

Nonexistence of sub-elliptic critical problems with Hardy-type potentials on Carnot group

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Abstract

Using the Pohozaev-type arguments, we prove the nonexistence results for sub-elliptic problems with critical Sobolev-Hardy exponents and Hardy-type potentials on the Carnot group.

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1 Introduction and the main results

In this paper, we are concerned with the following sub-elliptic problem:

$$\begin{cases} -\Delta_{\mathbb{G}}u - \gamma \frac{\psi^2 u}{d(z)^2} + \mu \frac{\psi^2 u}{d(z, a)^2} = K(z) \frac{\psi^\alpha |u|^{2_\alpha^* - 2} u}{d(z)^\alpha} + |u|^{q-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $-\Delta_{\mathbb{G}}$ is the sub-Laplacian operator on Carnot group \mathbb{G} , $\Omega \subset \mathbb{G}$ is a bounded domain with smooth boundary, $0, a \in \Omega$, d is the natural gauge and the geometrical function ψ is define by $\psi = |\nabla_{\mathbb{G}} d(z)|$, $K(z) \in C^1(\Omega)$, the parameters $\gamma \in (-\infty, \gamma_{\mathbb{G}})$, $\mu \in (0, +\infty)$ and $q \geq 2^*$.

We begin with some basic definitions and useful results for Carnot group, see [1, 3, 6, 13] for some details. A connected and simply connected Lie group (\mathbb{G}, \cdot) is a Carnot group of step k if its Lie algebra \mathfrak{g} admits a step k stratification. This means that there exist non-trivial linear subspaces V_1, \dots, V_k of \mathfrak{g} such that

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \dots \oplus V_k,$$

where $[V_1, V_i] = V_{i+1}$ for $i \in \{1, 2, \dots, k-1\}$ and $[V_1, V_k] = \{0\}$. Let $m_i = \dim(V_i)$ for $i = 1, \dots, k$, by means of the natural identification of \mathbb{G} with its Lie algebra via the exponential map, it is not restrictive to suppose that \mathbb{G} is a homogeneous Lie group on $\mathbb{R}^N := \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_k}$ equipped with a group-automorphisms (called dilations) $\delta_\gamma : \mathbb{G} \rightarrow \mathbb{G}$ of the form

$$\delta_\gamma(z) = (\gamma^1 z^{(1)}, \gamma^2 z^{(2)}, \dots, \gamma^k z^{(k)}),$$

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where $z = (z^{(1)}, z^{(2)}, \dots, z^{(k)}) \in \mathbb{G}$. On Carnot group \mathbb{G} , the topological dimension is defined by $N = \sum_{i=1}^k m_i$, and the homogeneous dimension is denoted that $Q = \sum_{i=1}^k i \cdot m_i$. Observe that $\mathbb{G} \cong \mathbb{R}^N$. In what follows, we assume that $Q \geq 3$.

Let X_1, X_2, \dots, X_m be the left invariant vector fields of V_1 , the operator

$$\Delta_{\mathbb{G}} = \sum_{i=1}^m X_i X_i$$

is called a sub-Laplacian on \mathbb{G} . We shall denote by $\nabla_{\mathbb{G}} = (X_1, \dots, X_m)$ the related horizontal gradient. Moreover, for any C^1 vector field $u = (u_1, u_2, \dots, u_m)$, we shall indicate by $\operatorname{div}_{\mathbb{G}} u := \sum_{i=1}^m X_i u_i$, the divergence with respect to the vector fields X_i 's. Finally, when $Q \geq 3$, the sub-Laplacian possess the following property: there exists a suitable homogeneous symmetry norm d on \mathbb{G} , which we shall refer to as the $\Delta_{\mathbb{G}}$ -gauge, such that

$$\Gamma(z) = \frac{C}{d(z)^{Q-2}}, \quad \forall z \in \mathbb{G}$$

is a fundamental solution of $-\Delta_{\mathbb{G}}$ with pole at 0, for a suitable constant $C > 0$. By definition, a homogeneous norm on \mathbb{G} is a continuous function $d(\cdot) : \mathbb{G} \rightarrow [0, \infty)$ such that: $d(\delta_{\gamma}(z)) = \gamma d(z)$ for every $\gamma > 0$ and every $z \in \mathbb{G}$, $d(z) = 0$ if and only if $z = 0$. We say that the homogeneous norm $d(\cdot)$ is symmetric if $d(z^{-1}) = d(z)$ for all $z \in \mathbb{G}$. If $d(\cdot)$ is a homogeneous norm on \mathbb{G} , then $d(z, y) = d(y^{-1} \circ z)$ is a pseudo-distance on \mathbb{G} . Note that any two continuous homogeneous norms are equivalent, i.e. within constant multiplicative constant factors of each other. As customary, we will denote by $B_d(z_0, \rho)$ the d -ball with center $z_0 \in \mathbb{G}$ and radius $\rho > 0$ given by

