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Existence of a solution for a strongly nonlinear elliptic perturbed problem in anisotropic Orlicz-Sobolev space

Hakima Ouyahya*, Rabab Elarabi

Équipe EDP et calcul scientifique, laboratoire de mathématiques et leurs interactions, Faculté des Sciences, Moulay Ismail University, Meknes, Morocco

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Abstract

This paper is devoted to studying the existence of a solution to the Dirichlet problem for a specific class of elliptical anisotropic equations of the type

$$\begin{cases} A(u) + g(x,u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

in the anisotropic Orlicz-Sobolev spaces, where A is a Leray-Lions operator $A(u) = \sum_{i=1}^{N} -\frac{\partial}{\partial x_i}(a_i(x, D^i u))$, the Carathéodory function $g(x,s)$ is a non-linear lower order term that verify some natural growth and sign conditions, where the data f is framed in anisotropic Orlicz-Sobolev spaces, and it is described by an Orlicz function that does not meet the Δ_2-condition. Within this framework, we prove the existence of a weak solution for our strongly nonlinear elliptic problem.

Keywords: Non-linear problem, anisotropic Orlicz-Sobolev spaces, Orlicz function, weak solution, perturbing term.
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1 Introduction

The purpose of this study is to investigate the existence of a weak solution to the nonlinear Dirichlet problem

$$\begin{cases} A(u) + g(x,u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N, N \geq 2$ is an integer and the operator $A(u) = \sum_{i=1}^{N} -\frac{\partial}{\partial x_i}(a_i(x, D^i u))$ is a Leray-Lions operator, $a_i(x,\zeta) : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function for all $i = 1, ..., N$, satisfying the non polynomial growth condition, the monotonicity and coercivity conditions in anisotropic Orlicz-Sobolev spaces $W^1 L_{\overrightarrow{M}}(\Omega)$ given

*Corresponding author
Email addresses: hakima.ouyahya@edu.umi.ac.ma (Hakima Ouyahya), rababelarabi162@gmail.com (Rabab Elarabi)

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by $W^1L_{\overrightarrow{M}}(\Omega) = \{u \in L_{M_0}/D^i u \in L_{M_i}, i = 1, 2..., N\}$, with $\overrightarrow{M}$ denoting a vector of N-Orlicz functions i.e, $\overrightarrow{M} = (M_1, ..., M_N)$, where M_i are N-Orlicz functions for all $i = 1, ..., N$ does not satisfy the Δ_2 condition. The perturbing term g is nonlinear lower order term that verify some natural growth and sign condition $g(x, \sigma)\sigma \geq 0$. The second term f belongs to $W^{-1}E_{\overrightarrow{M^*}}(\Omega)$.

The solvability of the problem (1.1) has been studied by many authors. For example, Browder demonstrated this result in classical Sobolev spaces in [15], and others have used this result in different frameworks ([3, 6, 16, 17]). In a different setting, Gossez et al. proved the existence result for problem (1.1) in Orlicz-Sobolev spaces in [20]. Sidi El Vally also proved the existence of a weak solution to this problem in Musielak Orlicz spaces in [28]. More recently, M. Bendahmane et al. proved the existence of this result in anisotropic Sobolev spaces using Hedberg-type approximation in [13].

The study of physical processes in anisotropic continuous medium has piqued the interest of researchers in the field of mathematical modeling. This has resulted in the formulation of equations that aim to describe these processes accurately. Recent research by Elarabi, Lahmi, and Ouyahya, as presented in their work ([18]), focuses on demonstrating the existence of a weak solution to a nonlinear Dirichlet problem in an anisotropic Orlicz-Sobolev space. This problem involves the equation $Au = f$ in the domain Ω. Further research in anisotropic Orlicz spaces can be found in the following works, as referenced in ([4, 5, 7, 8, 9, 10, 11, 12, 14, 24, 25, 26, 27]).

