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Existence of solutions for a strongly nonlinear $(p(x), q(x))$-elliptic systems via topological degree

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Abstract

This article is concerned with the study of the existence of a distributional solution for a strongly nonlinear $(p(x), q(x))-$elliptic systems. By means of the Berkovits degree theory, with suitable assumptions on the nonlinearities, we prove the existence of nontrivial solutions to our problem.

Keywords: Topological degree, Strangly nonlinear elliptic system, p(x)-Laplacian, Generalized Lebesgue and Sobolev spaces
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1 Introduction

In this article, we are interested in studying the existence of a distributional solution for the strongly nolinear elliptic system

$$\begin{cases} -\Delta_{p(x)}(u) = \lambda|u|^{r(x)-2}u + g(x, v, \nabla v) \quad \text{in } \Omega, \\ -\Delta_{q(x)}(v) = \mu|v|^{s(x)-2}v + h(x, u, \nabla u) \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $-\Delta_{p(x)}(u) = -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is $p(x)-$Laplacian, the functions $p, q, r, s \in C(\bar{\Omega})$ with $p(.), q(.)$ are log-Hölder continuous functions and λ, μ are a real parameters. We assume also that $2 < r^- \le r(x) \le r^+ < p^- \le p(x) \le p^+ < \infty$ and $2 < s^- \le s(x) \le s^+ < q^- \le q(x) \le q^+ < \infty$.

In recent years, the study of partial differential equations and variational problems involving variable exponent conditions is a very attractive topic and has been received considerable attention of many authors in this area of resarch (see [9, 10, 11, 13, 18, 20, 21, 24, 26, 27, 28, 29, 30, 31, 32, 33]). This is partly due to their various applications in various fields such as image processing [19], mathematical biology [16], elastic mechanics [40], stratigraphy problems

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[17] and electro-rheological fluids [1]. On the existence results for elliptic systems similar to (1.1), we refer [23, 35, 38] and references therein. In [4], the researchers proved the existence of weak solutions for the problem

$$\begin{cases} \Delta_{p(x)}(u) = \lambda|u|^{q(x)-2}u + f(x,u,\nabla u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

The proofs based on the recent Berkovits topological degree. In [14] Fan and Zhang considered the existence of weak solutions of the problem (1.2) with $\lambda = 0$ and f independent of ∇u. They present several existence results of weak solutions for problem. Their results are extensions of that of p-Laplacian problems. This same problem has been studied in [37]. Using critical point theory without the Ambrosetti-Rabinowitz condition, they obtain a couple of existence results of strong solutions. In [34], the authors solved the problem (1.2) with the right hand side is $\lambda f(x,u)$, under appropriate assumptions on f and g, they establish the existence and multiplicity of solutions. This existence is obtained by using the variational method. The similar problem that the problem (1.2) has been studied in [12]. The difference is that in [12] considered the case that the function $p(.) = q(.)$ and $f \equiv 0$. In [3], the authors studied the above problem with $\lambda = 0$. Using the topological degree theory for a class of demicontinuous operators of generalised (S_+) type, they obtain the existence results of at least weak solutions. The authors in [24] generalized these results to the system. More precisely, they studied the existence of solutions in the variational frame work by using the topological degree constructed by Kim and Hong [21]. For more details about this method, the reader can see [2, 6, 7, 25].

In our research, we concentrate our efforts to study the existence of distributional solutions for the system (1.1). This existence have been given by the topological degree method. Precisely, the existence of distributional solutions under suitable assumptions on the nonlinearities, has been discussed. These results are extensions of those in [4].

Our paper is structured as follows. In Section 2, we present some classes of mappings and topological degree, some basic properties of the variable exponent Lebesgue-Sobolev spaces and we collect several important properties of $p(x)-$Laplacian which will be later needed. Section 3 deals with the basic assumptions and the main results concerning the distributional solutions of system (1.1).

Notation. Throughout this paper, we will denoted by "$\to$" and "$\rightharpoonup$" the strong and weak convergence. We use $B_R(a)$ to denote the open ball in the Banach space X of radius $R > 0$ centered at a. The symbol "$\hookrightarrow$" means the continuous embedding.

2 Preliminaries

In order to discus system (1.1), we need some elementary results and theories on topological degree and on the variable-exponent Lebesgue-Soboleve spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. Firstly, we state some classes of mappings and topological degree, secondly, we recall basic properties of spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. Finally, we give some properties of $(p(x), q(x))$-Laplacian operators which will be used later.

2.1 Some classes of mappings and topological degree

Definition 2.1. Let X and Y be two real separable, reflexive Banach spaces and Ω a nonempty subset of X. A mapping $F : \Omega \subset X \to Y$ is

- bounded, if it takes any bounded set into a bounded set.
- demicontinuous, if for each $u \in \Omega$ and any sequence (u_n) in Ω, $u_n \to u$ implies $F(u_n) \rightharpoonup F(u)$.
- compact, if it is continuous and the image of any bounded set is relatively compact.

