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Nonexistence of sub-elliptic critical problems with Hardy-type potentials on Carnot group

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Abstract

Using the Pohozaev-type arguments, we prove the nonexistence results for sub-elliptic problems with critical Sobolev-Hardy exponents and Hardy-type potentials on the Carnot group.

Keywords: Nonexistence, Critical Sobolev-Hardy exponent, Pohozaev identity, Carnot group
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1 Introduction and the main results

In this paper, we are concerned with the following sub-elliptic problem:

$$\begin{cases} -\Delta_{\mathbb{G}} u - \gamma \dfrac{\psi^2 u}{\mathrm{d}(z)^2} + \mu \dfrac{\psi^2 u}{\mathrm{d}(z,a)^2} = K(z) \dfrac{\psi^\alpha |u|^{2^*_\alpha - 2} u}{\mathrm{d}(z)^\alpha} + |u|^{q-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $-\Delta_{\mathbb{G}}$ is the sub-Laplacian operator on Carnot group $\mathbb{G}$, $\Omega \subset \mathbb{G}$ is a bounded domain with smooth boundary, $0,\ a \in \Omega$, d is the natural gauge and the geometrical function ψ is define by $\psi = |\nabla_{\mathbb{G}} \mathrm{d}(z)|$, $K(z) \in C^1(\Omega)$, the parameters $\gamma \in (-\infty, \gamma_{\mathbb{G}})$, $\mu \in (0, +\infty)$ and $q \geq 2^*$.

We begin with some basic definitions and useful results for Carnot group, see [1, 3, 6, 13] for some details. A connected and simply connected Lie group $(\mathbb{G}, \cdot)$ is a Carnot group of step k if its Lie algebra $\mathfrak{g}$ admits a step k stratification. This means that there exist non-trivial linear subspaces $V_1, \cdots, V_k$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_k,$$

where $[V_1, V_i] = V_{i+1}$ for $i \in \{1, 2, \cdots, k-1\}$ and $[V_1, V_k] = \{0\}$. Let $m_i = \dim(V_i)$ for $i = 1, \cdots, k$, by means of the natural identification of $\mathbb{G}$ with its Lie algebra via the exponential map, it is not restrictive to suppose that $\mathbb{G}$ is a homogeneous Lie group on $\mathbb{R}^N := \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_k}$ equipped with a group-automorphisms (called dilations) $\delta_\gamma : \mathbb{G} \to \mathbb{G}$ of the form

$$\delta_\gamma(z) = (\gamma^1 z^{(1)}, \gamma^2 z^{(2)}, \cdots, \gamma^k z^{(k)}),$$

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where $z = (z^{(1)}, z^{(2)}, \cdots, z^{(k)}) \in \mathbb{G}$. On Carnot group $\mathbb{G}$, the topological dimension is defined by $N = \sum_{i=1}^{k} m_i$, and the homogeneous dimension is denoted that $Q = \sum_{i=1}^{k} i \cdot m_i$. Observe that $\mathbb{G} \cong \mathbb{R}^N$. In what follows, we assume that $Q \geq 3$.

Let X_1, X_2, $\cdots$,$\cdots$, X_m be the left invariant vector fields of V_1, the operator

$$\Delta_{\mathbb{G}} = \sum_{i=1}^{m} X_i X_i$$

is called a sub-Laplacian on $\mathbb{G}$. We shall denote by $\nabla_{\mathbb{G}} = (X_1, \cdots, X_m)$ the related horizontal gradient. Moreover, for any C^1 vector field $u = (u_1, u_2, \cdots, u_m)$, we shall indicate by $\mathrm{div}_{\mathbb{G}} u := \sum_{i=1}^{m} X_i u_i$, the divergence with respect to the vector fields X_i's. Finally, when $Q \geq 3$, the sub-Laplacian possess the following property: there exists a suitable homogeneous symmetry norm d on $\mathbb{G}$, which we shall refer to as the $\Delta_{\mathbb{G}}$-gauge, such that

$$\Gamma(z) = \frac{C}{\mathrm{d}(z)^{Q-2}}, \quad \forall z \in \mathbb{G}$$

is a fundamental solution of $-\Delta_{\mathbb{G}}$ with pole at 0, for a suitable constant $C > 0$. By definition, a homogeneous norm on $\mathbb{G}$ is a continuous function $\mathrm{d}(\cdot) : \mathbb{G} \to [0, \infty)$ such that: $\mathrm{d}(\delta_\gamma(z)) = \gamma \mathrm{d}(z)$ for every $\gamma > 0$ and every $z \in \mathbb{G}$, $\mathrm{d}(z) = 0$ if and only if $z = 0$. We say that the homogeneous norm $\mathrm{d}(\cdot)$ is symmetric if $\mathrm{d}(z^{-1}) = \mathrm{d}(z)$ for all $z \in \mathbb{G}$. If $\mathrm{d}(\cdot)$ is a homogeneous norm on $\mathbb{G}$, then $\mathrm{d}(z, y) = \mathrm{d}(y^{-1} \circ z)$ is a pseudo-distance on $\mathbb{G}$. Note that any two continuous homogeneous norms are equivalent, i.e. within constant multiplicative constant factors of each other. As customary, we will denote by $B_{\mathrm{d}}(z_0, \rho)$ the d-ball with center $z_0 \in \mathbb{G}$ and radius $\rho > 0$ given by

$$B_{\mathrm{d}}(z_0, \rho) = \{z \in \mathbb{G} : \mathrm{d}(z_0^{-1} \circ z) < \rho\}.$$

