilon^{8}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}) + \cdots$$

$$+\,\epsilon^{4r}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})\Bigg) \times \gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})$$

$$+ \cdots + \Bigg(\epsilon^{2r}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})$$

$$+\,\epsilon^{4r}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}) + \cdots$$

$$+\,\epsilon^{2rl}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})\Bigg)$$

$$\times\,\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})\Bigg)\beta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_j})$$

$$+\,\Bigg(\Bigg(\Bigg(\epsilon^{4}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})$$

$$+\,\epsilon^{8}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}) + \cdots$$

$$+\,\epsilon^{4r}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})\Bigg) \times \gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})$$

$$+\,\Bigg(\epsilon^{6}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})$$

$$+\,\epsilon^{12}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}) + \cdots$$

$$+\,\epsilon^{6r}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})\Bigg) \times \gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})$$

$$+ \cdots + \Bigg(\epsilon^{4r}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})$$

$$+\,\epsilon^{8r}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}) + \cdots$$

$$+\,\epsilon^{4r}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})\Bigg)$$

$$\times\,\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})\Bigg)\beta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_j}) + \cdots$$

$$+\,\Bigg(\Bigg(\Bigg(\epsilon^{2r}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})$$

$$+\,\epsilon^{4r}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}) + \cdots$$

$$+\,\epsilon^{2mr}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})\Bigg) \times \gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})$$

$$+\,\Bigg(\epsilon^{4r}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})$$

$$+\,\epsilon^{8r}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}) + \cdots$$

$$+\,\epsilon^{4mr}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})\Bigg) \times \gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})$$

$$+ \cdots + \Bigg(\epsilon^{2mr}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})$$

$$+ \epsilon^{4mr}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}) + \cdots$$

$$+ \epsilon^{2rml}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})\Bigg)$$

$$\times \gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})\Bigg)\beta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_j})\Bigg)\Bigg)\Bigg) \times d(x_{k_{j-1}}, x_{k_{h-1}}) + \cdots$$

$$+ \Bigg(\Bigg(\Bigg(\Bigg(\epsilon^{r}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})$$

$$+ \epsilon^{2r}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}) + \cdots$$

$$+ \epsilon^{2mr}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})\Bigg) \times \gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})$$

$$+ \Bigg(\epsilon^{2r}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})$$

$$+ \epsilon^{4r}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}) + \cdots$$

$$+ \epsilon^{lr}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})\Bigg) \times \gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h}) + \cdots$$

$$+ \Bigg(\epsilon^{rm}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})$$

$$+ \epsilon^{2mr}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}) + \cdots$$

$$+ \epsilon^{rml}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})\Bigg)$$

$$\times \gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})\Bigg)\beta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_j})$$

$$+ \Bigg(\Bigg(\Bigg(\epsilon^{4}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})$$

$$+ \epsilon^{8}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}) + \cdots$$

$$+ \epsilon^{r}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})\Bigg) \times \gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})$$

$$+ \Bigg(\epsilon^{6}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})$$

$$+ \epsilon^{12}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}) + \cdots$$

$$+ \epsilon^{6r}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})\Bigg) \times \gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h}) + \cdots$$

$$+ \Bigg(\epsilon^{4r}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})$$

$$+ \epsilon^{8r}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}) + \cdots$$

$$+ \epsilon^{4lr}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})\Bigg)$$

$$\times \gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})\Bigg)\beta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_j}) + \cdots$$

$$+ \Bigg(\Bigg(\epsilon^{mr}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})$$

$$+ \epsilon^{2mr}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}) + \cdots$$

$$+ \epsilon^{mrl}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})\Bigg) \times \gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})$$

$$+ \Bigg(\epsilon^{2mr}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})$$

$$+ \epsilon^{4mr}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}) + \cdots$$
$$+ \epsilon^{2mrl}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})\Big) \times \gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})$$
$$+ \cdots + \Big(\epsilon^{lrn}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})$$
$$+ \epsilon^{2lrn}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h}) + \cdots$$
$$+ \epsilon^{rlmn}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})\Big)$$
$$\times \gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_h})\Big)\beta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_j})\Big)\Big)\Big) \times d(x_{k_{j-1}}, x_{k_{h-1}}) + \cdots\Big\}$$
$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \alpha(d(x_{k_{j-1}}, x_{k_{h-1}}))\beta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_j})\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))$$
$$d(x_{k_{h-1}}, x_{k_h})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})\epsilon^{rlmn}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})d(x_{k_{j-1}}, x_{k_{h-1}}). \tag{3.11}$$

$$d(x_{k_j}, x_{k_h})^2 \leq \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \alpha(d(x_{k_{j-1}}, x_{k_{h-1}}))\beta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_j})\gamma(d(x_{k_{j-1}}, x_{k_{h-1}}))$$
$$d(x_{k_{h-1}}, x_{k_h})\delta(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{j-1}}, x_{k_h})\epsilon^{rlmn}(d(x_{k_{j-1}}, x_{k_{h-1}}))d(x_{k_{h-1}}, x_{k_j})d(x_{k_{j-1}}, x_{k_{h-1}}). \tag{3.12}$$

Therefore,

$$d(x_{k_j}, x_{k_h}) \leq \eta\vartheta \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \alpha(d(x_{k_j}, x_{k_{h-1}}))\beta(d(x_{k_j}, x_{k_{h-1}}))\gamma(d(x_{k_j}, x_{k_{h-1}}))$$
$$\delta(d(x_{k_j}, x_{k_{h-1}}))\epsilon^{rlmn}(d(x_{k_j}, x_{k_{h-1}}))d(x_{k_j}, x_{k_{h-1}})$$
$$= \eta\vartheta \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{l=0}^{\infty} \frac{\alpha(d(x_{k_j}, x_{k_{h-1}}))\beta(d(x_{k_j}, x_{k_{h-1}}))\gamma(d(x_{k_j}, x_{k_{h-1}}))\delta(d(x_{k_j}, x_{k_{h-1}}))}{1 - \epsilon^{rln}(d(x_{k_j}, x_{k_{h-1}}))} d(x_{k_j}, x_{k_{h-1}}). \tag{3.13}$$

Repeating the above processes again, we get;

$$d(x_{k_{j-1}}, x_{k_{h-1}}) \leq \eta\vartheta \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \alpha(d(x_{k_{j-1}}, x_{k_{h-2}}))\beta(d(x_{k_{j-1}}, x_{k_{h-2}}))\gamma(d(x_{k_{j-1}}, x_{k_{h-2}}))$$
$$\delta(d(x_{k_{j-1}}, x_{k_{h-2}}))\epsilon^{rlmn}(d(x_{k_{j-1}}, x_{k_{h-2}}))d(x_{k_{j-1}}, x_{k_{h-2}})$$
$$= \eta\vartheta \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{l=0}^{\infty} \frac{\alpha(d(x_{k_{j-1}}, x_{k_{h-2}}))\beta(d(x_{k_{j-1}}, x_{k_{h-2}}))\gamma(d(x_{k_{j-1}}, x_{k_{h-2}}))\delta(d(x_{k_{j-1}}, x_{k_{h-2}}))}{1 - \epsilon^{rln}(d(x_{k_{j-1}}, x_{k_{h-2}}))} d(x_{k_{j-1}}, x_{k_{h-2}}). \tag{3.14}$$