Definition 2.2. Let X be a real separable reflexive Banach space with dual space X^*. An operator $F : \Omega \subset X \to X^*$ is said to be

- of class (S_+), if for any sequence (u_n) in Ω with $u_n \rightharpoonup u$ and $\limsup\langle Fu_n, u_n - u\rangle \leq 0$, we have $u_n \to u$.
- quasimonotone, if for any sequence (u_n) in Ω with $u_n \rightharpoonup u$, we have $\limsup\langle Fu_n, u_n - u\rangle \geq 0$.

Definition 2.3. Let $T : \Omega_1 \subset X \to X^*$ be a bounded mapping such that $\Omega \subset \Omega_1$. For any mapping $F : \Omega \subset X \to X$, we say that

- F of class $(S_+)_T$, if for any sequence (u_n) in Ω with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ and $\limsup\langle Fu_n, y_n - y\rangle \leq 0$, we have $u_n \to u$.
- F has the property $(QM)_T$, if for any sequence (u_n) in Ω with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$, we have $\limsup\langle Fu_n, y_n - y\rangle \geq 0$.

Now, let $\mathcal{O}$ be the collection of all bounded open set in X. For any $\Omega \subset X$, we consider the following classes of operators:

$$\begin{aligned}
\mathcal{F}_1(\Omega) &:= \{F : \Omega \to X^* | F \text{ is bounded, demicontinuous and of class } (S_+)\},\\
\mathcal{F}_{T,B}(\Omega) &:= \{F : \Omega \to X | F \text{ is bounded, demicontinuous and of class } (S_+)_T\},\\
\mathcal{F}_T(\Omega) &:= \{F : \Omega \to X | F \text{ is demicontinuous and of class } (S_+)_T\},\\
\mathcal{F}_B(X) &:= \{F \in \mathcal{F}_{T,B}(\overline{G}) | G \in \mathcal{O}, T \in \mathcal{F}_1(\overline{G})\},\\
\mathcal{F}(X) &:= \{F \in \mathcal{F}_T(\overline{G}) | G \in \mathcal{O}, T \in \mathcal{F}_1(\overline{G})\},
\end{aligned}$$

where, $T \in \mathcal{F}_1(\overline{G})$ is calledc an essential inner map to F.

Lemma 2.4 ([5], Lemmas 2.2 and 2.4). Let $T \in \mathcal{F}_1(\overline{G}), G \in \mathcal{O}$, be continuous and $S : D_S \subset X^* \to X$ a bounded demicontinuous mapping such that $T(\overline{G}) \subset D_S$. Then the following statements are true:

- If S is quasimonotone, then $I + SoT \in \mathcal{F}_T(\overline{G})$, where I denote the identity operator.
- If S of class (S_+), then $SoT \in \mathcal{F}_T(\overline{G})$.

Definition 2.5. Let $F, S \in \mathcal{F}_T(\overline{G})$ and let G be a bounded open subset of a real reflexive Banach space X. The affine homotopy $\mathcal{H} : [0, 1] \times \overline{G} \to X$ given by

$$\mathcal{H}(\lambda, u) := (1 - \lambda)Fu + \lambda Su, \text{ for } (\lambda, u) \in [0, 1] \times \overline{G}$$

is called an admissible affine homotopy with the continuous essential inner map T.

Remark 2.6. [5] The above affine homotopy satisfies condition (S_+).

Now, we introduce the Berkovits topological degree for the class $\mathcal{F}_B(X)$. For more details see [5].

Theorem 2.7. There exists a unique degree function

$$\deg : \{(F, G, h) | G \in \mathcal{O}, T \in \mathcal{F}_1(\overline{G}), F \in \mathcal{F}_{T,B}(\overline{G}), h \notin F(\partial G)\} \to \mathbb{Z}$$

that satisfies the following properties:

- (Normalization) For any $h \in G$, we have $\deg(I, G, h) = 1$.
- (Additivity) Let $F \in \mathcal{F}_{T,B}(\overline{G})$. If G_1 and G_2 are two disjoint open subsets of G such that $h \notin F(\overline{G} \setminus (G_1 \cup G_2))$, then we have
$$\deg(F, G, h) = \deg(F, G_1, h) + \deg(F, G2, h).$$
- (Homotopy invariance) If $\mathcal{H} : [0, 1] \times \overline{G} \to X$ is a bounded admissible affine homotopy with a common continuous essential inner map and $h : [0, 1] \times X$ is a continuous path in X such that $h(\lambda) \notin \mathcal{H}(\lambda, \partial G)$ for all $\lambda \in [0, 1]$, then the value of $\deg(\mathcal{H}(\lambda, \cdot), G, h(\lambda))$ is constant for all $\lambda \in [0, 1]$.
- (Existence) If $\deg(F, G, h) \neq 0$, then the equation $Fu = h$ has a solution in G.