In this paper, we work in the Sobolev-Stein space $S_0^1(\Omega)$, defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{S_0^1(\Omega)} = (\int_\Omega |\nabla_{\mathbb{G}} u|^2 dz)^{\frac{1}{2}}$. On $S_0^1(\mathbb{G})$, the Hardy inequality is known as

$$\gamma_{\mathbb{G}} \int_{\mathbb{G}} \frac{\psi^2 |u|^2}{\mathrm{d}(z)^2} dz \leq \int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 dz, \quad \forall u \in C_0^\infty(\mathbb{G}), \tag{1.2}$$

where $\gamma_{\mathbb{G}} = (\frac{Q-2}{2})^2$ is the best constant in the above inequality. For $\alpha \in [0, 2)$, the following Hardy-Sobolev inequality

$$S_\alpha \Big(\int_{\mathbb{G}} \frac{\psi^\alpha |u|^{2_\alpha^*}}{\mathrm{d}(z)^\alpha} dz \Big)^{\frac{2}{2_\alpha^*}} \leq \int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 dz, \quad \forall u \in C_0^\infty(\mathbb{G}) \tag{1.3}$$

holds for some positive constant S_α. In here, $2_\alpha^* = \frac{2(Q-\alpha)}{Q-2}$ is called the critical exponent of the embedding $S_0^1(\mathbb{G}) \hookrightarrow L^{2_\alpha^*}(\mathbb{G}, \frac{\psi^\alpha}{\mathrm{d}(z)^\alpha} dz)$ and Q denotes the homogeneous dimension of the space $\mathbb{G}$ with respect to the dilation. Observe that $2^* = 2_0^* = \frac{2Q}{Q-2}$ is the critical Sobolev exponent.

Associated with problem (1.1), we consider the energy functional

$$I(u) = \frac{1}{2} \int_\Omega \Big(|\nabla_{\mathbb{G}} u|^2 - \gamma \frac{\psi^2 |u|^2}{\mathrm{d}(z)^2} + \mu \frac{\psi^2 |u|^2}{\mathrm{d}(z,a)^2} \Big) dz - \frac{1}{2_\alpha^*} \int_\Omega K(z) \frac{\psi^\alpha |u|^{2_\alpha^*}}{\mathrm{d}(z)^\alpha} dz - \frac{1}{q} \int_\Omega |u|^q dz.$$

From (1.2) and (1.3), we knows that I is well-defined and $I \in C^1(S_0^1(\Omega) \cap L^q(\Omega), \mathbb{R})$, and then its critical points correspond to solutions of (1.1).

Recently, the kind of sub-elliptic Dirichlet problem with singular potentials on Carnot group

$$-\Delta_{\mathbb{G}} u - \gamma \frac{\psi^2 u}{\mathrm{d}(z)^2} = f(z, u) \quad \text{in } \Omega \subset \mathbb{G}$$

have been widely studied. We refer to [1, 3, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] and the references therein. Garofalo et al [5, 6] establish the existence, regularity and nonexistence results for sub-elliptic problems on Heisenberg group. In [7], the author extends partially the existence and nonexistence results due to [4] again on Carnot group of the form

$$-\Delta_{\mathbb{G}} u - \mu \frac{\psi^2 u}{\mathrm{d}(z)^2} = \lambda u + |u|^{2^*-2} u \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega. \tag{1.4}$$

In this paper the author proved that: if $0 \le \mu \le \mu_{\mathbb{G}} - 1$, problem (1.4) has at least one positive solution $u \in S_0^1(\Omega)$ for any $0 < \lambda < \Lambda_1(\mu)$; if $\mu_{\mathbb{G}} - 1 < \mu < \mu_{\mathbb{G}}$, there exists $\lambda^* \in (0, \Lambda_1(\mu))$ such that problem (1.4) has at least one positive solution $u \in S_0^1(\Omega)$ for $\lambda^* < \lambda < \Lambda_1(\mu)$, where $\Lambda_1(\mu) > 0$ is the first eigenvalue of the operator $L_\mu := -\Delta_{\mathbb{G}} \cdot -\mu \frac{\psi^2 \cdot}{\mathrm{d}(z)^2}$. In particular, the author state some qualitative properties of ground state solutions to the following limit problem:

$$-\Delta_{\mathbb{G}} u - \mu \frac{\psi^2 u}{\mathrm{d}(z)^2} = |u|^{2^*-2} u, \quad u \in S^{1,2}(\mathbb{G}).$$