The objective of this study is to investigate the presence of a weak solution to problem (1.1) in anisotropic Orlicz-Sobolev spaces, without requiring the Δ_2 condition on the N-functions. By relaxing this assumption, we aim to expand our understanding of the existence of weak solutions and their behavior in more general function spaces.

This paper is structured as follows. Section 2 provides an introduction to the necessary preliminaries, notations, and functional spaces. Additionally, we discuss certain technical results that will be required in the subsequent sections. Section 3 covers the solvability of the main result.

2 Preliminaries

We can begin by recalling some definitions and properties from Orlicz spaces (as described in [2, 21, 23]). Additionally, we will present the anisotropic Orlicz-Sobolev spaces.

2.1 N-function.

Let $M : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be an N-function i.e. M is continuous, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \longrightarrow 0$ *as* $t \longrightarrow 0$ and $\frac{M(t)}{t} \longrightarrow +\infty$ *as* $t \longrightarrow +\infty$. Equivalently, M admits the representation $M(t) = \int_0^t m(x)dx$, where $m : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is non-decreasing and right-continuous function, with $m(t) > 0$ for $t > 0$, and $m(t) \longrightarrow +\infty$ as $t \longrightarrow +\infty$. The N-function M^* conjugate to M is defined by $M^*(t) = \int_0^t m^*(x)dx$, where $m^* : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is given by $m^*(t) = \sup\{s;\ m(s) \leq t\}$. The N-function M is said to satisfy the Δ_2-condition if, for some $k > 0$,

$$M(2t) \leq k\, M(t) \quad \text{for all } t \geq 0.$$

Moreover, we have the following Young's inequality

$$ts \leq M(t) + M^*(s) \ \text{ for all } t, s \geq 0.$$

Given two N-functions P and Q, we write $P \ll Q$ to indicate that P grows essentially less rapidly than Q i.e for each $\epsilon > 0$, $\frac{P(t)}{Q(\epsilon t)} \longrightarrow 0$ as $t \longrightarrow \infty$. This is the case if and only if

$$\lim_{t \longrightarrow \infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0.$$

Let Ω be an open subset of $\mathbb{R}^N$, $N \in \mathbb{N}$. The Orlicz class $\mathcal{L}_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that:

$$\int_\Omega M(u(x))\,\mathrm{d}x < +\infty \quad (\text{resp. } \int_\Omega M\Big(\frac{u(x)}{\lambda}\Big)\,\mathrm{d}x < +\infty \text{ for some } \lambda > 0).$$

Notice that $L_M(\Omega)$ is a Banach space under the so-called Luxemburg norm, namely

$$\|u\|_M = \inf\Big\{\lambda > 0 \,/\, \int_\Omega M\Big(\frac{u(x)}{\lambda}\Big)\,\mathrm{d}x \le 1\Big\}$$

and $\mathcal{L}_M(\Omega)$ is a convex subset of $L_M(\Omega)$. Indeed, $L_M(\Omega)$ is the linear hull of $\mathcal{L}_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\bar{\Omega}$ is denoted by $E_M(\Omega)$. The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the Δ_2-condition, for all t or for t large according to whether Ω has infinite measure or not. The dual of $E_M(\Omega)$ can be identified with $L_{M^*}(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)\,\mathrm{d}x$, and the dual norm on $L_{M^*}(\Omega)$ is equivalent to $\|.\|_{M^*}$. For two complementary N-functions M and M^*, if $u \in L_M(\Omega)$ and $v \in L_{M^*}(\Omega)$, we have the following Hölder's inequality

$$\int_\Omega |u(t)v(t)|dx \le 2 \parallel u \parallel_M \parallel v \parallel_{M^*} .$$