2.2 Notation and functional spaces

In this subsection, we list and recall some fact and results on variable exponent spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. See [8, 10, 15, 22, 39] for more details. Throughout the rest of the paper we consider a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$ with a Lipschitz boundary $\partial\Omega$. We denote

$$C_+(\overline{\Omega}) = \{f \in C(\overline{\Omega}) \mid \inf_{x\in\overline{\Omega}} f(x) > 1\},$$

$$f^- = \min_{x\in\overline{\Omega}} f(x), \quad f^+ = \max_{x\in\overline{\Omega}} f(x), \quad \text{for every } f \in C_+(\overline{\Omega}).$$

For each $p \in C_+(\overline{\Omega})$, we define the space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \Big\{u \mid u : \Omega \to \mathbb{R} \text{ is a measurable function, } \rho_{p(x)}(u) < \infty\Big\},$$

where $\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)}dx$, this space equipped with the Luxemburg norm

$$\|u\|_{p(x)} = \inf\{\lambda > 0 \mid \rho_{p(x)}(\frac{u}{\lambda}) \leq 1\},$$

and $(L^{p(x)}(\Omega), \|\cdot\|_{p(x)})$ becomes a Banach space.

Proposition 2.8. [22]

- The space $L^{p(x)}(\Omega)$ is a separable and reflexive Banach space.
- The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. Then for any $u \in L^{p(x)}(\Omega)$ and $w \in L^{p'(x)}(\Omega)$, we have the following Hölder inequality

$$\Big|\int_\Omega uw dx\Big| \leq \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)\|u\|_{p(x)}\|w\|_{p'(x)} \leq 2\|u\|_{p(x)}\|w\|_{p'(x)}. \tag{2.1}$$

- If $p_1, p_2 \in C_+(\overline{\Omega})$, $p_1(x) \leq p_2(x)$ for any $x \in \overline{\Omega}$, then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$

Proposition 2.9. [24, 39] If $u, u_n \in L^{p(x)}(\Omega)$, then the following assertions hold true:

$$\|u\|_{p(x)} < 1 \ (= 1, > 1) \Leftrightarrow \rho_{p(x)}(u) < 1 \ (= 1, > 1). \tag{2.2}$$

$$\|u\|_{p(x)} < 1 \Rightarrow \|u\|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq \|u\|_{p(x)}^{p^-}. \tag{2.3}$$

$$\|u\|_{p(x)} > 1 \Rightarrow \|u\|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq \|u\|_{p(x)}^{p^+}. \tag{2.4}$$

$$\lim_{n\to\infty} \|u_n - u\|_{p(x)} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{p(x)}(u_n - u) = 0. \tag{2.5}$$

$$\|u\|_{p(x)} \leq \rho_{p(x)}(u) + 1. \tag{2.6}$$

$$\rho_{p(x)}(u) \leq \|u\|_{p(x)}^{p^-} + \|u\|_{p(x)}^{p^+}. \tag{2.7}$$

Now, we define the space $W^{1,p(x)}(\Omega)$ as $W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \mid |\nabla u| \in L^{p(x)}(\Omega)\}$, equipped with the norm

$$\|u\|_{W^{1,p(x)}} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}. \tag{2.8}$$

Let $W_0^{1,p(x)}(\Omega)$ denote the subspace of $W^{1,p(x)}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.8).

Proposition 2.10. [8, 15, 22]

- The two spaces $W_0^{1,p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are a Banach spaces separable and reflexive.

- If $p(x)$ satisfies the log-Hölder continuity condition, i.e., there is a constant $\alpha > 0$ such that for every $x, y \in \Omega, x \notin y$ with $|x-y| \leq \frac{1}{2}$ one has
$$|p(x) - p(y)| \leq \frac{\alpha}{-\log|x-y|}, \tag{2.9}$$
then there exists a constant $C > 0$, such that
$$\|u\|_{p(x)} \leq C\|\nabla u\|_{p(x)}, \quad \forall u \in W_0^{1,p(x)}(\Omega). \tag{2.10}$$

- If $p \in C_+(\overline{\Omega})$ for any $x \in \overline{\Omega}$, then the imbedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ is compact.

Remark 2.11.

- By (2) of Lemma 2.10, we know that $\|\nabla u\|_{p(x)}$ and $\|u\|$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$.

- The dual space of $W_0^{1,p(x)}(\Omega)$ is $W^{-1,p'(x)}(\Omega)$, which endowed with the norm
$$\|u\|_{-1,p'(x)} = \inf\Big\{\|u_0\|_{p'(x)} + \sum_{i=1}^{N}\|u_i\|_{p'(x)}\Big\},$$
where the infinimum is taken on all possible decompositions $u = u_0 - divF$ with $u_0 \in L^{p'(x)}(\Omega)$ and $F = (u_1, \cdots, u_N) \in (L^{p'(x)}(\Omega))^N$.

Let us define $U = W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega)$ endowed with the norm $\|(u,v)\|_U = \max(\|u\|_{1,p(x)}, \|v\|_{1,q(x)})$ where $\|u\|_{1,p(x)} = \|\nabla u\|_{p(x)}$ and $(U, \|\cdot\|)$ is a Banach space, separable and reflexive.