Here the Sobolev-Stein space $S^{1,2}(\mathbb{G})$ is the completion of $C_0^\infty(\mathbb{G})$ with respect to the norm $(\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 dz)^{\frac{1}{2}}$. Inspired by the above works, by using the variational methods and the mountain-pass theorem, the existence of positive solution to the critical sub-elliptic system is established in [16]. Moreover, by means of the Moser iteration method, some asymptotic properties of its nontrivial solution at the singular point are verified.

However, there are few results about the nonexistence of sub-elliptic equation with multiple critical Sobolev-Hardy terms and multiple singular points. To the best of our knowledge, the problem of nonexistence of sub-elliptic solutions with multiple Hardy-type terms and critical exponent has never been considered before on the Carnot group in a non-Euclidean setting. We would like to point out that since an important feature of the sub-Laplacian is its degenerate property, it turn out from several technical reasons that studying our degenerate equation (1.1) is not directly by using a classical Pohozaev identity. The purpose of this paper is to prove Theorem 1.2, for this we recall the definition of δ_γ-starshaped domains.

Definition 1.1. Let Ω be a C^1 connected open set, $0 \in \Omega$. We say that Ω is a δ_γ-starshaped domain with respect to the origin if

$$\langle Z, \nu \rangle(z) \ge 0, \quad \forall z \in \partial\Omega,$$

where Z is the infinitesimal generator of the dilations δ_γ and $\nu = (\nu_1, \nu_2, \cdots, \nu_N)$ is the outward normal to Ω.

In the above definition, the smooth vector field Z is the infinitesimal generator of the one-parameter group of non-isotopic dilations δ_γ, that is, the vector field such that

$$\left[\frac{d}{d\gamma} u(\delta_\gamma(z))\right]_{\gamma=1} = Zu.$$

For a generic Carnot group of step k, Z has the following expression

$$Z = \sum_{i=1}^{k} \sum_{j=1}^{m_i} i z_j^{(i)} \frac{\partial}{\partial z_j^{(i)}}.$$

We recall that Z is characterized by the property that a function $u : \mathbb{G} \to \mathbb{R}$ is homogeneous of degree k with respect to δ_γ, i.e. $u(\delta_\gamma(z)) = \gamma^k u(z)$ if and only if $Zu = ku$. Moreover, the following properties hold for Z (see [6]):

$$[X_i, Z] = X_i, \quad \forall i = 1, 2, \cdots, m, \qquad \mathrm{div} Z = Q.$$

Now we state our main results as follows.

Theorem 1.2. Let Ω is a δ_γ-starshaped domain with respect to the origin in $\mathbb{G}$. Assume that $K \in C^1(\overline{\Omega})$ and $ZK(z) \le 0$, $Z\mathrm{d}(z,a) \le 0$ for a.e. $z \in \Omega$. Then problem (1.1) does not possess nontrivial nonnegative solution $u \in S_0^1(\Omega) \cap L^q(\Omega)$ such that $\frac{Zu}{\mathrm{d}(z)} \in L^2(\Omega)$ for any $\mu > 0$.

The result in Theorem 1.2 can be easily generalized to the following sub-elliptic critical problem with multiple singular points:

$$\begin{cases} -\Delta_{\mathbb{G}} u - \gamma \dfrac{\psi^2 u}{\mathrm{d}(z)^2} + \displaystyle\sum_{k=1}^{m} \mu_k \dfrac{\psi^2 u}{\mathrm{d}(z, a_k)^2} = K(z) \dfrac{\psi^\alpha |u|^{2_\alpha^*-2} u}{\mathrm{d}(z)^\alpha} + |u|^{q-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $a_k \in \Omega$, $1 \le k \le m$ and $m \in \mathbb{N}$. Thus we have the following result.

Theorem 1.3. Let Ω is a δ_γ-starshaped domain with respect to the origin in $\mathbb{G}$, and assume that $K \in C^1(\overline{\Omega})$, $ZK(z) \leq 0$ and $Z\mathrm{d}(z, a_k) \leq 0$ ($\forall k \in \{1, 2, \cdots, m\}$) for a.e. $z \in \Omega$. Then problem (1.5) does not possess nontrivial nonnegative solution $u \in S_0^1(\Omega) \cap L^q(\Omega)$ such that $\frac{Zu}{\mathrm{d}(z)} \in L^2(\Omega)$ for any $\mu_k > 0$