From inequality (3.12), we have;

$$1 \leq \lim_{j,h\to\infty} \inf \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \alpha(d(x_{k_j}, x_{k_{h-1}}))\beta(d(x_{k_j}, x_{k_{h-1}}))\gamma(d(x_{k_j}, x_{k_{h-1}}))\delta(d(x_{k_j}, x_{k_{h-1}}))\epsilon^{rlmn}(d(x_{k_j}, x_{k_{h-1}})), \tag{3.15}$$

or from inequality (3.13), we get;

$$\frac{1}{\eta\vartheta} \leq \lim_{j,h\to\infty} \inf \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \alpha(d(x_{k_j}, x_{k_{h-1}}))\beta(d(x_{k_j}, x_{k_{h-1}}))\gamma(d(x_{k_j}, x_{k_{h-1}}))\delta(d(x_{k_j}, x_{k_{h-1}}))\epsilon^{rlmn}(d(x_{k_j}, x_{k_{h-1}})), \tag{3.16}$$

which implies that $d(x_{k_j}, x_{k_{h-1}}) = 0$, so that

$$d(x_{k_j}, x_{k_h}) < \epsilon^*, \tag{3.17}$$

which is a contradiction. Hence $\{x_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in X. Since X is complete, there is a candidate $x^* \in X$ such that $x_k \to x^* \in X$. Now we show that x^* is a fixed point of T. Suppose that $Tx^* \neq x^*$ and $d(x_k, Tx^*) = d(Tx_{k-1}, Tx^*)$. From inequality (3.1), take $x = x_{k-1}$ and $y = x^*$ for all $k \in \mathbb{N}$, thus

$$\begin{aligned} d(x_k, Tx^*)^2 &= d(Tx_{k-1}, Tx^*)^2 \\ &\leq \alpha(d(x_{k-1}, x^*))d(x_{k-1}, x^*)^{A_\beta^{B_\gamma^{C_\delta^{D_\epsilon}}}}, \end{aligned} \tag{3.18}$$

where,

$$A_\beta := \frac{\beta(d(x_{k-1}, x^*))d(x_{k-1}, Tx_{k-1})}{1+\epsilon(d(x_{k-1}, x^*))d(x^*, Tx_{k-1})}, \quad B_\gamma := \frac{\gamma(d(x_{k-1}, x^*))d(x^*, Tx^*)}{1+\epsilon(d(x_{k-1}, x^*))d(x^*, Tx_{k-1})},$$
$$C_\delta := \frac{\delta(d(x_{k-1}, x^*))d(x_{k-1}, Tx^*)}{1+\epsilon(d(x_{k-1}, x^*))d(x^*, Tx_{k-1})}, \quad D_\epsilon := \frac{\epsilon(d(x_{k-1}, x^*))d(x^*, Tx_{k-1})}{1+\epsilon(d(x_{k-1}, x^*))d(x^*, Tx_{k-1})}.$$

Thus,

$$A_\beta := \frac{\beta(d(x_{k-1}, x^*))d(x_{k-1}, x_k)}{1+\epsilon(d(x_{k-1}, x^*))d(x^*, x_k)}, \quad B_\gamma := \frac{\gamma(d(x_{k-1}, x^*))d(x^*, Tx^*)}{1+\epsilon(d(x_{k-1}, x^*))d(x^*, x_k)},$$
$$C_\delta := \frac{\delta(d(x_{k-1}, x^*))d(x_{k-1}, Tx^*)}{1+\epsilon(d(x_{k-1}, x^*))d(x^*, x_k)}, \quad D_\epsilon := \frac{\epsilon(d(x_{k-1}, x^*))d(x^*, x_k)}{1+\epsilon(d(x_{k-1}, x^*))d(x^*, x_k)}.$$

Therefore,

$$A_\beta := \frac{\beta(d(x_{k-1}, x^*))d(x_{k-1}, x_k)}{1+\epsilon(d(x_{k-1}, x^*))d(x^*, x_k)} \leq \beta(d(x_{k-1}, x^*))d(x_{k-1}, x_k),$$
$$B_\gamma := \frac{\gamma(d(x_{k-1}, x^*))d(x^*, Tx^*)}{1+\epsilon(d(x_{k-1}, x^*))d(x^*, x_k)} \leq \gamma(d(x_{k-1}, x^*))d(x^*, Tx^*),$$
$$C_\delta := \frac{\delta(d(x_{k-1}, x^*))d(x_{k-1}, Tx^*)}{1+\epsilon(d(x_{k-1}, x^*))d(x^*, x_k)} \leq \delta(d(x_{k-1}, x^*))d(x_{k-1}, Tx^*),$$
$$D_\epsilon := \frac{\epsilon(d(x_{k-1}, x^*))d(x^*, x_k)}{1+\epsilon(d(x_{k-1}, x^*))d(x^*, x_k)} \leq \epsilon(d(x_{k-1}, x^*))d(x^*, x_k).$$

From inequality (3.12), we get,

$$d(x^*, Tx^*)^2 = 0, \tag{3.19}$$

a contradiction. Therefore, $x^* = Tx^*$. For the uniqueness, suppose that the fixed point of T in X that is $x^* = Tx^*$ for some $x^* \in Fix(T)$. Again, suppose if possible that there exists another fixed point of T in X, that is $y^* = Ty^*$ for some $y^* \in Fix(T)$. Now suppose that $x^* \neq y^* \implies d(x^*, y^*) > 0$. Therefore

$$\begin{aligned} d(x^*, y^*)^2 &= d(Tx^*, Ty^*)^2 \\ &\leq \alpha(d(x^*, y^*))d(x^*, y^*)^{A_\beta^{B_\gamma^{C_\delta^{D_\epsilon}}}} \\ &= \alpha(d(x^*, y^*)) \exp\left(A_\beta^{B_\gamma^{C_\delta^{D_\epsilon}}} \ln d(x^*, y^*)\right) \\ &= \alpha(d(x^*, y^*)) \sum_{m=0}^{\infty} \frac{A_\beta^{B_\gamma^{C_\delta^{D_\epsilon^m}}} \ln^m d(x^*, y^*)}{m!} \\ &= \alpha(d(x^*, y^*)) \sum_{m=0}^{\infty} \frac{\ln^m d(x^*, y^*)}{m!} \exp\left(B_\gamma^{C_\delta^{D_\epsilon^m}} \ln A_\beta\right), \end{aligned} \tag{3.20}$$

where,

$$A_\beta := \frac{\beta(d(x^*, y^*))d(x^*, Tx^*)}{1+\epsilon(d(x^*, y^*))d(y^*, Tx^*)}, \quad B_\gamma := \frac{\gamma(d(x^*, y^*))d(y^*, Ty^*)}{1+\epsilon(d(x^*, y^*))d(y^*, Tx^*)},$$
$$C_\delta := \frac{\delta(d(x^*, y^*))d(x^*, Ty^*)}{1+\epsilon(d(x^*, y^*))d(y^*, Tx^*)}, \quad D_\epsilon := \frac{\epsilon(d(x^*, y^*))d(y^*, Tx^*)}{1+\epsilon(d(x^*, y^*))d(y^*, Tx^*)},$$

for all distinct $x^*, y^* \in Fix(T)$. We can see clearly from inequality (3.20) that

$$d(x^*, y^*)^2 \leq \alpha(d(x^*, y^*)) \exp\Bigg(\ln(d(x^*, y^*))\Bigg). \tag{3.21}$$