We say that u_n converges to u for the modular convergence in $L_M(\Omega)$ if, for some $\lambda > 0$,

$$\int_\Omega M\Big(\frac{u_n(x) - u(x)}{\lambda}\Big)dx \longrightarrow 0.$$

Lemma 2.1 (cf. [20]). Let $(u_n), u \in L_M(\Omega)$, $v_n, v \in L_{M^*}(\Omega)$ such that:

$$u_n \longrightarrow u \text{ in } L_M(\Omega) \quad \text{for the convergence modular}$$

and

$$v_n \longrightarrow v \text{ in } L_{M^*}(\Omega) \text{ for the convergence modular}$$

then

$$\int_\Omega u_n v_n \longrightarrow \int_\Omega uv \text{ as } n \longrightarrow \infty.$$

2.2 Anisotropic Orlicz-Sobolev space

Let Ω be an open subset of $\mathbb{R}^N$, M_i be N-functions for $i = 1, ..., N$, we denote $\overrightarrow{M} = (M_1, ..., M_N)$ and $D^i = \frac{\partial \cdot}{\partial x_i}$. The anisotropic Orlicz space $L_{\overrightarrow{M}}$(resp. $E_{\overrightarrow{M}}$) is defined by

$$L_{\overrightarrow{M}}(\Omega) = \prod_{i=1}^{N} L_{M_i}(\Omega) \text{ (resp. } E_{\overrightarrow{M}}(\Omega) = \prod_{i=1}^{N} E_{M_i}(\Omega)),$$

endowed with the following norm

$$\|u\| = \sum_{i=1}^{N} \|u_i\|_{M_i}, \tag{2.1}$$

where $u = (u_1, ..., u_N)$. In order to introduce the anisotropic Orlicz-Sobolev spaces it will be interesting to define the function M_0 given by

$$M_0(x) = \max_{i=1,...,N} M_i(x) \tag{2.2}$$

Remark 2.2. It is easy to check that:

- (i) The function M_0 is an N-function.
- (ii) The following embedding $L_{M_0}(\Omega) \hookrightarrow L_{M_i}(\Omega)$ is continuous for all i.

This remark allows us to define the anisotropic Orlicz-Sobolev spaces $W^1L_{\overrightarrow{M}}(\Omega)$ and $W^1E_{\overrightarrow{M}}(\Omega))$

$$W^1L_{\overrightarrow{M}}(\Omega) = \{u \in L_{M_0}(\Omega);\ D^iu \in L_{M_i}(\Omega),\ i = 1, 2, ..., N\}$$

and

$$W^1E_{\overrightarrow{M}}(\Omega) = \{u \in E_{M_0}(\Omega);\ D^iu \in E_{M_i}(\Omega),\ i = 1, 2, ..., N\},$$

who are Banach spaces under the norm

$$\|u\|_{1,\overrightarrow{M}} = \|u\|_{M_0} + \sum_{i=1}^{N} \|D^iu\|_{M_i}. \tag{2.3}$$

The space $W_0^1E_{\overrightarrow{M}}(\Omega)$ is defined as the (norm) closure of the space $\mathcal{D}(\Omega)$ in $W^1E_{\overrightarrow{M}}(\Omega)$ and the space $W_0^1L_{\overrightarrow{M}}(\Omega)$ as the $\sigma(L_{\overrightarrow{M}}(\Omega), E_{\overrightarrow{M^*}}(\Omega))$ closure of $\mathcal{D}(\Omega)$ in $W^1L_{\overrightarrow{M}}(\Omega)$. Let $\overrightarrow{M^*} = (M_1^*, ..., M_N^*)$. The dual of the spaces $W_0^1L_{\overrightarrow{M}}(\Omega)$ and $W_0^1E_{\overrightarrow{M}}(\Omega)$ are defined respectively by

$$W^{-1}L_{\overrightarrow{M^*}}(\Omega) = \{f \in \mathcal{D}'(\Omega); f = -\sum_{1\le i\le N} D^if_i \text{ with } f_i \in L_{M_i^*}(\Omega)\}$$

and

$$W^{-1}E_{\overrightarrow{M^*}}(\Omega) = \{f \in \mathcal{D}'(\Omega); f = -\sum_{1\le i\le N} D^if_i \text{ with } f_i \in E_{M_i^*}(\Omega)\}.$$

These spaces are equipped by their usual quotient norms. It is easy to see that $W_0^1E_{\overrightarrow{M}}(\Omega)$ and $W^{-1}E_{\overrightarrow{M^*}}(\Omega)$ are separable Banach spaces.