2.3 Properties of $(p(x), q(x))$-Laplacian operators

Now, we discuss the $(p(x), q(x))$-Laplacian operator
$$-\Delta_{p(x)}u = -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u), \quad \text{and} \quad -\Delta_{q(x)}v = -\mathrm{div}(|\nabla v|^{q(x)-2}\nabla v).$$

We consider the following functional:
$$\mathcal{J}(u,v) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx + \int_\Omega \frac{|\nabla v|^{q(x)}}{q(x)}dx.$$

We know that (see [14]) $\mathcal{J} \in C^1(U, \mathbb{R})$ and the $(p(x), q(x))$-Laplacian operator is the derivative operator of $\mathcal{J}$ in the weak sense. Denote $T = \mathcal{J}' : U \to U^*$, then for any $(w, \psi) \in U$
$$\langle T(u,v), (w,\psi)\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla w dx + \int_\Omega |\nabla v|^{q(x)-2}\nabla v \nabla \psi dx, \quad \forall u, v \in U.$$

Theorem 2.12. [14]

- $T : U \to U^*$ is a continuous, bounded and strictly monotone operator.
- $T : U \to U^*$ is a mapping of type (S_+).
- $T : U \to U^*$ is a homeomorphism.

The proof of the above theorem can be found in [14].

3 Basic assumptions and the main results

In the present section, we study the existence of distributional solutions for the systems (1.1) based on the degree theory in Section 2, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a Lipschitz boundary $\partial\Omega$, $p, q \in C_+(\bar{\Omega})$ satisfies the log-Hölder continuity (2.9), $r, s \in C_+(\bar{\Omega})$, $2 < r^- \leq r(x) \leq r^+ < p^- \leq p(x) \leq p^+ < \infty$, $2 < s^- \leq s(x) \leq s^+ < q^- \leq q(x) \leq q^+ < \infty$ and $g, h : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are a real-valued functions such that

(A_1) (Continuity) g, h are the Carathéodory functions (i.e., $g(x, \cdot, \cdot)$ is continuous in (t_1, t_2) for almost every $x \in \Omega$ and $g(\cdot, t_1, t_2)$ is measurable in x for each $(t_1, t_2) \in \mathbb{R} \times \mathbb{R}^N$).

(A_2) (Growth) There exist a positive constants k_1, k_2, $b \in L^{p'(x)}(\Omega), d \in L^{q'(x)}(\Omega)$, $b(x), d(x) \geq 0$ and $\alpha, \beta \in C_+(\bar{\Omega})$ with $2 < \alpha^- \leq \alpha(x) \leq \alpha^+ < p^-$, $2 < \beta^- \leq \beta(x) \leq \beta^+ < q^-$, such that

$$|g(x, t_1, t_2)| \leq k_1(b(x) + |t_1|^{\alpha(x)-1} + |t_2|^{\alpha(x)-1}), \quad \text{and} \quad |h(x, \xi_1, \xi_2)| \leq k_2(d(x) + |\xi_1|^{\beta(x)-1} + |\xi_2|^{\beta(x)-1}).$$

Definition 3.1. We say that $(u, v) \in U$ is a distributional solution of the system (1.1) if for any $(w, \psi) \in U$ we have

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla w dx + \int_\Omega |\nabla v|^{q(x)-2}\nabla v \nabla \psi dx = \int_\Omega (\lambda|u|^{r(x)-2}u + g(x, v, \nabla v))w dx + \int_\Omega (\mu|v|^{s(x)-2}v + h(x, u, \nabla u))\psi dx. \tag{3.1}$$

Remark 3.2. Note that $\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla w dx + \int_\Omega |\nabla v|^{q(x)-2}\nabla v \nabla \psi dx = \langle T(u, v), (w, \psi)\rangle$ as defined in subsection 2.3, $\lambda|u|^{r(x)-2}u \in L^{p'(x)}(\Omega)$, $\mu|v|^{s(x)-2}v \in L^{q'(x)}(\Omega)$, $g(x, v, \nabla v) \in L^{q'(x)}(\Omega)$ and $h(x, u, \nabla u) \in L^{p'(x)}(\Omega)$ under $(u, v) \in U$, the assumptions A_2) and the given hypotheses about the exponents p, q, r and s becaose: $b \in L^{p'(x)}(\Omega)$ and $d \in L^{q'(x)}(\Omega)$, $\gamma(x) = (r(x) - 1)p'(x) \in C_+(\bar{\Omega})$ with $\gamma(x) < p(x)$, $\kappa(x) = (\beta(x) - 1)p'(x) \in C_+(\bar{\Omega})$ with $\kappa(x) < p(x)$, $\theta(x) = (s(x) - 1)q'(x) \in C_+(\bar{\Omega})$ with $\theta(x) < q(x)$ and $\delta(x) = (\alpha(x) - 1)p'(x) \in C_+(\bar{\Omega})$ with $\alpha(x) < q(x)$. Then, by the continuousembedding, we can conclude that $L^{p(x)} \hookrightarrow L^{\gamma(x)}$, $L^{p(x)} \hookrightarrow L^{\kappa(x)}$, $L^{q(x)} \hookrightarrow L^{\theta(x)}$ and $L^{q(x)} \hookrightarrow L^{\delta(x)}$. Hence, since $(w, \psi) \in L^{p(x)} \times L^{q(x)}$, we have

$$\int_\Omega (\lambda|u|^{r(x)-2}u + g(x, v, \nabla v))w dx + \int_\Omega (\mu|v|^{s(x)-2}v + h(x, u, \nabla u))\psi dx \in L^1(\Omega) \times L^1(\Omega).$$