So that

$$d(x^*, y^*)^2 \leq \alpha(d(x^*, y^*))(d(x^*, y^*)). \tag{3.22}$$

Thus,

$$d(x^*, y^*) \leq \alpha(d(x^*, y^*))d(x^*, y^*). \tag{3.23}$$

So, $\alpha(d(x^*, y^*)) \to 1 \implies d(x^*, y^*) = 0$. Therefore,

$$d(x^*, y^*) = 0, \tag{3.24}$$

which is a contradiction. Hence equation (3.24) shows that the sequence $\{x_k\}_{k\in\mathbb{N}}$ defined by $Tx_{k-1} = x_k$ converges to a unique fixed point of T in X and the proof is completed. □

Remark 3.2. (a) If $\gamma(d(x, y)) \neq 0$ and $B_\gamma = 0$, invoking the condition that $d^k(x, y) \leq d(x, y)$, $k \in \mathbb{N}$ and closed $x, y \in X$ hold, then the inequality (3.1) collapsed to Geraghty contraction inequality, i.e., $d(Tx, Ty) \leq \alpha(d(x, y))d(x, y)$ which is itself a generalization of results given in Rakotch [28] since the class of functions defined in Geraghty [13] is more general. That is Rakotch [28] considered $\alpha : \mathbb{R}_+ \to [0, 1)$ and is monotone decreasing or increasing. Obviously, such an α is clearly in the class S. Again if $\alpha : \mathbb{R}_+ \to [0, 1)$ and is continuous, however, such an α is clearly in the class S and this is the result of Boyd-Wong [3].

(b) The Remark (a) is itself generalizations of famous Banach contraction inequality. That is to say, if $\alpha(d(x, y)) = L$ (say), for $0 < L < 1$, then $d(Tx, Ty) \leq Ld(x, y)$, for all $x, y \in X$.

(c) Suppose $\frac{1}{1+\epsilon(d(x,y))d(y,Tx)} \leq 1$ and $\epsilon(d(x, y)) \in \mathcal{F}_{Ger}$ class, Then Theorem 3.1 surprisingly reduced to Theorem 3.1 in Okeke-Francis [21].

Corollary 3.3. Let (X, d) be a complete metric space and $T : X \to X$ such that there are $\{\alpha, \delta\} \in \mathcal{F}_{Ger}$ satisfying;

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y)^{C_\delta^{C_\delta^{C_\delta^{C_\delta}}}}, \tag{3.25}$$

where, $C_\delta := \dfrac{\delta(d(x, y))d(x, Ty)}{1 + \epsilon(d(x, y))d(y, Tx)}$; for all distinct close $x, y \in X$ and $\ln^k(d(x, y)) \leq d(x, y)$, $d^k(x, y) \leq d(x, y)$ for all $k \in \mathbb{N}$. Then the sequence $\{x_k\}_{k\in\mathbb{N}}$ defined by $Tx_{k-1} = x_k$ converges to a unique fixed point of T in X.

Proof . Suppose that $x_k = Tx_{k-1}$, then $x_{k+1} = Tx_k$ for all $k \in \mathbb{N}$. But $x_k \neq x_{k+1}$ and this implies that $d(x_k, x_{k+1}) > 0$, so that $d(x_k, x_{k+1}) = d(Tx_{k-1}, Tx_k)$. From inequality (3.25), take $x = x_{k-1}$ and $y = x_k$ for all $k \in \mathbb{N}$, thus, by Theorem 3.1 we' re done. □

Corollary 3.4. Let (X, d) be a complete metric space and $T : X \to X$ such that there are $\{\alpha, \beta, \gamma, \delta, \epsilon\} \in \mathcal{F}_{Ger}$ satisfying;

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y)^{C_\delta^{B_\gamma^{D_\epsilon^{C_\delta}}}}, \tag{3.26}$$

where, $B_\gamma := \dfrac{\gamma(d(x, y))d(y, Ty)}{1 + \epsilon(d(x, y))d(y, Tx)}$, $C_\delta := \dfrac{\delta(d(x, y))d(x, Ty)}{1 + \epsilon(d(x, y))d(y, Tx)}$; $D_\epsilon := \dfrac{\epsilon(d(x, y))d(y, Tx)}{1 + \epsilon(d(x, y))d(y, Tx)}$, for all distinct close $x, y \in X$ and $\ln^k(d(x, y)) \leq d(x, y)$, $d^k(x, y) \leq d(x, y)$ for all $k \in \mathbb{N}$. Then the sequence $\{x_k\}_{k\in\mathbb{N}}$ defined by $Tx_{k-1} = x_k$ converges to a unique fixed point of T in X.

Proof . For the uniqueness, suppose that the fixed point of T in X that is $x^* = Tx^*$ for some $x^* \in Fix(T)$. Again, suppose if possible that there exists another fixed point of T in X, that is $y^* = Ty^*$ for some $y^* \in Fix(T)$. Now suppose that $x^* \neq y^* \implies d(x^*, y^*) > 0$. Therefore

$$d(x^*, y^*) = d(Tx^*, Ty^*)$$

$$\leq \alpha(d(x^*,y^*))d(x^*,y^*)^{C_\delta^{B_\gamma^{D_\epsilon^{C_\delta}}}}, \tag{3.27}$$

where, $B_\gamma := \dfrac{\gamma(d(x^*,y^*))d(y^*,Ty^*)}{1+\epsilon(d(x^*,y^*))d(y^*,Tx^*)}$, $C_\delta := \dfrac{\delta(d(x^*,y^*))d(x^*,Ty^*)}{1+\epsilon(d(x^*,y^*))d(y^*,Tx^*)}$, $D_\epsilon := \dfrac{\epsilon(d(x^*,y^*))d(y^*,Tx^*)}{1+\epsilon(d(x^*,y^*))d(y^*,Tx^*)}$, for all distinct $x^*, y^* \in Fix(T)$. So that

$$B_\gamma := \frac{\gamma(d(x^*,y^*))d(y^*,y^*)}{1+\epsilon(d(x^*,y^*))d(y^*,x^*)},\ C_\delta := \frac{\delta(d(x^*,y^*))d(x^*,y^*)}{1+\epsilon(d(x^*,y^*))d(y^*,x^*)},\ D_\epsilon := \frac{\epsilon(d(x^*,y^*))d(y^*,x^*)}{1+\epsilon(d(x^*,y^*))d(y^*,x^*)}.$$

From inequality (3.27), we get;

$$d(x^*,y^*) \leq \alpha(d(x^*,y^*))d(x^*,y^*). \tag{3.28}$$

Solving inequality (3.28), we that $d(x^*,y^*) = 0$. or since $\alpha \in \mathcal{F}_{Ger}$, $1 \leq \alpha(d(x^*,y^*))$. Therefore, $x^* = y^*$. For the existence, interchange the roles of $A_\beta, B_\gamma, C_\delta$ and D_ϵ in inequality (3.1) we are done. □

Corollary 3.5. Let (X,d) be a complete metric space and $T : X \to X$ such that there are $\{\alpha,\beta,\gamma,\delta,\epsilon\} \in \mathcal{F}_{Ger}$ satisfying;

$$d(Tx,Ty) \leq \alpha(d(x,y))d(x,y)^{A_\beta^{D_\epsilon^{B_\gamma^{C_\delta}}}}; \tag{3.29}$$

where, $A_\beta := \dfrac{\beta(d(x,y))d(x,Tx)}{1+\epsilon(d(x,y))d(y,Tx)}$, $B_\gamma := \dfrac{\gamma(d(x,y))d(y,Ty)}{1+\epsilon(d(x,y))d(y,Tx)}$, $C_\delta := \dfrac{\delta(d(x,y))d(x,Ty)}{1+\epsilon(d(x,y))d(y,Tx)}$;

$D_\epsilon := \dfrac{\epsilon(d(x,y))d(y,Tx)}{1+\epsilon(d(x,y))d(y,Tx)}$, for all distinct $x, y \in X$ for all $k \in \mathbb{N}$. Then the sequence $\{x_k\}_{k\in\mathbb{N}}$ defined by $Tx_{k-1} = x_k$ converges to a unique fixed point of T in X.