$$B_d(z_0, \rho) = \{z \in \mathbb{G} : d(z_0^{-1} \circ z) < \rho\}.$$

In this paper, we work in the Sobolev-Stein space $S_0^1(\Omega)$, defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{S_0^1(\Omega)} = (\int_{\Omega} |\nabla_{\mathbb{G}} u|^2 dz)^{\frac{1}{2}}$. On $S_0^1(\mathbb{G})$, the Hardy inequality is known as

$$\gamma_{\mathbb{G}} \int_{\mathbb{G}} \frac{\psi^2 |u|^2}{d(z)^2} dz \leq \int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 dz, \quad \forall u \in C_0^\infty(\mathbb{G}), \tag{1.2}$$

where $\gamma_{\mathbb{G}} = (\frac{Q-2}{2})^2$ is the best constant in the above inequality. For $\alpha \in [0, 2)$, the following Hardy-Sobolev inequality

$$S_{\alpha} \left(\int_{\mathbb{G}} \frac{\psi^{\alpha} |u|^{2_{\alpha}^*}}{d(z)^{\alpha}} dz \right)^{\frac{2}{2_{\alpha}^*}} \leq \int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 dz, \quad \forall u \in C_0^\infty(\mathbb{G}) \tag{1.3}$$

holds for some positive constant S_{α} . In here, $2_{\alpha}^* = \frac{2(Q-\alpha)}{Q-2}$ is called the critical exponent of the embedding $S_0^1(\mathbb{G}) \hookrightarrow L^{2_{\alpha}^*}(\mathbb{G}, \frac{\psi^{\alpha}}{d(z)^{\alpha}} dz)$ and Q denotes the homogeneous dimension of the space \mathbb{G} with respect to the dilation. Observe that $2^* = 2_0^* = \frac{2Q}{Q-2}$ is the critical Sobolev exponent.

Associated with problem (1.1), we consider the energy functional

$$I(u) = \frac{1}{2} \int_{\Omega} \left(|\nabla_{\mathbb{G}} u|^2 - \gamma \frac{\psi^2 |u|^2}{d(z)^2} + \mu \frac{\psi^2 |u|^2}{d(z, a)^2} \right) dz - \frac{1}{2_{\alpha}^*} \int_{\Omega} K(z) \frac{\psi^{\alpha} |u|^{2_{\alpha}^*}}{d(z)^{\alpha}} dz - \frac{1}{q} \int_{\Omega} |u|^q dz.$$

From (1.2) and (1.3), we know that I is well-defined and $I \in C^1(S_0^1(\Omega) \cap L^q(\Omega), \mathbb{R})$, and then its critical points correspond to solutions of (1.1).

Recently, the kind of sub-elliptic Dirichlet problem with singular potentials on Carnot group

$$-\Delta_{\mathbb{G}} u - \gamma \frac{\psi^2 u}{d(z)^2} = f(z, u) \quad \text{in } \Omega \subset \mathbb{G}$$

have been widely studied. We refer to [1, 3, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] and the references therein. Garofalo et al [5, 6] establish the existence, regularity and nonexistence results for sub-elliptic problems on Heisenberg group. In [7], the author extends partially the existence and nonexistence results due to [4] again on Carnot group of the form

$$-\Delta_{\mathbb{G}} u - \mu \frac{\psi^2 u}{d(z)^2} = \lambda u + |u|^{2^*-2} u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{1.4}$$

In this paper the author proved that: if $0 \leq \mu \leq \mu_{\mathbb{G}} - 1$, problem (1.4) has at least one positive solution $u \in S_0^1(\Omega)$ for any $0 < \lambda < \Lambda_1(\mu)$; if $\mu_{\mathbb{G}} - 1 < \mu < \mu_{\mathbb{G}}$, there exists $\lambda^* \in (0, \Lambda_1(\mu))$ such that problem (1.4) has at least one positive solution $u \in S_0^1(\Omega)$ for $\lambda^* < \lambda < \Lambda_1(\mu)$, where $\Lambda_1(\mu) > 0$ is the first eigenvalue of the operator $L_\mu := -\Delta_{\mathbb{G}} \cdot -\mu \frac{\psi^2}{d(z)^2}$. In particular, the author state some qualitative properties of ground state solutions to the following limit problem:

$$-\Delta_{\mathbb{G}} u - \mu \frac{\psi^2 u}{d(z)^2} = |u|^{2^*-2} u, \quad u \in S^{1,2}(\mathbb{G}).$$