Furthermore, Theorem 1.3 can be generalized to the following sub-elliptic problem with multiple critical Sobolev-Hardy terms and multiple singular points:

$$\begin{cases} -\Delta_{\mathbb{G}} u - \gamma \dfrac{\psi^2 u}{\mathrm{d}(z)^2} + \displaystyle\sum_{k=1}^{m} \mu_k \dfrac{\psi^2 u}{\mathrm{d}(z, a_k)^2} = \displaystyle\sum_{j=1}^{l} K_j(z) \dfrac{\psi^{\alpha_j} |u|^{2^*_{\alpha_j}-2} u}{\mathrm{d}(z, b_j)^{\alpha_j}} + |u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $a_k,\ b_j \in \Omega$, $m,\ l \in \mathbb{N}$ and $2^*_{\alpha_j} = \frac{2(Q-\alpha_j)}{Q-2}$, $0 < \alpha_j < 2$, $1 \leq j \leq l$. Thus we have the following result.

Theorem 1.4. Let Ω is a δ_γ-starshaped domain with respect to the origin in $\mathbb{G}$, and $a_k,\ b_j \in \Omega$, $1 \leq k \leq m$, $1 \leq j \leq l$, $m,\ l \in \mathbb{N}$. Assume that $K \in C^1(\overline{\Omega})$, $ZK(z) \leq 0$, $Z\mathrm{d}(z, a_k) \leq 0$ and $K_j(z)Z\mathrm{d}(z, b_j) \geq 0$ for a.e. $z \in \Omega$. Then problem (1.6) does not possess nontrivial nonnegative solution $u \in S_0^1(\Omega) \cap L^q(\Omega)$ such that $\frac{Zu}{\mathrm{d}(z)} \in L^2(\Omega)$ for any $\mu_k > 0$ ($k \in \{1, 2, \cdots, m\}$).

A short overview of the article is in order. In Section 2 we prove a Pohozaev-type identity of our sub-elliptic singular problem and we deduce a nonexistence result on bound δ_γ-starshaped domain for $\mu > 0$ and $q \geq 2^*$.

2 Proof of the main results

For $(z, u) \in \Omega \times \mathbb{R}$, set

$$f(z, u) = \lambda \frac{\psi^2 u}{\mathrm{d}(z)^2} - \mu \frac{\psi^2 u}{\mathrm{d}(z, a)^2} + K(z) \frac{\psi^\alpha |u|^{2^*_\alpha - 2} u}{\mathrm{d}(z)^\alpha} + |u|^{q-2}u.$$

Then

$$F(z, u) = \int_0^u f(z, t) dt = \frac{\lambda}{2} \frac{\psi^2 |u|^2}{\mathrm{d}(z)^2} - \frac{\mu}{2} \frac{\psi^2 |u|^2}{\mathrm{d}(z, a)^2} + \frac{1}{2^*_\alpha} K(z) \frac{\psi^\alpha |u|^{2^*_\alpha}}{\mathrm{d}(z)^\alpha} + \frac{1}{q} |u|^q$$

and

$$ZF(z, u) = \langle Z, \nabla_z F(z, u) \rangle + f(z, u) Zu, \tag{2.1}$$

where

$$\begin{aligned} \langle Z, \nabla_z F(z, u) \rangle &= \lambda \psi Z \psi \frac{|u|^2}{\mathrm{d}(z)^2} - \lambda \frac{\psi^2 |u|^2}{\mathrm{d}(z)^3} Z\mathrm{d}(z) - \mu \psi Z \psi \frac{|u|^2}{\mathrm{d}(z, a)^2} + \mu \frac{\psi^2 |u|^2}{\mathrm{d}(z, a)^3} Z\mathrm{d}(z, a) \\ &\quad + \frac{1}{2^*_\alpha} ZK(z) \frac{\psi^\alpha |u|^{2^*_\alpha}}{\mathrm{d}(z)^\alpha} + \frac{1}{2^*_\alpha} K(z) \Big(\frac{\alpha \psi^{\alpha-1} Z\psi |u|^{2^*_\alpha}}{\mathrm{d}(z)^\alpha} - \alpha \frac{\psi^\alpha |u|^{2^*_\alpha}}{\mathrm{d}(z)^{\alpha+1}} Z\mathrm{d}(z) \Big) \\ &= -\lambda \frac{\psi^2 |u|^2}{\mathrm{d}(z)^2} + \mu \frac{\psi^2 |u|^2}{\mathrm{d}(z, a)^3} Z\mathrm{d}(z, a) + \frac{1}{2^*_\alpha} ZK(z) \frac{\psi^\alpha |u|^{2^*_\alpha}}{\mathrm{d}(z)^\alpha} - \frac{\alpha}{2^*_\alpha} K(z) \frac{\psi^\alpha |u|^{2^*_\alpha}}{\mathrm{d}(z)^\alpha}, \end{aligned} \tag{2.2}$$

where we have used that $Z\psi = 0$ and $Z\mathrm{d}(z) = \mathrm{d}(z)$.