This implies that the integral

$$\int_\Omega (\lambda|u|^{r(x)-2}u + g(x, v, \nabla v))w dx + \int_\Omega (\mu|v|^{s(x)-2}v + h(x, u, \nabla u))\psi dx$$

exists.

Lemma 3.3. Assume that the assumptions (A_1) and (A_2) hold. Then the operator $S : U \to U^*$ given by

$$\begin{cases} (u, V) \in U, \\ \langle S(u, v), (w, \psi)\rangle = -\int_\Omega (\lambda|u|^{r(x)-2}u + g(x, v, \nabla v))w dx - \int_\Omega (\mu|v|^{s(x)-2}v + h(x, u, \nabla u))\psi dx, \ \forall (w, \psi) \in U \end{cases}$$

is compact.

Proof . We divide the proof into three steps.

Step 1 Let $\varphi : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$, $\phi : W_0^{1,q(x)}(\Omega) \to L^{q'(x)}(\Omega)$ be two operators defined by

$$\varphi u(x) = -\lambda|u(x)|^{r(x)-2}u(x) \ \text{ for } u \in W_0^{1,p(x)} \ \text{ and } x \in \Omega,$$

and

$$\phi v(x) = -\mu|v(x)|^{s(x)-2}v \ \text{ for } v \in W_0^{1,q(x)} \ \text{ and } x \in \Omega.$$

In this step, we show that the operators φ and ϕ are continuous and bounded. It is clear that the operators φ and ϕ are continuous. Next, we show that φ and ϕ are bounded. Let $u \in W_0^{1,p(x)}(\Omega)$, by inequalities (2.6) and (2.7), we obtain

$$\begin{aligned}
\|\varphi u\|_{p'(x)} &\leq \rho_{p'(x)}(\varphi u) + 1 \\
&= \int_\Omega |\lambda|u|^{r(x)-1}|^{p'(x)} dx + 1 \\
&\leq (|\lambda|^{p'^-} + |\lambda|^{p'^+})\rho_{\gamma(x)} + 1 \\
&\leq (|\lambda|^{p'^-} + |\lambda|^{p'^+})(|u|^{\gamma^-}_{\gamma(x)} + |u|^{\gamma^+}_{\gamma(x)}) + 1.
\end{aligned}$$

Then, we have by (2.10) and $L^{p(x)} \hookrightarrow L^{\gamma(x)}$ that

$$\|\varphi u\|_{p'(x)} \leq \text{ const}\Big(\|u\|^{\gamma^-}_{1,p(x)} + \|u\|^{\gamma^+}_{1,p(x)}\Big) + 1,$$

that means φ is bounded on $W_0^{1,p(x)}$. Similarly, we can show that ϕ is bounded on $W_0^{1,q(x)}$.

Step 2 We define the operators $\vartheta : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$, $\chi : W_0^{1,q(x)}(\Omega) \to L^{q'(x)}(\Omega)$ by

$$\vartheta u(x) = -h(x, u, \nabla u) \ \text{ for } u \in W_0^{1,p(x)} \ \text{ and } x \in \Omega,$$

and

$$\chi w(x) = -g(x, w, \nabla w) \ \text{ for } w \in W_0^{1,q(x)} \ \text{ and } x \in \Omega.$$

We will show that ϑ and χ are bounded and continuous. For any $u \in W_0^{1,p(x)}(\Omega)$, we have, by the inequalities (2.6) and (2.7) and the condition (A_2) that

$$\begin{aligned}
\|\vartheta u\|_{p'(x)} &\leq \rho_{p'(x)}(\vartheta u) + 1 \\
&= \int_\Omega |h(x, u(x), \nabla u(x))|^{p'(x)} + 1 \\
&\leq \text{const}\Big(\int_\Omega \Big(|d| + |u|^{\beta(x)-1} + |\nabla u|^{\beta(x)-1}\Big)^{p'(x)} dx\Big) \\
&\leq \text{const}\Big(\rho_{p'(x)}(d) + \rho_{\kappa(x)}(u) + \rho_{\kappa(x)}(\nabla u)\Big) + 1 \\
&\leq \text{const}\Big(\|d\|^{p'^-}_{p'(x)} + \|d\|^{p'^+}_{p'(x)} + \|u\|^{\kappa^-}_{\kappa(x)} + \|u\|^{\kappa^+}_{\kappa(x)} + \|\nabla u\|^{\kappa^-}_{\kappa(x)} + \|\nabla u\|^{\kappa^+}_{\kappa(x)}\Big) + 1.
\end{aligned}$$