Here the Sobolev-Stein space $S^{1,2}(\mathbb{G})$ is the completion of $C_0^\infty(\mathbb{G})$ with respect to the norm $(\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 dz)^{\frac{1}{2}}$. Inspired by the above works, by using the variational methods and the mountain-pass theorem, the existence of positive solution to the critical sub-elliptic system is established in [16]. Moreover, by means of the Moser iteration method, some asymptotic properties of its nontrivial solution at the singular point are verified.

However, there are few results about the nonexistence of sub-elliptic equation with multiple critical Sobolev-Hardy terms and multiple singular points. To the best of our knowledge, the problem of nonexistence of sub-elliptic solutions with multiple Hardy-type terms and critical exponent has never been considered before on the Carnot group in a non-Euclidean setting. We would like to point out that since an important feature of the sub-Laplacian is its degenerate property, it turn out from several technical reasons that studying our degenerate equation (1.1) is not directly by using a classical Pohozaev identity. The purpose of this paper is to prove Theorem 1.2, for this we recall the definition of δ_γ -starshaped domains.

Definition 1.1. Let Ω be a C^1 connected open set, $0 \in \Omega$. We say that Ω is a δ_γ -starshaped domain with respect to the origin if

$$\langle Z, \nu \rangle(z) \geq 0, \quad \forall z \in \partial\Omega,$$

where Z is the infinitesimal generator of the dilations δ_γ and $\nu = (\nu_1, \nu_2, \dots, \nu_N)$ is the outward normal to Ω .

In the above definition, the smooth vector field Z is the infinitesimal generator of the one-parameter group of non-isotopic dilations δ_γ , that is, the vector field such that

$$\left[\frac{d}{d\gamma} u(\delta_\gamma(z)) \right]_{\gamma=1} = Zu.$$

For a generic Carnot group of step k , Z has the following expression

$$Z = \sum_{i=1}^k \sum_{j=1}^{m_i} i z_j^{(i)} \frac{\partial}{\partial z_j^{(i)}}.$$

We recall that Z is characterized by the property that a function $u : \mathbb{G} \rightarrow \mathbb{R}$ is homogeneous of degree k with respect to δ_γ , i.e. $u(\delta_\gamma(z)) = \gamma^k u(z)$ if and only if $Zu = ku$. Moreover, the following properties hold for Z (see [6]):

$$[X_i, Z] = X_i, \quad \forall i = 1, 2, \dots, m, \quad \text{div} Z = Q.$$

Now we state our main results as follows.

Theorem 1.2. Let Ω is a δ_γ -starshaped domain with respect to the origin in \mathbb{G} . Assume that $K \in C^1(\bar{\Omega})$ and $ZK(z) \leq 0$, $Zd(z, a) \leq 0$ for a.e. $z \in \Omega$. Then problem (1.1) does not possess nontrivial nonnegative solution $u \in S_0^1(\Omega) \cap L^q(\Omega)$ such that $\frac{Zu}{d(z)} \in L^2(\Omega)$ for any $\mu > 0$.

The result in Theorem 1.2 can be easily generalized to the following sub-elliptic critical problem with multiple singular points:

$$\begin{cases} -\Delta_{\mathbb{G}} u - \gamma \frac{\psi^2 u}{d(z)^2} + \sum_{k=1}^m \mu_k \frac{\psi^2 u}{d(z, a_k)^2} = K(z) \frac{\psi^\alpha |u|^{2_\alpha^* - 2} u}{d(z)^\alpha} + |u|^{q-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $a_k \in \Omega$, $1 \leq k \leq m$ and $m \in \mathbb{N}$. Thus we have the following result.