Proposition 2.3. [18] Let Ω be a bounded open subset of $\mathbb{R}^N$. Then there exists a constant $C = C(\Omega, N)$ such that for all $u \in W_0^1L_{\overrightarrow{M}}(\Omega)$

$$\int_\Omega \sum_{i=1}^{N} M_i(u(x))dx \le C \int_\Omega \sum_{i=1}^{N} M_i(CD^iu(x))dx \tag{2.4}$$

Lemma 2.4 ([18]). Suppose that Ω has the segment property. Then

- $\mathcal{D}(\Omega)$ is $\sigma(\prod_{i=1}^{N} L_{M_i}(\Omega), \prod_{i=1}^{N} E_{M_i^*}(\Omega))$ dense in $W_0^1L_{\overrightarrow{M}}(\Omega)$.
- $(W_0^1L_{\overrightarrow{M}}(\Omega), W_0^1E_{\overrightarrow{M}}(\Omega), W^{-1}L_{\overrightarrow{M^*}}(\Omega), W^{-1}E_{\overrightarrow{M^*}}(\Omega))$ constitutes a complementary system.

Let T the mapping from $D(T) \subset W_0^1L_{\overrightarrow{M}}(\Omega)$ into $W^{-1}L_{\overrightarrow{M^*}}(\Omega)$ defined by

$$<Tu, v> = \int_\Omega \sum_{i=1}^{N} a_i(x, D^iu)D^iv\, dx, \quad \forall v \in W_0^1L_{\overrightarrow{M}}(\Omega)$$

where $D(T) = \{u \in W_0^1L_{\overrightarrow{M}}(\Omega);\ Au \in L_{\overrightarrow{M^*}}(\Omega)\}$.which the corresponding mapping T in the following complementary system (Y, Y_0, Z, Z_0) where

$$(Y, Y_0, Z, Z_0) = (W_0^1L_{\overrightarrow{M}}(\Omega), W_0^1E_{\overrightarrow{M}}(\Omega), W^{-1}L_{\overrightarrow{M^*}}(\Omega), W^{-1}E_{\overrightarrow{M^*}}(\Omega))$$

satisfies the following conditions $i) - iv)$ (similar to the conditions of Proposition 1. and Proposition 5. in [19]) with respect to some $\overline{u} \in Y_0$ and $f \in Z_0$:

(i) (finite continuity) i.e. $Y_0 \subset D(T)$ and T is continuous from each finite-dimensional subspace of Y_0 into Z for $\sigma(Z, Y_0)$,

(ii) (Sequential pseudo-monotonicity) for any sequence $u_n \subset D(T)$ such that $u_n \longrightarrow u$ in Y for $\sigma(Y, Z_0), Tu_n \longrightarrow \chi \in Z$ for $\sigma(Z, Y_0)$ and $\limsup <u_n, Tu_n> \le <u, \chi>$ it follows that $u \in D(T), Tu = \chi$ and $<u_n, Tu_n> \longrightarrow <u, Tu>$,

(iii) Tu remains bounded in Z whenever $u \in D(T)$ remains bounded in Y and $<u, Tu>$ remains bounded from the above,

(iv) $<u, Tu - f> > 0$ when $u \in D(T)$ has sufficiently large norm in Y.

3 Essential assumptions and main result

Let Ω is an open and bounded set of $\mathbb{R}^N$ satisfying the segment property and the differential operator $A : W_0^1 L_{\overrightarrow{M}}(\Omega) \longrightarrow W^{-1} L_{\overrightarrow{M^*}}(\Omega)$ in divergence form

$$A(u) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}(a_i(x, D^i u)). \tag{3.1}$$

We make the following assumptions:

$A_1)$ $a_i(x,\zeta) : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ are a Carathéodory functions for all $i = 1, ..., N$ such that $a_i(x, 0) = 0$.