Proof . Suppose that $x_k = Tx_{k-1}$, then $x_{k+1} = Tx_k$ for all $k \in \mathbb{N}$. But $x_k \neq x_{k+1}$ and this implies that $d(x_k,x_{k+1}) > 0$, so that $d(x_k,x_{k+1}) = d(Tx_{k-1},Tx_k)$. From inequality (3.29), take $x = x_{k-1}$ and $y = x_k$ for all $k \in \mathbb{N}$, thus

$$d(Tx_{k-1},Tx_k) \leq \alpha(d(x_{k-1},x_k))d(x_{k-1},x_k)^{A_\beta^{D_\gamma^{B_\delta^{C_\epsilon}}}}, \tag{3.30}$$

where,

$$A_\beta := \frac{\beta(d(x_{k-1},x_k))d(x_{k-1},Tx_{k-1})}{1+\epsilon(d(x_{k-1},x_k))d(x_k,Tx_{k-1})},\ B_\gamma := \frac{\gamma(d(x_{k-1},x_k))d(x_k,Tx_k)}{1+\epsilon(d(x_{k-1},x_k))d(x_k,Tx_{k-1})},$$
$$C_\delta := \frac{\delta(d(x_{k-1},x_k))d(x_{k-1},Tx_k)}{1+\epsilon(d(x_{k-1},x_k))d(x_k,Tx_{k-1})},\ D_\epsilon := \frac{\epsilon(d(x_{k-1},x_k))d(x_k,Tx_{k-1})}{1+\epsilon(d(x_{k-1},x_k))d(x_k,Tx_{k-1})}.$$

Thus,

$$A_\beta := \beta(d(x_{k-1},x_k))d(x_{k-1},x_k),\quad B_\gamma := \gamma(d(x_{k-1},x_k))d(x_k,x_{k+1}),$$
$$C_\delta := \delta(d(x_{k-1},x_k))d(x_{k-1},x_{k+1}),\quad D_\epsilon := \epsilon(d(x_{k-1},x_k))d(x_k,x_k) = 0.$$

So inequality (3.30) becomes,

$$d(Tx_{k-1},Tx_k) \leq \alpha(d(x_{k-1},x_k))d(x_{k-1},x_k). \tag{3.31}$$

We need only to show that the existence of such an α in $\mathcal{F}_{Ger}$ is equivalent to say that for $x_n \to x^*$ in X, with x^* a unique fixed point of T, iff for any two sub-sequences x_{h_n} and x_{h_k} of x_n with $x_{h_n} \neq x_{h_k}$, we have that $\Omega_n \to 1$ only if $d_n \to 0$, where we take for any pair of sequences x_n and y_n with $x_n \neq y_n$, we write $d_n = d(x_n,y_n)$ and $\Omega_n = \frac{d(T(x_n),T(y_n))}{d(x_n,y_n)}$. First assume such an α exists. Let $\{x_{h_n}\}$ and $\{x_{k_n}\}$ be two subsequences with $x_{h_n} \neq x_{k_n}$. Assume that $\Omega_n \to 1$ as $n \to \infty$. Then it follows from inequality (3.31) that $\alpha(d(x_{h_n},x_{k_n})) \to 1$. But then since $\alpha \in \mathcal{F}_{Ger}$, we have $d(x_{h_n},x_{k_n}) \to 0$. Next assume that the sequential condition holds. Define $\alpha : \mathbb{R}^+ \to \mathbb{R}$ as follows $\alpha(t) = \sup\{\frac{d(T(x_n),T(x_m))}{d(x_n,x_m)} : d(x_n,x_m) \geq t\}$. From inequality (3.31), the quotients are all below 1 and so α is defined for all $t > 0$ and $\alpha \leq 1$. Now assume that $\alpha(t_n) \to 1$ for $t_n \in \mathbb{R}^+$. We may further assume without loss of generality that $1 - \frac{1}{n} \leq \alpha(t_n) \leq 1$. We must show $t_n \to 0$. But $\alpha(t_n)$ is the above least upper bound. So there is for each $n > 0$ a pair $\{x_{h_n}\}$ and $\{x_{k_n}\}$ in $\{x_n\}$ with $d(x_{h_n},x_{k_n}) \geq t_n$ and $1 - \frac{1}{n} < \frac{d(T(x_{h_n}),T(x_{k_n}))}{d(x_{h_n},x_{k_n})} \leq \alpha(t_n)$. So the sequence $\Omega_n \to 1$ as $n \to \infty$. But then by the sequential condition above, $d(x_{h_n},x_{k_n}) \to 0$. So $t_n \to 0$, as was shown. This completes the proof of Corollary 3.5. □

Remark 3.6. Corollary 3.5 is a generalization of Theorem 2.7 as in Geraghty [13]. That is take a sequential operator sequence of the form $x_k = Tx_{k-1}$, then the structure in Corollary 3.5 collapsed into Theorem 1.3 in [13].

The above result is also true with an infinite type contraction as the following result shows;

Proposition 3.7. Let (X, d) be a complete metric space and $T : X \to X$ such that there are $\{\alpha, \beta, \gamma, \delta, \epsilon\} \in \mathcal{F}_{Ger}$ satisfying;

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y)^{A_\beta^{D_\epsilon^{B_\gamma^{C_\delta^{C_\delta^{\cdot^{\cdot^{\cdot}}}}}}}}, \tag{3.32}$$

where,

$$A_\beta := \frac{\beta(d(x, y))d(x, Tx)}{1 + \epsilon(d(x, y))d(y, Tx)},\ B_\gamma := \frac{\gamma(d(x, y))d(y, Ty)}{1 + \epsilon(d(x, y))d(y, Tx)},\ C_\delta := \frac{\delta(d(x, y))d(x, Ty)}{1 + \epsilon(d(x, y))d(y, Tx)},\ D_\epsilon := \frac{\epsilon(d(x, y))d(y, Tx)}{1 + \epsilon(d(x, y))d(y, Tx)}$$

for all distinct $x, y \in X$ for all $k \in \mathbb{N}$. Then the sequence $\{x_k\}_{k\in\mathbb{N}}$ defined by $Tx_{k-1} = x_k$ converges to a unique fixed point of T in X.