Theorem 1.3. Let Ω is a δ_γ -starshaped domain with respect to the origin in \mathbb{G} , and assume that $K \in C^1(\bar{\Omega})$, $ZK(z) \leq 0$ and $Zd(z, a_k) \leq 0$ ($\forall k \in \{1, 2, \dots, m\}$) for a.e. $z \in \Omega$. Then problem (1.5) does not possess nontrivial nonnegative solution $u \in S_0^1(\Omega) \cap L^q(\Omega)$ such that $\frac{Zu}{d(z)} \in L^2(\Omega)$ for any $\mu_k > 0$

Furthermore, Theorem 1.3 can be generalized to the following sub-elliptic problem with multiple critical Sobolev-Hardy terms and multiple singular points:

$$\begin{cases} -\Delta_{\mathbb{G}}u - \gamma \frac{\psi^2 u}{d(z)^2} + \sum_{k=1}^m \mu_k \frac{\psi^2 u}{d(z, a_k)^2} = \sum_{j=1}^l K_j(z) \frac{\psi^{\alpha_j} |u|^{2_{\alpha_j}^* - 2} u}{d(z, b_j)^{\alpha_j}} + |u|^{q-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $a_k, b_j \in \Omega$, $m, l \in \mathbb{N}$ and $2_{\alpha_j}^* = \frac{2(Q-\alpha_j)}{Q-2}$, $0 < \alpha_j < 2$, $1 \leq j \leq l$. Thus we have the following result.

Theorem 1.4. Let Ω is a δ_γ -starshaped domain with respect to the origin in \mathbb{G} , and $a_k, b_j \in \Omega$, $1 \leq k \leq m$, $1 \leq j \leq l$, $m, l \in \mathbb{N}$. Assume that $K \in C^1(\bar{\Omega})$, $ZK(z) \leq 0$, $Zd(z, a_k) \leq 0$ and $K_j(z)Zd(z, b_j) \geq 0$ for a.e. $z \in \Omega$. Then problem (1.6) does not possess nontrivial nonnegative solution $u \in S_0^1(\Omega) \cap L^q(\Omega)$ such that $\frac{Zu}{d(z)} \in L^2(\Omega)$ for any $\mu_k > 0$ ($k \in \{1, 2, \dots, m\}$).

A short overview of the article is in order. In Section 2 we prove a Pohozaev-type identity of our sub-elliptic singular problem and we deduce a nonexistence result on bound δ_γ -starshaped domain for $\mu > 0$ and $q \geq 2^*$.

2 Proof of the main results

For $(z, u) \in \Omega \times \mathbb{R}$, set

$$f(z, u) = \lambda \frac{\psi^2 u}{d(z)^2} - \mu \frac{\psi^2 u}{d(z, a)^2} + K(z) \frac{\psi^\alpha |u|^{2_\alpha^* - 2} u}{d(z)^\alpha} + |u|^{q-2} u.$$

Then

$$F(z, u) = \int_0^u f(z, t) dt = \frac{\lambda}{2} \frac{\psi^2 |u|^2}{d(z)^2} - \frac{\mu}{2} \frac{\psi^2 |u|^2}{d(z, a)^2} + \frac{1}{2_\alpha^*} K(z) \frac{\psi^\alpha |u|^{2_\alpha^*}}{d(z)^\alpha} + \frac{1}{q} |u|^q$$

and

$$ZF(z, u) = \langle Z, \nabla_z F(z, u) \rangle + f(z, u)Zu, \tag{2.1}$$

where

$$\begin{aligned} \langle Z, \nabla_z F(z, u) \rangle &= \lambda \psi Z\psi \frac{|u|^2}{d(z)^2} - \lambda \frac{\psi^2 |u|^2}{d(z)^3} Zd(z) - \mu \psi Z\psi \frac{|u|^2}{d(z, a)^2} + \mu \frac{\psi^2 |u|^2}{d(z, a)^3} Zd(z, a) \\ &+ \frac{1}{2_\alpha^*} ZK(z) \frac{\psi^\alpha |u|^{2_\alpha^*}}{d(z)^\alpha} + \frac{1}{2_\alpha^*} K(z) \left(\frac{\alpha \psi^{\alpha-1} Z\psi |u|^{2_\alpha^*}}{d(z)^\alpha} - \alpha \frac{\psi^\alpha |u|^{2_\alpha^*}}{d(z)^{\alpha+1}} Zd(z) \right) \\ &= -\lambda \frac{\psi^2 |u|^2}{d(z)^2} + \mu \frac{\psi^2 |u|^2}{d(z, a)^3} Zd(z, a) + \frac{1}{2_\alpha^*} ZK(z) \frac{\psi^\alpha |u|^{2_\alpha^*}}{d(z)^\alpha} - \frac{\alpha}{2_\alpha^*} K(z) \frac{\psi^\alpha |u|^{2_\alpha^*}}{d(z)^\alpha}, \end{aligned} \tag{2.2}$$

where we have used that $Z\psi = 0$ and $Zd(z) = d(z)$.