$A_2)$ There exist N-functions M_i $(i = 1, ..., N)$, a function $a_0 \in E_{M_i^*}$ and positive constant c such that for a.e. $x \in \Omega$, for all $\zeta \in \mathbb{R}$ and all $i = 1, 2, \ldots, N$, $|a_i(x,\zeta)| \leq a_0(x) + (M_i^*)^{-1} M_i(c|\zeta|)$.

$A_3)$ For a.e x in Ω and $\zeta, \zeta' \in \mathbb{R}$ with $\zeta \neq \zeta'$, $[a_i(x,\zeta) - a_i(x,\zeta')](\zeta - \zeta') > 0$.

$A_4)$ There exist functions $b_i \in E_{M_i^*}(\Omega)$, $\delta(x) \in L^1(\Omega)$ and positive constant d such that for some fixed element φ in $W_0^1 E_{\overrightarrow{M}}(\Omega)$, $(a_i(x,\zeta))(\zeta - D^i\varphi(x)) \geq d M_i(|\zeta|) - b_i(x)\zeta - \delta(x)$.

$B_1)$ $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the sign condition: $g(x, s)s \geq 0$.

$B_2)$ For each $r \geq 0$, there exist $h_r \in L^1(\Omega)$ such that

$$|g(x, s)| \leq h_r \ \text{ for all } s \in \mathbb{R}, \text{ for a.a } x \in \Omega \text{ and for all } s \in \mathbb{R} \text{ with } |s| \leq r. \tag{3.2}$$

Let $H \subset W_0^1 L_{\overrightarrow{M}}(\Omega)$ be a convex set, then:

$C_1)$ For each $u \in H \cap L^\infty(\Omega)$, there exists a sequence $u_n \in H \cap L^\infty(\Omega) \cap W_0^1 E_{\overrightarrow{M}}(\Omega)$ such that $u_n \to u$ for $\sigma(L_{M_i}(\Omega), L_{M_i^*}(\Omega))$, with $\|u_n\|_\infty$ bounded.

$C_2)$ For each $u \in H$, there exists a sequence $u_n \in H \cap L^\infty(\Omega)$ and a constant k such that $u_n \to u$ for $\sigma(L_{M_i}(\Omega), L_{M_i^*}(\Omega))$ and $|u_n(x)| \leq k|u(x)|$ for a.e $x \in \Omega$.

Lemma 3.1. (cf. [18]) Suppose that $A_1)$ and $A_2)$ hold true. Then the mapping

$$\begin{array}{rccc} \phi : & E_{\overrightarrow{M}}(\Omega) & \longrightarrow & L_{\overrightarrow{M^*}}(\Omega) \\ w = (w_j)_{1\leq j\leq N} & & \longmapsto & (a_i(x, D^i w_i)_{1\leq i\leq N} \end{array}$$

is finitely continuous from $\Pi_{i=1}^N E_{M_i}(\Omega)$ to the $\sigma(\Pi_{i=1}^N L_{M_i^*}(\Omega), \Pi_{i=1}^N E_{M_i}(\Omega))$ topology of $\Pi_{i=1}^N L_{M_i^*}(\Omega)$.

3.1 Existence results

Our main result is the following theorem.

Theorem 3.2. Let $f \in W^{-1} E_{\overrightarrow{M^*}}(\Omega)$. Assume that $A_1) - A_4)$ and $B_1) - B_2)$ are satisfied. Then there exists $u \in W_0^1 L_{\overrightarrow{M}}(\Omega)$ such that

$$\begin{cases} g(x, u) \in L^1(\Omega), \ g(x, u)u \in L^1(\Omega) \\ \\ < A(u), v > + \displaystyle\int_\Omega g(x, u) v dx =< A(u), v > \qquad \forall v \in W_0^1 L_{\overrightarrow{M}}(\Omega) \cap L^\infty(\Omega) \end{cases} \tag{3.3}$$

where

$$<Tu, v> = \int_\Omega \sum_{i=1}^{N} a_i(x, D^i u) D^i v \, dx, \quad \forall v \in W_0^1 L_{\overrightarrow{M}}(\Omega)$$

The proof of Theorem 3.2 is devided into several steps: we show first the existence of solutions to the approximate problem of (3.3) and a priori estimates, the convergence of approximate solution and then passing to the limit in the approximate problems will yeild the main result.