Proof of Theorem 1.2. Due to the lack os regularity of solution at 0 and $a \in \Omega$, we begin by considering approximating domains $\Omega_{\rho_n} = \Omega \backslash (B_{\mathrm{d}}(0, \rho_n) \cup B_{\mathrm{d}}(a, \rho_n))$, where $B_{\mathrm{d}}(z, \rho)$ denotes the d-ball with center at z and radius ρ, here $\rho_n \to 0$ as $n \to \infty$. Clearly, $\Omega_{\rho_n} \to \Omega$ as $n \to \infty$. Multiplying equation (1.1) by Zu and integrating over Ω_{ρ_n}, we get

$$-\int_{\Omega_{\rho_n}} \Delta_{\mathbb{G}} u Zu \, dz = \int_{\Omega_{\rho_n}} f(z, u) Zu \, dz. \tag{2.3}$$

For the left hand side of (2.3), the following Rellich-type identity holds for u on Ω_{ρ_n} (see [6]):

$$-\int_{\Omega_{\rho_n}} \Delta_{\mathbb{G}} u Zu \, dz = \frac{2-Q}{2} \int_{\Omega_{\rho_n}} |\nabla_{\mathbb{G}} u|^2 dz + \frac{1}{2} \int_{\partial\Omega_{\rho_n}} |\nabla_{\mathbb{G}} u|^2 \langle Z, \nu \rangle d\sigma - \int_{\partial\Omega_{\rho_n}} \langle \nabla_{\mathbb{G}} u, \nu_{\mathbb{G}} \rangle Zu d\sigma, \tag{2.4}$$

where $\nu_{\mathbb{G}}=(\nu_{\mathbb{G}}^1,\cdots,\nu_{\mathbb{G}}^N)$ denotes the vector field with components $\nu_{\mathbb{G}}^i=\langle X_i,\nu\rangle$, here $\nu=(\nu_1,\cdots,\nu_N)$ is the outward normal to $\partial\Omega$.

Concerning the right hand side of (2.3), by (2.1) and (2.2), it is easy to see that

$$\begin{aligned}\int_{\Omega_{\rho_n}} f(z,u)Zudz&=\int_{\Omega_{\rho_n}} Z(F(z,u))dz-\int_{\Omega_{\rho_n}}\langle Z,\nabla_z F(z,u)\rangle dz\\
&=-\int_{\Omega_{\rho_n}}\mathrm{div}ZF(z,u)dz+\int_{\partial\Omega_{\rho_n}}F(z,u)\langle Z,\nu\rangle d\sigma-\int_{\Omega_{\rho_n}}\langle Z,\nabla_z F(z,u)\rangle dz\\
&=-Q\int_{\Omega_{\rho_n}}\Big[\frac{\lambda}{2}\frac{\psi^2|u|^2}{\mathrm{d}(z)^2}-\frac{\mu}{2}\frac{\psi^2|u|^2}{\mathrm{d}(z,a)^2}+\frac{1}{2_\alpha^*}K(z)\frac{\psi^\alpha|u|^{2_\alpha^*}}{\mathrm{d}(z)^\alpha}+\frac{1}{q}|u|^q\Big]dz\\
&\quad+\lambda\int_{\Omega_{\rho_n}}\frac{\psi^2|u|^2}{\mathrm{d}(z)^2}dz-\mu\int_{\Omega_{\rho_n}}\frac{\psi^2|u|^2}{\mathrm{d}(z,a)^3}Z\mathrm{d}(z,a)dz\\
&\quad-\frac{1}{2_\alpha^*}\int_{\Omega_{\rho_n}}ZK(z)\frac{\psi^\alpha|u|^{2_\alpha^*}}{\mathrm{d}(z)^\alpha}dz+\frac{\alpha}{2_\alpha^*}\int_{\Omega_{\rho_n}}K(z)\frac{\psi^\alpha|u|^{2_\alpha^*}}{\mathrm{d}(z)^\alpha}dz+\int_{\partial\Omega_{\rho_n}}F(z,u)\langle Z,\nu\rangle d\sigma\\
&=-\frac{Q-2}{2}\int_{\Omega_{\rho_n}}\lambda\frac{\psi^2|u|^2}{\mathrm{d}(z)^2}dz+\frac{Q-2}{2}\int_{\Omega_{\rho_n}}\mu\frac{\psi^2|u|^2}{\mathrm{d}(z,a)^2}dz-\mu\int_{\Omega_{\rho_n}}\frac{\psi^2|u|^2}{\mathrm{d}(z,a)^3}Z\mathrm{d}(z,a)dz\\
&\quad-\frac{Q}{q}\int_{\Omega_{\rho_n}}|u|^q dz-\frac{Q-\alpha}{2_\alpha^*}\int_{\Omega_{\rho_n}}K(z)\frac{\psi^\alpha|u|^{2_\alpha^*}}{\mathrm{d}(z)^\alpha}dz-\frac{1}{2_\alpha^*}\int_{\Omega_{\rho_n}}ZK(z)\frac{\psi^\alpha|u|^{2_\alpha^*}}{\mathrm{d}(z)^\alpha}dz\\
&\quad+\int_{\partial\Omega_{\rho_n}}F(z,u)\langle Z,\nu\rangle d\sigma.\end{aligned}\tag{2.5}$$