Proof of Theorem 1.2. Due to the lack of regularity of solution at 0 and $a \in \Omega$, we begin by considering approximating domains $\Omega_{\rho_n} = \Omega \setminus (B_d(0, \rho_n) \cup B_d(a, \rho_n))$, where $B_d(z, \rho)$ denotes the d-ball with center at z and radius ρ , here $\rho_n \rightarrow 0$ as $n \rightarrow \infty$. Clearly, $\Omega_{\rho_n} \rightarrow \Omega$ as $n \rightarrow \infty$. Multiplying equation (1.1) by Zu and integrating over Ω_{ρ_n} , we get

$$-\int_{\Omega_{\rho_n}} \Delta_{\mathbb{G}}u Zu dz = \int_{\Omega_{\rho_n}} f(z, u)Zu dz. \tag{2.3}$$

For the left hand side of (2.3), the following Rellich-type identity holds for u on Ω_{ρ_n} (see [6]):

$$-\int_{\Omega_{\rho_n}} \Delta_{\mathbb{G}}u Zu dz = \frac{2-Q}{2} \int_{\Omega_{\rho_n}} |\nabla_{\mathbb{G}}u|^2 dz + \frac{1}{2} \int_{\partial\Omega_{\rho_n}} |\nabla_{\mathbb{G}}u|^2 \langle Z, \nu \rangle d\sigma - \int_{\partial\Omega_{\rho_n}} \langle \nabla_{\mathbb{G}}u, \nu_{\mathbb{G}} \rangle Zu d\sigma, \tag{2.4}$$

where $\nu_{\mathbb{G}} = (\nu_{\mathbb{G}}^1, \dots, \nu_{\mathbb{G}}^N)$ denotes the vector field with components $\nu_{\mathbb{G}}^i = \langle X_i, \nu \rangle$, here $\nu = (\nu_1, \dots, \nu_N)$ is the outward normal to $\partial\Omega$.

Concerning the right hand side of (2.3), by (2.1) and (2.2), it is easy to see that

$$\begin{aligned}
 \int_{\Omega_{\rho_n}} f(z, u)Zudz &= \int_{\Omega_{\rho_n}} Z(F(z, u))dz - \int_{\Omega_{\rho_n}} \langle Z, \nabla_z F(z, u) \rangle dz \\
 &= - \int_{\Omega_{\rho_n}} \operatorname{div}ZF(z, u)dz + \int_{\partial\Omega_{\rho_n}} F(z, u)\langle Z, \nu \rangle d\sigma - \int_{\Omega_{\rho_n}} \langle Z, \nabla_z F(z, u) \rangle dz \\
 &= -Q \int_{\Omega_{\rho_n}} \left[\frac{\lambda}{2} \frac{\psi^2|u|^2}{d(z)^2} - \frac{\mu}{2} \frac{\psi^2|u|^2}{d(z, a)^2} + \frac{1}{2_{\alpha}^*} K(z) \frac{\psi^{\alpha}|u|^{2_{\alpha}^*}}{d(z)^{\alpha}} + \frac{1}{q}|u|^q \right] dz \\
 &\quad + \lambda \int_{\Omega_{\rho_n}} \frac{\psi^2|u|^2}{d(z)^2} dz - \mu \int_{\Omega_{\rho_n}} \frac{\psi^2|u|^2}{d(z, a)^3} Z d(z, a) dz \\
 &\quad - \frac{1}{2_{\alpha}^*} \int_{\Omega_{\rho_n}} ZK(z) \frac{\psi^{\alpha}|u|^{2_{\alpha}^*}}{d(z)^{\alpha}} dz + \frac{\alpha}{2_{\alpha}^*} \int_{\Omega_{\rho_n}} K(z) \frac{\psi^{\alpha}|u|^{2_{\alpha}^*}}{d(z)^{\alpha}} dz + \int_{\partial\Omega_{\rho_n}} F(z, u)\langle Z, \nu \rangle d\sigma \\
 &= -\frac{Q-2}{2} \int_{\Omega_{\rho_n}} \lambda \frac{\psi^2|u|^2}{d(z)^2} dz + \frac{Q-2}{2} \int_{\Omega_{\rho_n}} \mu \frac{\psi^2|u|^2}{d(z, a)^2} dz - \mu \int_{\Omega_{\rho_n}} \frac{\psi^2|u|^2}{d(z, a)^3} Z d(z, a) dz \\
 &\quad - \frac{Q}{q} \int_{\Omega_{\rho_n}} |u|^q dz - \frac{Q-\alpha}{2_{\alpha}^*} \int_{\Omega_{\rho_n}} K(z) \frac{\psi^{\alpha}|u|^{2_{\alpha}^*}}{d(z)^{\alpha}} dz - \frac{1}{2_{\alpha}^*} \int_{\Omega_{\rho_n}} ZK(z) \frac{\psi^{\alpha}|u|^{2_{\alpha}^*}}{d(z)^{\alpha}} dz \\
 &\quad + \int_{\partial\Omega_{\rho_n}} F(z, u)\langle Z, \nu \rangle d\sigma. \tag{2.5}
 \end{aligned}$$