Proof . Suppose that $x_k = Tx_{k-1}$, then $x_{k+1} = Tx_k$ for all $k \in \mathbb{N}$. But $x_k \neq x_{k+1}$ and this implies that $d(x_k, x_{k+1}) > 0$, so that $d(x_k, x_{k+1}) = d(Tx_{k-1}, Tx_k)$. From inequality (3.32), take $x = x_{k-1}$ and $y = x_k$ for all $k \in \mathbb{N}$, thus

$$d(Tx_{k-1}, Tx_k) \leq \alpha(d(x_{k-1}, x_k))d(x_{k-1}, x_k)^{A_\beta^{D_\epsilon^{B_\gamma^{C_\delta^{C_\delta^{\cdot^{\cdot^{\cdot}}}}}}}}, \tag{3.33}$$

where,

$$A_\beta := \frac{\beta(d(x_{k-1}, x_k))d(x_{k-1}, Tx_{k-1})}{1 + \epsilon(d(x_{k-1}, x_k))d(x_k, Tx_{k-1})}, \quad B_\gamma := \frac{\gamma(d(x_{k-1}, x_k))d(x_k, Tx_k)}{1 + \epsilon(d(x_{k-1}, x_k))d(x_k, Tx_{k-1})},$$
$$C_\delta := \frac{\delta(d(x_{k-1}, x_k))d(x_{k-1}, Tx_k)}{1 + \epsilon(d(x_{k-1}, x_k))d(x_k, Tx_{k-1})}, \quad D_\epsilon := \frac{\epsilon(d(x_{k-1}, x_k))d(x_k, Tx_{k-1})}{1 + \epsilon(d(x_{k-1}, x_k))d(x_k, Tx_{k-1})}.$$

Thus,

$$A_\beta := \beta(d(x_{k-1}, x_k))d(x_{k-1}, x_k), \quad B_\gamma := \gamma(d(x_{k-1}, x_k))d(x_k, x_{k+1}),$$
$$C_\delta := \delta(d(x_{k-1}, x_k))d(x_{k-1}, x_{k+1}), \quad D_\epsilon := \epsilon(d(x_{k-1}, x_k))d(x_k, x_k) = 0.$$

So inequality (3.33) becomes,

$$d(Tx_{k-1}, Tx_k) \leq \alpha(d(x_{k-1}, x_k))d(x_{k-1}, x_k). \tag{3.34}$$

Following the proof of Corollary 3.5, we see that the sequence $\{x_k\}_{k\in\mathbb{N}}$ defined by $Tx_{k-1} = x_k$ converges to a unique fixed point of T in X. □

Remark 3.8. Proposition 3.7 is an infinite version of Theorem 1.3 in [13]. i.e., take a sequential operator sequence of the form $x_k = Tx_{k-1}$, then the structure in Proposition 3.7 collapsed into Theorem 1.3 in [13]. Hence $d(Tx_{k-1}, Tx_k) \leq \alpha(d(x_{k-1}, x_k))d(x_{k-1}, x_k)$. Put $k - 1 = m$ and $k = n$, thus $d(Tx_m, Tx_n) \leq \alpha(d(x_m, x_n))d(x_m, x_n)$ which is the inequality in Theorem 1.3 of [13].

Corollary 3.9. Let (X, d) be a complete metric space and $T : X \to X$ such that there are $\{\alpha, \beta, \gamma, \delta, \epsilon\} \in \mathcal{F}_{Ger}$ with $\alpha < 1$ satisfying;

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y)^{A_\beta^{C_\delta^{D_\epsilon^{B_\gamma}}}}, \tag{3.35}$$

where,

$$A_\beta := \frac{\beta(d(x, y))d(x, Tx)}{1 + \epsilon(d(x, y))d(y, Tx)},\ B_\gamma := \frac{\gamma(d(x, y))d(y, Ty)}{1 + \epsilon(d(x, y))d(y, Tx)},\ C_\delta := \frac{\delta(d(x, y))d(x, Ty)}{1 + \epsilon(d(x, y))d(y, Tx)},\ D_\epsilon := \frac{\epsilon(d(x, y))d(y, Tx)}{1 + \epsilon(d(x, y))d(y, Tx)}$$

for all distinct close $x, y \in X$ and $\ln^k(d(x, y)) \leq d(x, y)$, $d^k(x, y) \leq d(x, y)$ for all $k \in \mathbb{N}$ and $x, y \in X$. Then the sequence $\{x_k\}_{k\in\mathbb{N}}$ defined by $Tx_{k-1} = x_k$ converges to a unique fixed point of T in X.

Proof . From Corollary 3.5, inequality (3.35) we have

$$d(x_k, x_{k+1}) \leq \alpha(d(x_{k-1}, x_k))d(x_{k-1}, x_k)^{\beta(d(x_{k-1},x_k))d(x_{k-1},x_k)}. \tag{3.36}$$

Therefore,

$$d(x_k, x_{k+1}) \leq \sum_{n=0}^{\infty} \alpha(d(x_{k-1}, x_k))\beta^n(d(x_{k-1}, x_k))d(x_{k-1}, x_k). \tag{3.37}$$

So that

$$\begin{aligned} d(x_k, x_{k+1}) &\leq \sum_{n=0}^{\infty} \alpha(d(x_{k-1}, x_k))\beta^n(d(x_{k-1}, x_k))d(x_{k-1}, x_k) \\ &\leq \sum_{n=0}^{\infty} \alpha(d(x_{k-1}, x_k))\beta^n(d(x_{k-1}, x_k))d(x_{k-2}, x_{k-1}) \\ &\vdots \\ &< d(x_1, x_0). \end{aligned} \tag{3.38}$$

Thus, $\{d(x_k, x_{k+1})\}_{k\in\mathbb{N}}$ is a non-increasing sequence and bounded below for which the sequence converges to some real number $\ell \geq 0$ (say). If $\ell > 0$, then we see that from inequality (3.37), we get

$$1 \leq \lim_{k\to\infty} \inf \sum_{n=0}^{\infty} \alpha(d(x_{k-1}, x_k))\beta^n(d(x_{k-1}, x_k)). \tag{3.39}$$

So that $\lim_{k\to\infty} d(x_{k-1}, x_k) = 0$, a contradiction. Therefore,

$$\lim_{k\to\infty} d(x_k, x_{k+1}) = 0. \tag{3.40}$$

The Cauchyness of $\{x_k\}_{k\in\mathbb{N}}$ is the same in inequality (3.16) of Theorem 3.1. Hence $\{x_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in X. Due to completeness of X, there is a candidate $x^* \in X$ such that $x_k \to x^* \in X$. Now we show that x^* is a fixed point of T. Suppose that $Tx^* \neq x^*$ and $d(x_k, Tx^*) = d(Tx_{k-1}, Tx^*)$. From inequality (3.35), take $x = x_{k-1}$ and $y = x^*$ for all $k \in \mathbb{N}$, thus

$$\begin{aligned} d(x_k, Tx^*) &= d(Tx_{k-1}, Tx^*) \\ &\leq \alpha(d(x_{k-1}, x^*))d(x_{k-1}, x^*)^{A_\beta^{C_\delta^{D_\epsilon^{B_\gamma}}}}, \end{aligned} \tag{3.41}$$

where;

$$A_\beta := \frac{\beta(d(x_{k-1}, x^*))d(x_{k-1}, Tx_{k-1})}{1 + \epsilon(d(x_{k-1}, x^*))d(x^*, Tx_{k-1})}, \quad B_\gamma := \frac{\gamma(d(x_{k-1}, x^*))d(x^*, Tx^*)}{1 + \epsilon(d(x_{k-1}, x^*))d(x^*, Tx_{k-1})},$$
$$C_\delta := \frac{\delta(d(x_{k-1}, x^*))d(x_{k-1}, Tx^*)}{1 + \epsilon(d(x_{k-1}, x^*))d(x^*, Tx_{k-1})} \quad D_\epsilon := \frac{\epsilon(d(x_{k-1}, x^*))d(x^*, Tx_{k-1})}{1 + \epsilon(d(x_{k-1}, x^*))d(x^*, Tx_{k-1})}.$$