Step 1: Approximate problem

Let $g_n(x,u) = \begin{cases} g(x,u) & \text{if } |g(x,u)| \leq n \\ ksign(g(x,u)) & \text{if } |g(x,u)| > n \end{cases}$ and we define the operator $G_n : W_0^1 L_{\overrightarrow{M}}(\Omega) \longrightarrow \mathbb{R}$ by $G_n(v) = \int_\Omega g_n(x,u) v dx$.

Remark 3.3. It is easy to check that $G_n : W_0^1 L_{\overrightarrow{M}}(\Omega) \longrightarrow W^{-1} L_{\overrightarrow{M^*}}(\Omega)$ is well defined.

Now we show that the mapping $T + G_n : D(T) \subset W_0^1 L_{\overrightarrow{M}}(\Omega) \longrightarrow W^{-1} L_{\overrightarrow{M^*}}(\Omega)$ satisfies $i) - iv)$. The finite continuite of $T + G_n$ follows from Lemma 3.1. Let us now show the condition (ii). Let $u_k \to u$ weakly in $W_0^1 L_{\overrightarrow{M}}(\Omega)$ for $\sigma(W_0^1 L_{\overrightarrow{M}}(\Omega), W^{-1} E_{\overrightarrow{M^*}}(\Omega))$ such that $(T + G_n)u_k \to h$ weakly in $W^{-1} L_{\overrightarrow{M^*}}(\Omega)$ for $\sigma(W^{-1} L_{\overrightarrow{M^*}}(\Omega), W_0^1 E_{\overrightarrow{M}}(\Omega))$ and $\lim_{n\to+\infty} < (T + G_n)u_k, u_k - u > \leq 0$. Thus, there exists a subsequence still denoted by u_k such that ; $u_n \to u$ a.e in Ω. Since g_n is continous, $g_n(x,u_k) \to g_n(x,u)$ a.e in Ω. By using the dominated convergence Lebesgue theorem, we have $g_n(x,u_k) \to g_n(x,u)$ in $L_{M_i^*}$, since $u_k \rightharpoonup u$ weakly in L_{M_i}, $\forall i = 1, ..., N$. We obtain

$$\int_\Omega g_n(x,u_k)(u_k - u) dx \to 0.$$

Hence, we get $Tu_k \rightharpoonup h - G_n u$ weakly in $W^{-1} L_{\overrightarrow{M^*}}(\Omega)$. Since T is pseudo-monotone, we have $h = Tu + G_n u$. On the other hand, since G_n is bounded, it easy to show that $T + G_n$ is bounded. Now, we show that the mapping $T + G_n$ satisfies the condition (iv). Let $f \in W^{-1} E_{\overrightarrow{M^*}}(\Omega)$, such that $f = \sum_{i=1}^{N} D^i f_i$, $f_i \in E_{M_i^*}$. We have

$$< u, (T + G_n)u - f > \quad = \quad < u, Tu - f + G_n u > = < u, Tu - f > + < u, G_n u > .$$

By using the sign condition, we deduce our result. Finally, the operator $T + G_n$ satisfies (i)-(iv). Then, we can apply [18, theorem 3.2] to deduce the existence of a weak solution to the approximate problem. Thereforem there exists $u_n \in W_0^1 L_{\overrightarrow{M}}(\Omega)$ solution of the approximate problem

$$< Tu_n, u_n > + \int_\Omega g_n(x,u_n) u_n dx = < f, u_n > \quad \text{for all } v \in W_0^1 L_{\overrightarrow{M}}(\Omega). \tag{3.4}$$