Hence, we have by the continuous embedding $L^{p(x)} \hookrightarrow L^{\kappa(x)}$ and (2.10) that

$$\|\vartheta u\|_{p'(x)} \leq \text{ const}\Big(\|d\|^{p'^-}_{p'(x)} + \|d\|^{p'^+}_{p'(x)} + \|u\|^{\kappa^-}_{1,p(x)} + \|u\|^{\kappa^+}_{1,p(x)}\Big) + 1.$$

Consequently, ϑ is bounded on $W_0^{1,p(x)}$. Similarly, we can show that χ is bounded on $W_0^{1,q(x)}$. Now, we prove that the operators ϑ and χ are continuous. To this purpose, let (u_n, v_n) converge to (u, v) in U. Then

$$u_n \to u \text{ and } \nabla u_n \to \nabla u \text{ in } W_0^{1,p(x)}, \quad \text{and} \quad v_n \to v \text{ and } \nabla v_n \to \nabla v \text{ in } W_0^{1,q(x)}.$$

Hence there exist two subsequences denote again by (u_n), (v_n) and measurable functions ω_1 (resp. ω_2) in $L^{p(x)}(\Omega)$ (resp. in $L^{q(x)}(\Omega)$) and ϖ_1 (resp.ϖ_2) in $(L^{p(x)}(\Omega))^N$ (resp. in $(L^{q(x)}(\Omega))^N$), such that

$$u_n(x) \to u(x) \text{ and } \nabla u_n(x) \to \nabla u(x),$$

$$v_n(x) \to w(x) \text{ and } \nabla v_n(x) \to \nabla v(x),$$

$$|u_n(x)| \leq \omega_1(x),\ |\nabla u_n(x)| \leq |\varpi_1(x)| \text{ and } |v_n(x)| \leq \omega_2(x),\ |\nabla v_n(x)| \leq |\varpi_2(x)|,$$

for almost all $x \in \Omega$ and all $n \in N$. From (A_1) and (A_2), we obtain

$$h(x, u_n(x), \nabla u_n(x)) \to h(x, u(x), \nabla u(x)) \text{ for almost all } x \in \Omega,$$

and

$$|h(x, u_n(x), \nabla u_n(x))| \leq \text{ const}\Big(d(x) + |\omega_1(x)|^{\beta(x)-1} + |\varpi_1(x)|^{\beta(x)-1}\Big),$$

for almost all $x \in \Omega$ and all $n \in N$ and $d + |\omega_1|^{\beta(x)-1} + |\varpi_1|^{\beta(x)-1} \in L^{p'(x)}(\Omega)$. Taking into account the equality

$$\rho_{p'(x)}(\vartheta u_n - \vartheta u) = \int_\Omega |h(x, u_n(x), \nabla u_n(x)) - h(x, u(x), \nabla u(x))|^{p'(x)} dx,$$

then, from the equivalence (2.5) and the Lebesgue dominated convergence theorem, we obtain $\vartheta u_n \to \vartheta u$ in $L^{p'(x)}(\Omega)$, that is, ϑ is continuous. Similarly, we obtain that χ is continuous.

Step 3 Let $\mathcal{I}_1^* : L^{p'(x)}(\Omega) \to W_0^{1,p'(x)}(\Omega)$, $\mathcal{I}_2^* : L^{q'(x)}(\Omega) \to W_0^{1,q'(x)}(\Omega)$, be the adjoint operators of the operators $\mathcal{I}_1 : W_0^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$, $\mathcal{I}_2 : W_0^{1,q(x)}(\Omega) \to L^{q(x)}(\Omega)$, respectively.

Then we define $\mathcal{I}_1^* o\varphi : W_0^{1,p(x)}(\Omega) \to W_0^{1,p'(x)}(\Omega)$, $\mathcal{I}_2^* o\phi : W_0^{1,q(x)}(\Omega) \to W_0^{1,q'(x)}(\Omega)$, $\mathcal{I}_1^* o\vartheta : W_0^{1,p(x)}(\Omega) \to W_0^{1,p'(x)}(\Omega)$, and $\mathcal{I}_2^* o\chi : W_0^{1,q(x)}(\Omega) \to W_0^{1,q'(x)}(\Omega)$. On another hand, as the operators $\mathcal{I}_1$ and $\mathcal{I}_2$ are compact, then $\mathcal{I}_1^*$ and $\mathcal{I}_2^*$ are compact. Therefore, the compositions $\mathcal{I}_1^* o\varphi$, $\mathcal{I}_2^* o\phi$, $\mathcal{I}_1^* o\vartheta$ and $\mathcal{I}_2^* o\chi$ are compact. We conclude that $S = \mathcal{I}_1^* o\varphi + \mathcal{I}_2^* o\phi + \mathcal{I}_1^* o\vartheta + \mathcal{I}_2^* o\chi$ is compact, which completes the proof of Lemma3.3. □