Then, inserting (2.4) and (2.5) into (2.3), we deduce that

$$\begin{aligned}\int_{\partial\Omega_{\rho_n}}\Big[\frac{1}{2}|\nabla_{\mathbb{G}}u|^2\langle Z,\nu\rangle-\langle\nabla_{\mathbb{G}}u,\nu_{\mathbb{G}}\rangle Zu-F(z,u)\langle Z,\nu\rangle\Big]d\sigma&=\frac{Q-2}{2}\int_{\Omega_{\rho_n}}|\nabla_{\mathbb{G}}u|^2dz-\frac{Q-2}{2}\int_{\Omega_{\rho_n}}\lambda\frac{\psi^2|u|^2}{\mathrm{d}(z)^2}dz\\
&\quad+\frac{Q-2}{2}\int_{\Omega_{\rho_n}}\mu\frac{\psi^2|u|^2}{\mathrm{d}(z,a)^2}dz-\mu\int_{\Omega_{\rho_n}}\frac{\psi^2|u|^2}{\mathrm{d}(z,a)^3}Z\mathrm{d}(z,a)dz\\
&\quad-\frac{Q}{q}\int_{\Omega_{\rho_n}}|u|^q dz-\frac{Q-\alpha}{2_\alpha^*}\int_{\Omega_{\rho_n}}K(z)\frac{\psi^\alpha|u|^{2_\alpha^*}}{\mathrm{d}(z)^\alpha}dz\\
&\quad-\frac{1}{2_\alpha^*}\int_{\Omega_{\rho_n}}ZK(z)\frac{\psi^\alpha|u|^{2_\alpha^*}}{\mathrm{d}(z)^\alpha}dz.\end{aligned}\tag{2.6}$$

Now, we computing the integral over $\partial\Omega_{\rho_n}=\partial\Omega\cup\partial B_{\mathrm{d}}(0,\rho_n)\cup\partial B_{\mathrm{d}}(a,\rho_n)$ in right side of (2.6). First, on $\partial B_{\mathrm{d}}(0,\rho_n)$, since $\nu=-\frac{\nabla\mathrm{d}}{|\nabla\mathrm{d}|}$, we have $\langle Z,\nu\rangle=-\frac{Z\mathrm{d}}{|\nabla\mathrm{d}|}=-\frac{\mathrm{d}}{|\nabla\mathrm{d}|}$ and $|\langle\nabla_{\mathbb{G}}u,\nu_{\mathbb{G}}\rangle|=|\langle\nabla_{\mathbb{G}}u,\frac{\nabla_{\mathbb{G}}\mathrm{d}}{|\nabla\mathrm{d}|}\rangle|\le\psi|\frac{\nabla_{\mathbb{G}}u}{|\nabla\mathrm{d}|}|\le c|\frac{\nabla_{\mathbb{G}}u}{|\nabla\mathrm{d}|}|$, which and Federer's coarea formula [2] imply that

$$\begin{aligned}\int_{\partial B_{\mathrm{d}}(0,\rho_n)}|\nabla_{\mathbb{G}}u|^2|\langle Z,\nu\rangle|d\sigma&=\rho_n\int_{\partial B_{\mathrm{d}}(0,\rho_n)}|\nabla_{\mathbb{G}}u|^2\frac{1}{|\nabla\mathrm{d}|}d\sigma\\
&=\int_0^{\rho_n}ds\int_{\partial B_{\mathrm{d}}(0,\rho_n)}|\nabla_{\mathbb{G}}u|^2\frac{1}{|\nabla\mathrm{d}|}d\sigma\\
&=\int_{B_{\mathrm{d}}(0,\rho_n)}|\nabla_{\mathbb{G}}u|^2dz\to 0\ \text{ as } n\to\infty,\end{aligned}\tag{2.7}$$