Step 2: Apriori estimates

Taking u_n as a test function we deduce: $< Tu_n, u_n > + \int_\Omega g_n(x,u_n) u_n dx = < f, u_n >$, in virtue of (iii) and B_1), we deduce the following estimates:

$$\|u_n\|_{W_0^1 L_{\overrightarrow{M}}(\Omega)} \leq C_0 \tag{3.5}$$

$$\|Tu_n\|_{W^{-1} L_{\overrightarrow{M^*}}(\Omega)} \leq C_1 \tag{3.6}$$

$$\int_\Omega g_n(x,u_n) dx \leq C_2, \tag{3.7}$$

where C_i is a positive constant not depending on n.

Step 3: The convergence of approximate solution

In virtue of $(3.5), (3.6)$ and (3.7), we deduce that:

$$u_n \to u \text{ in } W_0^1 L_{\overrightarrow{M}}(\Omega) \ \text{ for } \sigma(W_0^1 L_{\overrightarrow{M}}(\Omega), W^{-1} E_{\overrightarrow{M^*}}(\Omega))$$

$$Tu_n \rightharpoonup \chi \text{ in } W^{-1} L_{\overrightarrow{M^*}}(\Omega) \ \text{ for } \sigma(W^{-1} L_{\overrightarrow{M^*}}(\Omega), W_0^1 E_{\overrightarrow{M}}(\Omega))$$

$$u_n \to u \text{ a.e in } \Omega.$$

and $g_n(x,u_n) \to g(x,u)$ a.e in Ω. Now, it remains to show that $g(x,u) \in L^1(\Omega)$. Let $\beta > 0$, we get

$$|g_n(x,u_n)| \leq \sup_{|t|\leq\beta} |g(x,t)| + \beta^{-1}|g_n(x,u_n)u_n| \leq h_r(x) + \delta^{-1}|g_n(x,u_n)u_n|$$

and let E is a measurable subset of Ω and $\epsilon > 0$, we get

$$\int_E |g_n(x,u_n)|dx \leq \int_E h_r(x)dx + \delta^{-1}\int_E g_n(x,u_n)u_n dx \leq \int_E h_r(x)dx + \delta^{-1}C_2.$$

where C_2 is the constant of (3.7). For $|E|$ is sufficiently small and $\delta = \dfrac{2C_2}{\epsilon}$, we get

$$\int_E |g_n(x,u_n)|dx \leq \int_E h_r(x)dx + \delta^{-1}C_2 \leq \epsilon.$$

Then, by using Vitali's theorem, we obtain $g_n(x,u_n) \to g(x,u)$ in $L^1(\Omega)$, then $g(x,u) \in L^1(\Omega)$. On the other hand, by using (3.7) and Fatou Lemma. We have

$$\int_\Omega \lim_{n\to+\infty} \inf g_n(x,u_n)u_n dx = \int_\Omega g(x,u)u dx \leq \lim_{n\to+\infty} \inf \int_\Omega g_n(x,u_n)u_n dx \leq C_2.$$

thus, $\int_\Omega g(x,u)udx \leq C_2$. Then, it follows that $g(x,u)u \in L^1(\Omega)$.

Step 4: Passing to the limit

By passing to limit in (3.3), we deduce

$$< \chi, v > + \int_\Omega g(x,u)v dx = < f, v >, \forall v \in W_0^1 L_{\overrightarrow{M}}(\Omega) \cap L^\infty(\Omega).$$

To complete the proof, it remains to show that $\chi = Tu$. Indeed, taking $v = u_n$ in (3.3) and by using Fatou's Lemma and (C_1) and (C_2) we deduce:

$$\lim_{n\to+\infty} \sup < Tu_n, u_n > \leq < f, v > - \int_\Omega g(x,u)u dx = < \chi, u > . \tag{3.8}$$

Hence, we conclude by (ii), $\chi = Tu$. Finally, $< Tu, v > + \int_\Omega g(x,u)v dx = < f, v >$.

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