Thus,

$$A_\beta := \beta(d(x_{k-1}, x^*))d(x_{k-1}, x_k), \quad B_\gamma := \gamma(d(x_{k-1}, x^*))d(x^*, Tx^*),$$
$$C_\delta := \delta(d(x_{k-1}, x^*))d(x_{k-1}, Tx^*), \quad D_\epsilon := \epsilon(d(x_{k-1}, x^*))d(x^*, x_k).$$

From inequality (3.12), we get;

$$d(x^*, Tx^*) = 0, \tag{3.42}$$

a contradiction. Therefore, $x^* = Tx^*$. For the uniqueness, suppose that the fixed point of T in X that is $x^* = Tx^*$ for some $x^* \in Fix(T)$. Again, suppose if possible that there exists another fixed point of T in X, that is $y^* = Ty^*$ for some $y^* \in Fix(T)$. Now suppose that $x^* \neq y^* \implies d(x^*, y^*) > 0$. Therefore

$$d(x^*, y^*) = d(Tx^*, Ty^*)$$

$$\leq \alpha(d(x^*,y^*))d(x^*,y^*)^{A_\beta^{C_\delta^{D_\epsilon^{B_\gamma}}}}, \tag{3.43}$$

where,

$$A_\beta := \frac{\beta(d(x^*,y^*))d(x^*,Tx^*)}{1+\epsilon(d(x^*,y^*))d(y^*,Tx^*)}, \quad B_\gamma := \frac{\gamma(d(x^*,y^*))d(y^*,Ty^*)}{1+\epsilon(d(x^*,y^*))d(y^*,Tx^*)},$$
$$C_\delta := \frac{\delta(d(x^*,y^*))d(x^*,Ty^*)}{1+\epsilon(d(x^*,y^*))d(y^*,Tx^*)}, \quad D_\epsilon := \frac{\epsilon(d(x^*,y^*))d(y^*,Tx^*)}{1+\epsilon(d(x^*,y^*))d(y^*,Tx^*)},$$

for all distinct $x^*, y^* \in Fix(T)$. Inequality (3.43) shows that

$$d(x^*,y^*) \leq \alpha(d(x^*,y^*)). \tag{3.44}$$

Therefore, the sequence $\{x_k\}_{k\in\mathbb{N}}$ defined by $Tx_{k-1} = x_k$ converges to a unique fixed point of T in X. □

Corollary 3.10. Let (X,d) be a complete metric space and $T : X \to X$ such that there are $\{\alpha,\beta,\gamma,\delta,\epsilon\} \in \mathcal{F}_{Ger}$ and there exists $\theta \in \mathbb{R}^+$ satisfying;

$$d(Tx,Ty) \lneqq \alpha(d(x,y))d(x,y)^{A_\beta^{(\theta D_\epsilon)^{B_\gamma^{C_\delta}}}}, \tag{3.45}$$

where,

$$A_\beta := \frac{\beta(d(x,y))d(x,Tx)}{1+\epsilon(d(x,y))d(y,Tx)}, \; B_\gamma := \frac{\gamma(d(x,y))d(y,Ty)}{1+\epsilon(d(x,y))d(y,Tx)}, \; C_\delta := \frac{\delta(d(x,y))d(x,Ty)}{1+\epsilon(d(x,y))d(y,Tx)}, \; D_\epsilon := \frac{\epsilon(d(x,y))d(y,Tx)}{1+\epsilon(d(x,y))d(y,Tx)}$$

for all distinct $x, y \in X$ for all $k \in \mathbb{N}$. Then the sequence $\{x_k\}_{k\in\mathbb{N}}$ defined by $Tx_{k-1} = x_k$ converges to a unique fixed point of T in X.

Proof . We give the proof of Corollary 3.10 in its full strength. We prove in two cases:

(a) Observe first that if $\theta = 0$, then inequality (3.45) reduced to

$$d(Tx,Ty) < \alpha(d(x,y))d(x,y). \tag{3.46}$$

First assume that $x_n \to x^*$ in X and let $\{x_{h_n}\}$ and $\{x_{k_n}\}$ be any two subsequences of $\{x_h\}$. Then clearly $d_n = d(x_{h_n}, y_{h_n}) \to 0$ so the condition is satisfied. Next assume the condition is satisfied for a given initial point x_0 in X Then $d_n = d(x_n, y_n)$ is a decreasing sequence of nonnegative real numbers and so approaches some $\epsilon \geq 0$. Assume $\epsilon > 0$. Then letting $h_n = n$ and $k_n = n+1$, we have that $d_n \to \epsilon > 0$ while $\Omega_n \to 1$ as $n \to \infty$. So the condition is violated. Thus $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Now assume the given sequence of iterates $\{x_n\}$ is not Cauchy. Then there exists some $\epsilon > 0$ such that every tail $\{x_n\}_{n\geq N}$ of the sequence has diameter $D_N = \sup_{n,m\geq N} d(x_n, x_m) > \epsilon$. Given this ϵ, we will construct a pair of subsequences violating the condition. For any $n > 0$, let N_n be large that $d(x_m, x_{m+1}) < \frac{1}{n}$ for all $m \geq N_n$, as is possible that $d(x_m, x_{m+1}) \to 0$. Let $h_n \geq N_n$ be the lowest integer such that for some $k_n > h_n$, $d(x_{h_n}, x_{h_k}) > \epsilon$. Such pairs exist by the above diameter condition. Next choose k_n to be the least such integer above h_n. Then either $k_n - 1 = h_n$ or $d(x_{h_n}, x_{h_k - 1}) \leq \epsilon$. In either case we have $\epsilon \leq d_n = d(x_{h_n}, x_{h_k-1}) < \epsilon + \frac{1}{n}$. Moreover, using the triangular inequality, from inequality (3.46), we have

$$1 \geq \Omega_n = \frac{d(T(x_{h_n}), T(y_{k_n}))}{d(x_n, y_n)} \geq \frac{d_n - \frac{2}{n}}{d_n}. \tag{3.47}$$

So $\Omega_n \to 1$ while $d_n \to \epsilon$, again violating the condition. So $\{x_n\}$ must be a Cauchy sequence in X and as X is complete, we have $x_n \to x^*$ for some x^* in X. Then by the usual arguments, x^* is a unique fixed point of T and the proof of (a) is complete.

(b) if $\theta \neq 0$, then inequality (3.45) becomes;

$$d(Tx,Ty) \lneqq \alpha(d(x,y))d(x,y)^{A_\beta^{(\theta D_\epsilon)^{B_\gamma^{C_\delta}}}}, \tag{3.48}$$

where,

$$A_\beta := \frac{\beta(d(x,y))d(x,Tx)}{1+\epsilon(d(x,y))d(y,Tx)}, \; B_\gamma := \frac{\gamma(d(x,y))d(y,Ty)}{1+\epsilon(d(x,y))d(y,Tx)}, \; C_\delta := \frac{\delta(d(x,y))d(x,Ty)}{1+\epsilon(d(x,y))d(y,Tx)}, \; D_\epsilon := \frac{\epsilon(d(x,y))d(y,Tx)}{1+\epsilon(d(x,y))d(y,Tx)}$$

for all distinct $x, y \in X$ for all $k \in \mathbb{N}$. Suppose that $x_k = Tx_{k-1}$, then $x_{k+1} = Tx_k$ for all $k \in \mathbb{N}$. But $x_k \neq x_{k+1}$ and this implies that $d(x_k, x_{k+1}) > 0$, so that $d(x_k, x_{k+1}) = d(Tx_{k-1}, Tx_k)$. From inequality (3.48), take $x = x_{k-1}$ and $y = x_k$ for all $k \in \mathbb{N}$, thus