Theorem 3.4 ([36], Theorem 26A). Let the operator equation

$$Au = b, \;\; u \in X \tag{3.2}$$

together with the corresponding Galerkin equations

$$a(u_n, w_k) = \langle b, w_k \rangle, \;\; k = 1, \cdots, n, \tag{3.3}$$

where $A : X \to X^*$ is a monotone, coercive, and hemicontinuous operator on the real, separable, reflexive B-space X. Assume $\{w_1, w_2, \cdots\}$ is a basis in X. Then the following assertions hold:

1. Solution set. For each $b \in X^*$, equation (3.2) has a solution. The solution set of (3.2) is bounded, convex, and closed.
2. Galerkin method. If $\dim X = \infty$, then for each $n \in \mathbb{N}$, the Galerkin equation (3.3) has a solution $u_n \in X_n$ and the sequence (u_n) has a weakly convergent subsequence

$$u_n \rightharpoonup u \text{ in } X \text{ as } n \to \infty,$$

   where u is a solution of the original equation (3.2).
3. Uniqueness. If the operator A is strictly monotone, then equation (3.2) (resp. equation (3.3)) is uniquely solvable in$X$ (resp. X_n).
4. Inverse operator. If A is strictly monotone, then the inverse operator $A^{-1} : X^* \to X$ exists. This operator is strictly monotone, demicontinuous, and bounded.
   If A is uniformly monotone, then A^{-1} is continuous.
   If A is strongly monotone, then A^{-1} is Lipschitz continuous.
5. Strong convergence of the Galerkin method. Let $\dim X = \infty$. If the operator A is strictly monotone, then the sequence of Galerkin solutions (u_n) converges weakly in X to the unique solution u of equation (3.2).
   If A is uniformly monotone, then (u_n) converges strongly in X to the unique solution u of (3.2).
6. Nonseparable spaces. IfX is not separable, then the assertions 1, 3, and 4 remain true.

Theorem 3.5. Suppose that the assumptions (A_1) and (A_2) hold true. Then problem (1.1) has least one distributional solution (u, v) in U.

Proof . Let $(u, v) \in U, (w, \psi) \in U$, we define the operator S as defined in Lemma 3.3 and the operator T as defined in subsection 2.3

$$\langle S(u,v),(w,\psi)\rangle = -\int_\Omega (\lambda|u|^{r(x)-2}u + g(x,v,\nabla v))w dx - \int_\Omega (\mu|v|^{s(x)-2}v + h(x,u,\nabla u))\psi dx,$$

$$\langle T(u,v),(w,\psi)\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla w dx + \int_\Omega |\nabla v|^{q(x)-2}\nabla v \nabla \psi dx.$$

Then $(u, v) \in U$ is a distributional solution of (1.1) if and only if

$$T(u,v) = -S(u,v). \tag{3.4}$$

According to the properties of the operator T seen in Theorem 2.12 and by using the Minty-Browder Theorem 3.4, the inverse operator $L = T^{-1} : U^* \to U$ is bounded, continuous and satisfies condition (S_+). On another side, thanks to Lemma 3.3, the operator S is bounded, continuous and quasimonotone. Consequently, equation (3.4) is equivalent to

$$(u,v) = L(w,\psi) \text{ and } (w,\psi) + SoL(w,\psi) = 0. \tag{3.5}$$

Folowing the terminology of [36], the equation $(w, \psi) + SoL(w, \psi) = 0$ is an abstract Hammerstein equation in the reflexive space $W^{-1,p'(x)}(\Omega) \times W^{-1,q'(x)}(\Omega)$. Since the equation (3.4) is equivalent to (3.5), then to solve (3.4), it is thus enough to solve (3.5). To solve (3.5), we will using the degree theory introduced in subsection 2.1. For this , we first show that the set

$$\Sigma = \{(w,\psi) \in U^* | (w,\psi) + tSoL(w,\psi) = 0 \text{ for some } t \in [0,1]\}$$

is bounded. Indeed, let us up $(u, v) = L(w, \psi)$ for all $(w, \psi) \in \Sigma$, then $\|L(w, \psi)\|_U = \|(u, v)\|_U = \max(\|\nabla u\|_{p(x)}, \|\nabla v\|_{q(x)})$. If $\|\nabla u\|_{p(x)} \le 1$ and $\|\nabla v\|_{q(x)} \le 1$. Then $\|L(w, \psi)\|_U \le 1$, that means $\{L(w, \psi) : (w, \psi) \in \Sigma\}$ is bounded. If $\|\nabla u\|_{p(x)} > 1$ and $\|\nabla v\|_{q(x)} > 1$, then by using the assumption (A_2), the inequalities (2.1), (2.7), the implication (2.4) and the Young inequality, we obtain the estimate