Then, inserting (2.4) and (2.5) into (2.3), we deduce that

$$\begin{aligned}
 \int_{\partial\Omega_{\rho_n}} \left[\frac{1}{2} |\nabla_{\mathbb{G}} u|^2 \langle Z, \nu \rangle - \langle \nabla_{\mathbb{G}} u, \nu_{\mathbb{G}} \rangle Zu - F(z, u)\langle Z, \nu \rangle \right] d\sigma &= \frac{Q-2}{2} \int_{\Omega_{\rho_n}} |\nabla_{\mathbb{G}} u|^2 dz - \frac{Q-2}{2} \int_{\Omega_{\rho_n}} \lambda \frac{\psi^2|u|^2}{d(z)^2} dz \\
 &\quad + \frac{Q-2}{2} \int_{\Omega_{\rho_n}} \mu \frac{\psi^2|u|^2}{d(z, a)^2} dz - \mu \int_{\Omega_{\rho_n}} \frac{\psi^2|u|^2}{d(z, a)^3} Z d(z, a) dz \\
 &\quad - \frac{Q}{q} \int_{\Omega_{\rho_n}} |u|^q dz - \frac{Q-\alpha}{2_{\alpha}^*} \int_{\Omega_{\rho_n}} K(z) \frac{\psi^{\alpha}|u|^{2_{\alpha}^*}}{d(z)^{\alpha}} dz \\
 &\quad - \frac{1}{2_{\alpha}^*} \int_{\Omega_{\rho_n}} ZK(z) \frac{\psi^{\alpha}|u|^{2_{\alpha}^*}}{d(z)^{\alpha}} dz. \tag{2.6}
 \end{aligned}$$

Now, we computing the integral over $\partial\Omega_{\rho_n} = \partial\Omega \cup \partial B_d(0, \rho_n) \cup \partial B_d(a, \rho_n)$ in right side of (2.6). First, on $\partial B_d(0, \rho_n)$, since $\nu = -\frac{\nabla d}{|\nabla d|}$, we have $\langle Z, \nu \rangle = -\frac{Zd}{|\nabla d|} = -\frac{d}{|\nabla d|}$ and $|\langle \nabla_{\mathbb{G}} u, \nu_{\mathbb{G}} \rangle| = |\langle \nabla_{\mathbb{G}} u, \frac{\nabla d}{|\nabla d|} \rangle| \leq \psi \frac{|\nabla_{\mathbb{G}} u|}{|\nabla d|} \leq c \frac{|\nabla_{\mathbb{G}} u|}{|\nabla d|}$, which and Federer’s coarea formula [2] imply that

$$\begin{aligned}
 \int_{\partial B_d(0, \rho_n)} |\nabla_{\mathbb{G}} u|^2 |\langle Z, \nu \rangle| d\sigma &= \rho_n \int_{\partial B_d(0, \rho_n)} |\nabla_{\mathbb{G}} u|^2 \frac{1}{|\nabla d|} d\sigma \\
 &= \int_0^{\rho_n} ds \int_{\partial B_d(0, \rho_n)} |\nabla_{\mathbb{G}} u|^2 \frac{1}{|\nabla d|} d\sigma \\
 &= \int_{B_d(0, \rho_n)} |\nabla_{\mathbb{G}} u|^2 dz \rightarrow 0 \text{ as } n \rightarrow \infty, \tag{2.7}
 \end{aligned}$$