$$
\begin{aligned}
&\left|\int_{\partial B_{\mathrm{d}}(0,\rho_n)}\langle\nabla_{\mathbb{G}}u,\nu_{\mathbb{G}}\rangle Zud\sigma\right|\le c\left(\int_{\partial B_{\mathrm{d}}(0,\rho_n)}\frac{|\nabla_{\mathbb{G}}u||Zu|}{|\nabla \mathrm{d}|}d\sigma\right)\\
&\le c\rho_n\left(\int_{\partial B_{\mathrm{d}}(0,\rho_n)}\frac{|\nabla_{\mathbb{G}}u|^2}{|\nabla \mathrm{d}|}d\sigma\right)^{\frac{1}{2}}\left(\int_{\partial B_{\mathrm{d}}(0,\rho_n)}\frac{|Zu|^2}{\mathrm{d}(z)^2|\nabla \mathrm{d}|}d\sigma\right)^{\frac{1}{2}}\\
&=c\left(\rho_n\int_{\partial B_{\mathrm{d}}(0,\rho_n)}\frac{|\nabla_{\mathbb{G}}u|^2}{|\nabla \mathrm{d}|}d\sigma\right)^{\frac{1}{2}}\left(\rho_n\int_{\partial B_{\mathrm{d}}(0,\rho_n)}\frac{|Zu|^2}{\mathrm{d}(z)^2|\nabla \mathrm{d}|}d\sigma\right)^{\frac{1}{2}}\\
&=c\left(\int_0^{\rho_n}ds\int_{\partial B_{\mathrm{d}}(0,\rho_n)}\frac{|\nabla_{\mathbb{G}}u|^2}{|\nabla \mathrm{d}|}d\sigma\right)^{\frac{1}{2}}\left(\int_0^{\rho_n}ds\int_{\partial B_{\mathrm{d}}(0,\rho_n)}\frac{|Zu|^2}{\mathrm{d}(z)^2|\nabla \mathrm{d}|}d\sigma\right)^{\frac{1}{2}}\\
&=c\left(\int_{B_{\mathrm{d}}(0,\rho_n)}|\nabla_{\mathbb{G}}u|^2dz\right)^{\frac{1}{2}}\left(\int_{B_{\mathrm{d}}(0,\rho_n)}\left(\frac{Zu}{\mathrm{d}(z)}\right)^2dz\right)^{\frac{1}{2}}\to 0 \text{ as } n\to\infty,
\end{aligned}
\tag{2.8}
$$

and

$$
\begin{aligned}
\left|\int_{\partial B_{\mathrm{d}}(0,\rho_n)}F(z,u)\langle Z,\nu\rangle d\sigma\right|&=\rho_n\int_{\partial B_{\mathrm{d}}(0,\rho_n)}F(z,u)\frac{1}{|\nabla \mathrm{d}|}d\sigma\\
&=\int_0^{\rho_n}ds\int_{\partial B_{\mathrm{d}}(0,\rho_n)}F(z,u)\frac{1}{|\nabla \mathrm{d}|}d\sigma\\
&=\int_{B_{\mathrm{d}}(0,\rho_n)}F(z,u)dz\to 0 \text{ as } n\to\infty.
\end{aligned}
\tag{2.9}
$$

Similarly, on $\partial B_{\mathrm{d}}(a,\rho_n)$, we have

$$
\int_{\partial B_{\mathrm{d}}(a,\rho_n)}|\nabla_{\mathbb{G}}u|^2|\langle Z,\nu\rangle|d\sigma\to 0 \text{ as } n\to\infty,
\tag{2.10}
$$

$$
\left|\int_{\partial B_{\mathrm{d}}(a,\rho_n)}\langle\nabla_{\mathbb{G}}u,\nu_{\mathbb{G}}\rangle Zud\sigma\right|\to 0 \text{ as } n\to\infty,
\tag{2.11}
$$

and

$$
\left|\int_{\partial B_{\mathrm{d}}(a,\rho_n)}F(z,u)\langle Z,\nu\rangle d\sigma\right|\to 0 \text{ as } n\to\infty.
\tag{2.12}
$$

On $\partial\Omega$, it is easy to get that

$$
\int_{\partial\Omega}\langle\nabla_{\mathbb{G}}u,\nu_{\mathbb{G}}\rangle Zud\sigma=\int_{\partial\Omega}|\nabla_{\mathbb{G}}u|^2\langle Z,\nu\rangle d\sigma.
\tag{2.13}
$$

Moreover, by using the assumption $u=0$ on $\partial\Omega$, we get

$$
\int_{\partial\Omega}F(z,u)\langle Z,\nu\rangle d\sigma=0.
\tag{2.14}
$$

Then, from (2.7)- (2.14) it follows that

$$
\begin{aligned}
&\int_{\partial\Omega_{\rho_n}}\Big[\frac{1}{2}|\nabla_{\mathbb{G}}u|^2\langle Z,\nu\rangle-\langle\nabla_{\mathbb{G}}u,\nu_{\mathbb{G}}\rangle Zu-F(z,u)\langle Z,\nu\rangle\Big]d\sigma\\
&=\int_{\partial\Omega\cup\partial B_{\mathrm{d}}(0,\rho_n)\cup\partial B_{\mathrm{d}}(a,\rho_n)}\Big[\frac{1}{2}|\nabla_{\mathbb{G}}u|^2\langle Z,\nu\rangle-\langle\nabla_{\mathbb{G}}u,\nu_{\mathbb{G}}\rangle Zu-F(z,u)\langle Z,\nu\rangle\Big]d\sigma\\
&\to\frac{1}{2}\int_{\partial\Omega}|\nabla_{\mathbb{G}}u|^2\langle Z,\nu\rangle d\sigma-\int_{\partial\Omega}\langle\nabla_{\mathbb{G}}u,\nu_{\mathbb{G}}\rangle Zud\sigma\\
&=-\frac{1}{2}\int_{\partial\Omega}|\nabla_{\mathbb{G}}u|^2\langle Z,\nu\rangle d\sigma \quad\text{as } n\to\infty.
\end{aligned}
$$