$$\begin{aligned}
\|L(w,\psi)\|_U^{\min(p^-,q^-)} &= \|(u,v)\|_U^{\min(p^-,q^-)} \\
&\le \rho_{p(x)}(\nabla u) + \rho_{q(x)}(\nabla v) \\
&= \langle T(u,v),(u,v)\rangle \\
&= \langle (w,\psi), L(w,\psi)\rangle \\
&= -t\langle SoL(w,\psi), L(w,\psi)\rangle \\
&= t\bigg(\int_\Omega \Big(\lambda|u|^{r(x)-2}u + g(x,v,\nabla v)\Big)u dx + \int_\Omega \Big(\mu|v|^{s(x)-2}v + h(x,u,\nabla u)\Big)v dx\bigg) \\
&\le \text{const } \Big(\|u\|_{r(x)}^{r^-} + \|u\|_{r(x)}^{r^+} + \|b\|_{p'(x)}\|v\|_{p(x)} + \frac{1}{\alpha'^-}\rho_{\alpha(x)}(v) + \frac{1}{\alpha^-}\rho_{\alpha(x)}(u) \\
&\quad + \frac{1}{\alpha'^-}\rho_{\alpha(x)}(\nabla v) + \frac{1}{\alpha^-}\rho_{\alpha(x)}(u) + \|v\|_{s(x)}^{s^-} + \|v\|_{s(x)}^{s^+} + \|d\|_{q'(x)}\|u\|_{q(x)} \\
&\quad + \frac{1}{\beta'^-}\rho_{\beta(x)}(u) + \frac{1}{\beta^-}\rho_{\beta(x)}(v) + \frac{1}{\beta'^-}\rho_{\beta(x)}(\nabla u) + \frac{1}{\beta^-}\rho_{\beta(x)}(v)\Big) \\
&\le \text{const } \Big(\|u\|_{r(x)}^{r^-} + \|u\|_{r(x)}^{r^+} + \|v\|_{p(x)} + \|v\|_{\alpha(x)}^{\alpha^+} + \|\nabla v\|_{\alpha(x)}^{\alpha^+} \\
&\quad + \|v\|_{s(x)}^{s^-} + \|v\|_{s(x)}^{s^+} + \|u\|_{q(x)} + \|u\|_{\beta(x)}^{\beta^+} + \|\nabla u\|_{\beta(x)}^{\beta^+}\Big),
\end{aligned}$$

then, thanks to $L^{p(x)} \hookrightarrow L^{r(x)}$, $L^{p(x)} \hookrightarrow L^{\alpha(x)}$, $L^{q(x)} \hookrightarrow L^{s(x)}$ and $L^{q(x)} \hookrightarrow L^{\beta(x)}$ and (2.10), we get

$$\|L(w,\psi)\|_U^{\min(p^-,q^-)} \le \text{const } (\|L(w,\psi)\|_U^{\max(r^+,s^+)} + \|L(w,\psi)\|_U + \|L(w,\psi)\|_U^{\max(\alpha^+,\beta^+)}).$$

If $\|\nabla u\|_{p(x)} > 1$ and $\|\nabla v\|_{q(x)} \le 1$ (resp. if $\|\nabla u\|_{p(x)} \le 1$ and $\|\nabla v\|_{q(x)} > 1$), we can also get that $\|L(w, \psi)\|_V$ is bounded. Consequently $\{L(w, \psi)|(w, \psi) \in \Sigma\}$ is bounded. Since the operator S is bounded, it is obvious from (3.5) that the set Σ is bounded in U^*. However, there exists $\eta > 0$ such that

$$\|(w,\psi)\|_{U^*} < \eta \ \text{ for all } (w,\psi) \in \Sigma,$$

which says that

$$(w,\psi)+tSoL(w,\psi)\neq 0 \text{ for all } (w,\psi)\in\partial\Sigma_\eta(0) \text{ and all } t\in[0,1],$$

where $\Sigma_\eta(0)$ is the ball of radus η and center 0 in U^*. By Lemma 2.4, we conclude that

$$I+SoL\in\mathcal{F}_L(\overline{\Sigma_\eta(0)}) \text{ and } I=ToL\in\mathcal{F}_L(\overline{\Sigma_\eta(0)}).$$

Since the operators I,S and L are bounded, then $I+SoL$ is bounded. We conclude that

$$I+SoL\in\mathcal{F}_{L,B}(\overline{\Sigma_\eta(0)}) \text{ and } I\in\mathcal{F}_{L,B}(\overline{\Sigma_\eta(0)}).$$

Next, we consider the homotopy $\mathcal{H}:[0,1]\times\overline{\Sigma_\eta(0)}\to U^*$ given by

$$\mathcal{H}(t,w,\psi):=(w,\psi)+tSoL(w,\psi) \text{ for } (t,w,\psi)\in[0,1]\times\overline{\Sigma_\eta(0)}.$$

Hence, according to the properties of the degree deg stated in Theorem 2.7, we obtain

$$\deg(I+SoL,\Sigma_\eta(0),0)=\deg(I,\Sigma_\eta(0),0)=1,$$

which implies that $\exists(w,\psi)\in\Sigma_\eta(0)$ such that

$$(w,\psi)+SoL(w,\psi)=0.$$

Which implies that $(u,v)=L(w,\psi)$ is a distributional solution of (1.1). This completes the proof. □

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