$$\begin{aligned}
 & \left| \int_{\partial B_d(0, \rho_n)} \langle \nabla_{\mathbb{G}} u, \nu_{\mathbb{G}} \rangle Z u d\sigma \right| \leq c \left(\int_{\partial B_d(0, \rho_n)} \frac{|\nabla_{\mathbb{G}} u| |Z u|}{|\nabla d|} d\sigma \right) \\
 & \leq c \rho_n \left(\int_{\partial B_d(0, \rho_n)} \frac{|\nabla_{\mathbb{G}} u|^2}{|\nabla d|} d\sigma \right)^{\frac{1}{2}} \left(\int_{\partial B_d(0, \rho_n)} \frac{|Z u|^2}{d(z)^2 |\nabla d|} d\sigma \right)^{\frac{1}{2}} \\
 & = c \left(\rho_n \int_{\partial B_d(0, \rho_n)} \frac{|\nabla_{\mathbb{G}} u|^2}{|\nabla d|} d\sigma \right)^{\frac{1}{2}} \left(\rho_n \int_{\partial B_d(0, \rho_n)} \frac{|Z u|^2}{d(z)^2 |\nabla d|} d\sigma \right)^{\frac{1}{2}} \tag{2.8} \\
 & = c \left(\int_0^{\rho_n} ds \int_{\partial B_d(0, \rho_n)} \frac{|\nabla_{\mathbb{G}} u|^2}{|\nabla d|} d\sigma \right)^{\frac{1}{2}} \left(\int_0^{\rho_n} ds \int_{\partial B_d(0, \rho_n)} \frac{|Z u|^2}{d(z)^2 |\nabla d|} d\sigma \right)^{\frac{1}{2}} \\
 & = c \left(\int_{B_d(0, \rho_n)} |\nabla_{\mathbb{G}} u|^2 dz \right)^{\frac{1}{2}} \left(\int_{B_d(0, \rho_n)} \left(\frac{Z u}{d(z)} \right)^2 dz \right)^{\frac{1}{2}} \rightarrow 0 \text{ as } n \rightarrow \infty,
 \end{aligned}$$

and

$$\begin{aligned}
 \left| \int_{\partial B_d(0, \rho_n)} F(z, u) \langle Z, \nu \rangle d\sigma \right| &= \rho_n \int_{\partial B_d(0, \rho_n)} F(z, u) \frac{1}{|\nabla d|} d\sigma \\
 &= \int_0^{\rho_n} ds \int_{\partial B_d(0, \rho_n)} F(z, u) \frac{1}{|\nabla d|} d\sigma \tag{2.9} \\
 &= \int_{B_d(0, \rho_n)} F(z, u) dz \rightarrow 0 \text{ as } n \rightarrow \infty.
 \end{aligned}$$

Similarly, on $\partial B_d(a, \rho_n)$, we have

$$\int_{\partial B_d(a, \rho_n)} |\nabla_{\mathbb{G}} u|^2 |\langle Z, \nu \rangle| d\sigma \rightarrow 0 \text{ as } n \rightarrow \infty, \tag{2.10}$$

$$\left| \int_{\partial B_d(a, \rho_n)} \langle \nabla_{\mathbb{G}} u, \nu_{\mathbb{G}} \rangle Z u d\sigma \right| \rightarrow 0 \text{ as } n \rightarrow \infty, \tag{2.11}$$

and

$$\left| \int_{\partial B_d(a, \rho_n)} F(z, u) \langle Z, \nu \rangle d\sigma \right| \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{2.12}$$

On $\partial\Omega$, it is easy to get that

$$\int_{\partial\Omega} \langle \nabla_{\mathbb{G}} u, \nu_{\mathbb{G}} \rangle Z u d\sigma = \int_{\partial\Omega} |\nabla_{\mathbb{G}} u|^2 \langle Z, \nu \rangle d\sigma. \tag{2.13}$$

Moreover, by using the assumption $u = 0$ on $\partial\Omega$, we get

$$\int_{\partial\Omega} F(z, u) \langle Z, \nu \rangle d\sigma = 0. \tag{2.14}$$

Then, from (2.7)- (2.14) it follows that

$$\begin{aligned}
 & \int_{\partial\Omega, \rho_n} \left[\frac{1}{2} |\nabla_{\mathbb{G}} u|^2 \langle Z, \nu \rangle - \langle \nabla_{\mathbb{G}} u, \nu_{\mathbb{G}} \rangle Z u - F(z, u) \langle Z, \nu \rangle \right] d\sigma \\
 &= \int_{\partial\Omega \cup \partial B_d(0, \rho_n) \cup \partial B_d(a, \rho_n)} \left[\frac{1}{2} |\nabla_{\mathbb{G}} u|^2 \langle Z, \nu \rangle - \langle \nabla_{\mathbb{G}} u, \nu_{\mathbb{G}} \rangle Z u - F(z, u) \langle Z, \nu \rangle \right] d\sigma \\
 &\rightarrow \frac{1}{2} \int_{\partial\Omega} |\nabla_{\mathbb{G}} u|^2 \langle Z, \nu \rangle d\sigma - \int_{\partial\Omega} \langle \nabla_{\mathbb{G}} u, \nu_{\mathbb{G}} \rangle Z u d\sigma \\
 &= -\frac{1}{2} \int_{\partial\Omega} |\nabla_{\mathbb{G}} u|^2 \langle Z, \nu \rangle d\sigma \text{ as } n \rightarrow \infty.
 \end{aligned}$$