Now, by the integrability of the functions $\frac{\psi^2|u|^2}{\mathrm{d}(z)^2}$, $\frac{\psi^2|u|^2}{\mathrm{d}(z,a)^2}$, $|u|^q$ and $\frac{\psi^\alpha|u|^{2^*_\alpha}}{\mathrm{d}(z)^\alpha}$, and letting $n\to\infty$ in (2.6), we get the following identity on the whole Ω:

$$\begin{aligned}-\frac{1}{2}\int_{\partial\Omega}|\nabla_{\mathbb{G}}u|^2\langle Z,\nu\rangle d\sigma=&\frac{Q-2}{2}\int_\Omega|\nabla_{\mathbb{G}}u|^2dz-\frac{Q-2}{2}\int_\Omega\lambda\frac{\psi^2|u|^2}{\mathrm{d}(z)^2}dz\\&+\frac{Q-2}{2}\int_\Omega\mu\frac{\psi^2|u|^2}{\mathrm{d}(z,a)^2}dz-\mu\int_\Omega\frac{\psi^2|u|^2}{\mathrm{d}(z,a)^3}Z\mathrm{d}(z,a)dz-\frac{Q}{q}\int_\Omega|u|^qdz\\&-\frac{Q-\alpha}{2^*_\alpha}\int_\Omega K(z)\frac{\psi^\alpha|u|^{2^*_\alpha}}{\mathrm{d}(z)^\alpha}dz-\frac{1}{2^*_\alpha}\int_\Omega ZK(z)\frac{\psi^\alpha|u|^{2^*_\alpha}}{\mathrm{d}(z)^\alpha}dz.\end{aligned}\tag{2.15}$$

On the other hand, since $u\in S^1_0(\Omega)\cap L^q(\Omega)$ is a solution of (1.1), one has

$$\int_\Omega|\nabla_{\mathbb{G}}u|^2dz=\int_\Omega f(z,u)udz=\int_\Omega\Big(\lambda\frac{\psi^2|u|^2}{\mathrm{d}(z)^2}-\mu\frac{\psi^2|u|^2}{\mathrm{d}(z,a)^2}+K(z)\frac{\psi^\alpha|u|^{2^*_\alpha}}{\mathrm{d}(z)^\alpha}+|u|^q\Big)dz.\tag{2.16}$$

Then, it follows from (2.15), (2.16) and $\frac{Q-2}{2}=\frac{Q-\alpha}{2^*_\alpha}$ that

$$\begin{aligned}\frac{1}{2}\int_{\partial\Omega}|\nabla_{\mathbb{G}}u|^2\langle Z,\nu\rangle d\sigma=&\mu\int_\Omega\frac{\psi^2|u|^2}{\mathrm{d}(z,a)^3}Z\mathrm{d}(z,a)dz\\&+\Big(\frac{Q}{q}-\frac{Q-2}{2}\Big)\int_\Omega|u|^qdx+\frac{1}{2^*_\alpha}\int_\Omega ZK(z)\frac{\psi^\alpha|u|^{2^*_\alpha}}{\mathrm{d}(z)^\alpha}dz.\end{aligned}\tag{2.17}$$

Since Ω is δ_γ-starshaped domain at 0, we have $\langle Z,\nu\rangle>0$ for all $z\in\partial\Omega$. Again $q\ge 2^*$, thus $\frac{Q}{q}-\frac{Q-2}{2}\le\frac{Q}{2^*}-\frac{Q-2}{2}=0$. So, from (2.17) we conclude that $u\equiv 0$ for all $\mu>0$. By sub-elliptic unique continuation argument (see [6, Cor. 10.7]), we have that u to be nonnegative, thus $u\equiv 0$ in Ω. That is the problem (1.1) does not possess nontrivial solution in $S^1_0(\Omega)\cap L^q(\Omega)$ for any $\mu>0$.

Proof of Theorems 1.3 and 1.4. The proof of Theorems 1.3 and 1.4 can be obtained from Theorem 1.2, we omit it here.

3 Conclusion

Using Pohozaev's methods, we obtain nonexistence results for critical sub-Laplacian problems on Carnot groups with Hardy-type potentials. These results generalize the corresponding results in Euclidean spaces.

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