$$d(Tx_{k-1}, Tx_k) \lneqq \alpha(d(x_{k-1}, x_k))d(x_{k-1}, x_k)^{A_\beta^{(\theta D_\epsilon)^{B_\gamma^{C_\delta}}}}; \tag{3.49}$$

where,

$$A_\beta := \frac{\beta(d(x_{k-1}, x_k))d(x_{k-1}, Tx_{k-1})}{1 + \epsilon(d(x_{k-1}, x_k))d(x_k, Tx_{k-1})}, \quad B_\gamma := \frac{\gamma(d(x_{k-1}, x_k))d(x_k, Tx_k)}{1 + \epsilon(d(x_{k-1}, x_k))d(x_k, Tx_{k-1})},$$
$$C_\delta := \frac{\delta(d(x_{k-1}, x_k))d(x_{k-1}, Tx_k)}{1 + \epsilon(d(x_{k-1}, x_k))d(x_k, Tx_{k-1})}, \quad D_\epsilon := \frac{\epsilon(d(x_{k-1}, x_k))d(x_k, Tx_{k-1})}{1 + \epsilon(d(x_{k-1}, x_k))d(x_k, Tx_{k-1})}.$$

Thus,

$$A_\beta := \beta(d(x_{k-1}, x_k))d(x_{k-1}, x_k), \quad B_\gamma := \gamma(d(x_{k-1}, x_k))d(x_k, x_{k+1}),$$
$$C_\delta := \delta(d(x_{k-1}, x_k))d(x_{k-1}, x_{k+1}), \quad D_\epsilon := \epsilon(d(x_{k-1}, x_k))d(x_k, x_k) = 0.$$

So inequality (3.49) becomes,

$$d(Tx_{k-1}, Tx_k) \lneqq \alpha(d(x_{k-1}, x_k))d(x_{k-1}, x_k). \tag{3.50}$$

We need only to show that the existence of such an α in $\mathcal{F}_{Ger}$ is equivalent to say that for $x_n \to x^*$ in X, with x^* a unique fixed point of T, if and only if for any two subsequences x_{h_n} and x_{h_k} of x_n with $x_{h_n} \neq x_{h_k}$, we have that $\Omega_n \to 1$ only if $d_n \to 0$, where we take for any pair of sequences x_n and y_n with $x_n \neq y_n$, we write $d_n = d(x_n, y_n)$ and $\Omega_n = \frac{d(T(x_n), T(y_n))}{d(x_n, y_n)}$. First assume such an α exists. Let $\{x_{h_n}\}$ and $\{x_{k_n}\}$ be two subsequences with $x_{h_n} \neq x_{k_n}$. Assume that $\Omega_n \to 1$ as $n \to \infty$. Then it follows from inequality (3.50) that $\alpha(d(x_{h_n}, x_{k_n})) \to 1$. But then since $\alpha \in \mathcal{F}_{Ger}$, we have $d(x_{h_n}, x_{k_n}) \to 0$. Next assume that the sequential condition holds. Define $\alpha : \mathbb{R}^+ \to \mathbb{R}$ as follows $\alpha(t) = \sup\{\frac{d(T(x_n), T(x_m))}{d(x_n, x_m)} : d(x_n, x_m) \geq t\}$. From inequality (3.50) the quotients are all below 1 and so α is defined for all $t > 0$ and $\alpha < 1$. Now assume that $\alpha(t_n) \to 1$ for $t_n \in \mathbb{R}^+$. We may further assume without loss of generality that $1 - \frac{1}{n} \leq \alpha(t_n) < 1$. We must show $t_n \to 0$. But $\alpha(t_n)$ is the above least upper bound. So there is for each $n > 0$ a pair $\{x_{h_n}\}$ and $\{x_{k_n}\}$ in $\{x_n\}$ with $d(x_{h_n}, x_{k_n}) \geq t_n$ and

$$1 - \frac{1}{n} < \frac{d(T(x_{h_n}), T(x_{k_n}))}{d(x_{h_n}, x_{k_n})} < \alpha(t_n). \tag{3.51}$$

So the sequence $\Omega_n \to 1$ as $n \to \infty$. But then by the sequential condition above, $d(x_{h_n}, x_{k_n}) \to 0$. So $t_n \to 0$, as was shown. This completes the proof of Corollary 3.10. □

Remark 3.11. Observe that Corollary 3.10 generalized Geraghty [13]. To be more direct, just as in case (a) of our proof, take $\theta = 0$, then Corollary (3.10) reduced to Theorem 2.1 in [13]. Again, if $\theta \neq 0$, then take $x_k = Tx_{k-1}$, then inequality (3.45) collapsed to $d(Tx_{k-1}, Tx_k) \lneqq \alpha(d(x_{k-1}, x_k))d(x_{k-1}, x_k)$ which agree with the inequality in Theorem 1.3 of [13].

Corollary 3.10 is also true if inequality (3.45) is replace with an infinite type as the following result shows.

Proposition 3.12. Let (X, d) be a complete metric space and $T : X \to X$ such that there are $\{\alpha, \beta, \gamma, \delta, \epsilon\} \in \mathcal{F}_{Ger}$ and there exists $\theta \in \mathbb{R}^+$ satisfying;

$$d(Tx, Ty) \lneqq \alpha(d(x, y))d(x, y)^{A_\beta^{(\theta D_\epsilon)^{B_\gamma^{C_\delta^{C_\delta^{\cdot^{\cdot^{\cdot}}}}}}}}, \tag{3.52}$$

where,

$$A_\beta := \frac{\beta(d(x, y))d(x, Tx)}{1 + \epsilon(d(x, y))d(y, Tx)}, \; B_\gamma := \frac{\gamma(d(x, y))d(y, Ty)}{1 + \epsilon(d(x, y))d(y, Tx)}, \; C_\delta := \frac{\delta(d(x, y))d(x, Ty)}{1 + \epsilon(d(x, y))d(y, Tx)}, \; D_\epsilon := \frac{\epsilon(d(x, y))d(y, Tx)}{1 + \epsilon(d(x, y))d(y, Tx)}$$

for all distinct $x, y \in X$ for all $k \in \mathbb{N}$. Then the sequence $\{x_k\}_{k \in \mathbb{N}}$ defined by $Tx_{k-1} = x_k$ converges to a unique fixed point of T in X.

Proof . Observe first that if $\theta = 0$, then inequality (3.52) reduced to

$$d(Tx, Ty) < \alpha(d(x, y))d(x, y). \tag{3.53}$$

Copy every other thing in the proof of Corollary 3.10 and we are done. □

Remark 3.13. Take $\theta = 0$, then observe that Proposition 3.16 reduced to Theorem 1.1 of [13], i.e., $d(Tx, Ty) < \alpha(d(x, y))d(x, y)$.