Now, by the integrability of the functions $\frac{\psi^2|u|^2}{d(z)^2}$, $\frac{\psi^2|u|^2}{d(z,a)^2}$, $|u|^q$ and $\frac{\psi^\alpha|u|^{2_\alpha^*}}{d(z)^\alpha}$, and letting $n \rightarrow \infty$ in (2.6), we get the following identity on the whole Ω :

$$\begin{aligned}
 -\frac{1}{2} \int_{\partial\Omega} |\nabla_{\mathbb{G}}u|^2 \langle Z, \nu \rangle d\sigma &= \frac{Q-2}{2} \int_{\Omega} |\nabla_{\mathbb{G}}u|^2 dz - \frac{Q-2}{2} \int_{\Omega} \lambda \frac{\psi^2|u|^2}{d(z)^2} dz \\
 &+ \frac{Q-2}{2} \int_{\Omega} \mu \frac{\psi^2|u|^2}{d(z,a)^2} dz - \mu \int_{\Omega} \frac{\psi^2|u|^2}{d(z,a)^3} Z d(z,a) dz - \frac{Q}{q} \int_{\Omega} |u|^q dz \\
 &- \frac{Q-\alpha}{2_\alpha^*} \int_{\Omega} K(z) \frac{\psi^\alpha|u|^{2_\alpha^*}}{d(z)^\alpha} dz - \frac{1}{2_\alpha^*} \int_{\Omega} ZK(z) \frac{\psi^\alpha|u|^{2_\alpha^*}}{d(z)^\alpha} dz.
 \end{aligned}
 \tag{2.15}$$

On the other hand, since $u \in S_0^1(\Omega) \cap L^q(\Omega)$ is a solution of (1.1), one has

$$\int_{\Omega} |\nabla_{\mathbb{G}}u|^2 dz = \int_{\Omega} f(z,u)u dz = \int_{\Omega} \left(\lambda \frac{\psi^2|u|^2}{d(z)^2} - \mu \frac{\psi^2|u|^2}{d(z,a)^2} + K(z) \frac{\psi^\alpha|u|^{2_\alpha^*}}{d(z)^\alpha} + |u|^q \right) dz.
 \tag{2.16}$$

Then, it follows from (2.15), (2.16) and $\frac{Q-2}{2} = \frac{Q-\alpha}{2_\alpha^*}$ that

$$\begin{aligned}
 \frac{1}{2} \int_{\partial\Omega} |\nabla_{\mathbb{G}}u|^2 \langle Z, \nu \rangle d\sigma &= \mu \int_{\Omega} \frac{\psi^2|u|^2}{d(z,a)^3} Z d(z,a) dz \\
 &+ \left(\frac{Q}{q} - \frac{Q-2}{2} \right) \int_{\Omega} |u|^q dx + \frac{1}{2_\alpha^*} \int_{\Omega} ZK(z) \frac{\psi^\alpha|u|^{2_\alpha^*}}{d(z)^\alpha} dz.
 \end{aligned}
 \tag{2.17}$$

Since Ω is δ_γ -starshaped domain at 0, we have $\langle Z, \nu \rangle > 0$ for all $z \in \partial\Omega$. Again $q \geq 2^*$, thus $\frac{Q}{q} - \frac{Q-2}{2} \leq \frac{Q}{2^*} - \frac{Q-2}{2} = 0$. So, from (2.17) we conclude that $u \equiv 0$ for all $\mu > 0$. By sub-elliptic unique continuation argument (see [6, Cor. 10.7]), we have that u to be nonnegative, thus $u \equiv 0$ in Ω . That is the problem (1.1) does not possess nontrivial solution in $S_0^1(\Omega) \cap L^q(\Omega)$ for any $\mu > 0$.

Proof of Theorems 1.3 and 1.4. The proof of Theorems 1.3 and 1.4 can be obtained from Theorem 1.2, we omit it here.

3 Conclusion

Using Pohozaev’s methods, we obtain nonexistence results for critical sub-Laplacian problems on Carnot groups with Hardy-type potentials. These results generalize the corresponding results in Euclidean spaces.

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