Corollary 3.14. Let (X, d) be a complete metric space and $T : X \to X$ such that there are $\{\alpha, \beta, \gamma, \delta, \epsilon\} \in \mathcal{F}_{Ger}$ and there exists $\theta \in \mathbb{R}^+$ satisfying;

$$d(Tx, Ty) \le \alpha(d(x, y))d(x, y)^{A_\beta^{(\theta D_\epsilon)^{B_\gamma^{C_\delta}}}}, \tag{3.54}$$

where,

$$A_\beta := \frac{\beta(d(x, y))d(x, Tx)}{1 + \epsilon(d(x, y))d(y, Tx)},\ B_\gamma := \frac{\gamma(d(x, y))d(y, Ty)}{1 + \epsilon(d(x, y))d(y, Tx)},\ C_\delta := \frac{\delta(d(x, y))d(x, Ty)}{1 + \epsilon(d(x, y))d(y, Tx)},\ D_\epsilon := \frac{\epsilon(d(x, y))d(y, Tx)}{1 + \epsilon(d(x, y))d(y, Tx)}$$

for all distinct $x, y \in X$ for all $k \in \mathbb{N}$. Then the sequence $\{x_k\}_{k\in\mathbb{N}}$ defined by $Tx_{k-1} = x_k$ converges to a unique fixed point of T in X.

Proof . We prove as follows:

(a) if $\theta = 0$, then inequality (3.54) becomes;

$$d(Tx, Ty) \le \alpha(d(x, y))d(x, y). \tag{3.55}$$

This is Geraghty contraction map or inequality (3.55) or Theorem 2.8 see [13] for the proof.

(b) if $\theta \neq 0$, then inequality (3.54) becomes;

$$d(Tx, Ty) \le \alpha(d(x, y))d(x, y)^{A_\beta^{(\theta D_\epsilon)^{B_\gamma^{C_\delta}}}}; \tag{3.56}$$

where,

$$A_\beta := \frac{\beta(d(x, y))d(x, Tx)}{1 + \epsilon(d(x, y))d(y, Tx)},\ B_\gamma := \frac{\gamma(d(x, y))d(y, Ty)}{1 + \epsilon(d(x, y))d(y, Tx)},\ C_\delta := \frac{\delta(d(x, y))d(x, Ty)}{1 + \epsilon(d(x, y))d(y, Tx)},\ D_\epsilon := \frac{\epsilon(d(x, y))d(y, Tx)}{1 + \epsilon(d(x, y))d(y, Tx)}$$

for all distinct $x, y \in X$ for all $k \in \mathbb{N}$. Suppose that $x_k = Tx_{k-1}$, then $x_{k+1} = Tx_k$ for all $k \in \mathbb{N}$. But $x_k \neq x_{k+1}$ and this implies that $d(x_k, x_{k+1}) > 0$, so that $d(x_k, x_{k+1}) = d(Tx_{k-1}, Tx_k)$. From inequality (3.56), take $x = x_{k-1}$ and $y = x_k$ for all $k \in \mathbb{N}$, thus

$$d(Tx_{k-1}, Tx_k) \le \alpha(d(x_{k-1}, x_k))d(x_{k-1}, x_k)^{A_\beta^{D_\gamma^{B_\delta^{C_\epsilon}}}}; \tag{3.57}$$

where,

$$A_\beta := \frac{\beta(d(x_{k-1}, x_k))d(x_{k-1}, Tx_{k-1})}{1 + \epsilon(d(x_{k-1}, x_k))d(x_k, Tx_{k-1})}, \quad B_\gamma := \frac{\gamma(d(x_{k-1}, x_k))d(x_k, Tx_k)}{1 + \epsilon(d(x_{k-1}, x_k))d(x_k, Tx_{k-1})},$$
$$C_\delta := \frac{\delta(d(x_{k-1}, x_k))d(x_{k-1}, Tx_k)}{1 + \epsilon(d(x_{k-1}, x_k))d(x_k, Tx_{k-1})}, \quad D_\epsilon := \frac{\epsilon(d(x_{k-1}, x_k))d(x_k, Tx_{k-1})}{1 + \epsilon(d(x_{k-1}, x_k))d(x_k, Tx_{k-1})}.$$

Thus,

$$A_\beta := \beta(d(x_{k-1}, x_k))d(x_{k-1}, x_k), \quad B_\gamma := \gamma(d(x_{k-1}, x_k))d(x_k, x_{k+1}),$$
$$C_\delta := \delta(d(x_{k-1}, x_k))d(x_{k-1}, x_{k+1}), \quad D_\epsilon := \epsilon(d(x_{k-1}, x_k))d(x_k, x_k) = 0.$$

So inequality (3.56) becomes;

$$d(Tx_{k-1}, Tx_k) \le \alpha(d(x_{k-1}, x_k))d(x_{k-1}, x_k). \tag{3.58}$$

It follows Corollary 3.5 above or this is Theorem 2.7. Therefore, the sequence $\{x_k\}_{k\in\mathbb{N}}$ defined by $Tx_{k-1} = x_k$ converges to a unique fixed point of T in X. □

Remark 3.15. As $\theta = 0$, then Corollary 3.14 becomes Corollary 3.1 in [13] which is itself the generalization of result in [28]. This is so because such an α is clearly in the class $\mathcal{F}_{Ger}$.

Proposition 3.16. Let (X, d) be a complete metric space and $T : X \to X$ such that there are $\{\alpha, \beta, \gamma, \delta, \epsilon\} \in \mathcal{F}_{Ger}$ and there exists $\theta \in \mathbb{R}^+$ satisfying;

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y)^{{A_\beta^{(\theta D_\epsilon)}}^{{B_\gamma}^{{C_\delta}^{{C_\delta}^{\cdot^{\cdot^{\cdot}}}}}}}, \tag{3.59}$$

where,

$$A_\beta := \frac{\beta(d(x, y))d(x, Tx)}{1 + \epsilon(d(x, y))d(y, Tx)},\ B_\gamma := \frac{\gamma(d(x, y))d(y, Ty)}{1 + \epsilon(d(x, y))d(y, Tx)},\ C_\delta := \frac{\delta(d(x, y))d(x, Ty)}{1 + \epsilon(d(x, y))d(y, Tx)},\ D_\epsilon := \frac{\epsilon(d(x, y))d(y, Tx)}{1 + \epsilon(d(x, y))d(y, Tx)}$$

for all distinct $x, y \in X$ for all $k \in \mathbb{N}$. Then the sequence $\{x_k\}_{k \in \mathbb{N}}$ defined by $Tx_{k-1} = x_k$ converges to a unique fixed point of T in X.

Proof . It follows from the proof of Corollary 3.14 and we conclude that the sequence $\{x_k\}_{k \in \mathbb{N}}$ defined by $Tx_{k-1} = x_k$ converges to a unique fixed point of T in X. □

Remark 3.17. As $\theta = 0$, then Proposition 3.16 becomes Corollary 3.3 in [13] which is itself the generalization of result in Boyd-Wong [3]. See some results in [33]. This is so because such an α is clearly in the class $\mathcal{F}_{Ger}$. However, if $\theta \neq 0$, then take $x_k = Tx_{k-1}$, then inequality (3.59) collapsed to $d(Tx_{k-1}, Tx_k) \gneqq \alpha(d(x_{k-1}, x_k))d(x_{k-1}, x_k)$ which agree with the inequality in Corollary 3.3 of [13].

Conjecture 3.18. Can conclusion of Corollary 3.3 hold if we replace inequality (3.25) with an infinite type? i.e.,

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y)^{{C_\delta}^{{C_\delta}^{{C_\delta}^{{C_\delta}^{\cdot^{\cdot^{\cdot}}}}}}}, \tag{3.60}$$

where, $C_\delta := \dfrac{\delta(d(x, y))d(x, Ty)}{1 + \epsilon(d(x, y))d(y, Tx)}$.

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