Int. J. Nonlinear Anal. Appl. In Press, 1–12
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2024.29098.4058

Coincidence and common fixed point theorems for generalized $\alpha\mathcal{T}$-contraction via tri-simulation function with an application

Sahil Arora

Department of Mathematics, K.R.M.D.A.V. College, Nakodar-144040, Punjab, India

(Communicated by Oluwatosin Temitope Mewomo)

Abstract

In this manuscript, we introduce the concept of generalized $\alpha\mathcal{T}$-contractive pair of mappings with the assistance of a tri-simulation function and use this concept to establish some coincidence and common fixed point theorems via α-permissible mapping. We also give an illustrative example which yields the main result. Also, many existing results in the frame of metric spaces are established. We also apply our main theorem to derive coincidence and common fixed point results for $\alpha\mathcal{T}$-contractive mapping with the assistance of α-permissible function.

Keywords: Tri-simulation function, Coincident point, Common fixed point, Generalized $\alpha\mathcal{T}$-contraction, $\alpha-$permissible function
2020 MSC: Primary 47H10; Secondary 54H25

1 Introduction

The investigation of metric fixed point theory assumes a vital job in light of the fact that it discovers applications in numerous essential zones such as solution of differential equations, integral equations and so forth. The perception of standard metric space is a major weapon in functional analysis and topology. The Banach contraction rule is a one of the predominant outcomes in analysis and has continuously been at the front line of making and providing remarkable speculations for its researchers.

The crucial perception of this paper is the tri-simulation function which is defined by Gubran et al.[10] to associate some fixed point results. In 2012, Samet et al.[17] presented fixed point results for a new category of $(\alpha - \psi)$-contractive functions. In 2014, Popescu [15] introduced the perception of triangular $\alpha-$orbital admissible function and exhibited several fixed point results with the aid of generalized α-Geraghty contraction and triangular α-orbital admissible function. In 2015, Khojasteh et al. [14] introduced $\mathcal{Z}$-contraction with respect to a simulation function, which generalizes the Banach contraction rule in [3] by combining various types of non-linear contractions. From there on, Roldán et al.[16] and Argoubi et al.[1] modified the thought of simulation function and demonstrated some common fixed point theorems utilizing the newly larger class of simulation functions. In 2016, Karapinar [13] introduced the notion of α-admissible $\mathcal{Z}$-contraction with the aid of simulation function and established fixed point results with the assistance of triangular $\alpha-$orbital admissible mapping in the framework of complete metric space. In 2017, Gubran et al. [9] established common fixed point results for a pair of self-mappings via simulation function. In 2018, Aydi et al. [2] proved fixed point results for α-admissible $\mathcal{Z}$-contraction by using triangular $\alpha-$orbital admissible mapping

Email address: drprofsahilarora@gmail.com (Sahil Arora)

Received: November 2022 *Accepted:* March 2024

in the context of complete quasi metric space. Afterwards, Chandok et al.[7] exhibited some results via simulation map for Geraghty type contractive functions. In 2020, Bonab et al.[4] established tripled fixed point results with the assistance of vector-valued metrics in the frame of generalized metric spaces. In 2021, Gubran et al. [10] introduced a tri-simulation function involving three variables which is also designed to unify various contractions. In 2022, Hosseinzadeh et al.[12] proved n-tuple fixed point results on partially ordered cone metric spaces with the aassistance of α-series. Afterwards, many authors obtained several interesting results in different kind of metric spaces, for example, see ([5],[6],[8],[11],[18]).

Now, we recollect some elementary results which are used in the sequel.

2 Preliminaries

In 2012, Samet et al. [17] presented the thought of α-admissible function and (ψ, α)-contractive type mappings and established fixed point results for such mappings as pursues:

Definition 2.1. [17] Let $\mathcal{Q} : \mathcal{H} \to \mathcal{H}$ and $\alpha : \mathcal{H} \times \mathcal{H} \to [0, +\infty)$. Then, $\mathcal{Q}$ is named as $\alpha-$admissible if $\alpha(\Omega, \mho) \geq 1$, then $\alpha(\mathcal{Q}\Omega, \mathcal{Q}\mho) \geq 1$, for each Ω, $\mho \in \mathcal{H}$.

Definition 2.2. [17] Let Ψ be the class of maps $\psi : [0, \infty) \to [0, \infty)$ fulfils the accompanying properties:

(i)ψ is upper semi-continuous, strictly increasing;

(ii)$\{\psi^f(\ell)\}_{n \in \mathbb{Z}_+}$ tends to 0 as $f \to \infty$, for all $\ell > 0$;

(iii)$\psi(\ell) < \ell$, for every $\ell > 0$.

These functions are known as comparison functions.

Definition 2.3. [17] Let $\mathcal{Q} : \mathcal{H} \to \mathcal{H}$ be a given self-mapping in a metric space $(\mathcal{H}, d)$. Then, $\mathcal{Q}$ is termed as $(\psi, \alpha)-$ contractive type mapping, if there exist two functions $\psi \in \Psi$ and $\alpha : \mathcal{H} \times \mathcal{H} \to [0, +\infty)$ such that

$$\alpha(\Omega, \wp) d(\mathcal{Q}\Omega, \mathcal{Q}\wp) \leq \psi(d(\Omega, \wp)),$$

for all $\Omega, \wp \in \mathcal{H}$.

In 2014, Popescu [15] introduced the concept of triangular $\alpha-$orbital admissible as follows:

Definition 2.4. [15] Let $\mathcal{Q} : \mathcal{H} \to \mathcal{H}$ be a map and $\alpha : \mathcal{H} \times \mathcal{H} \to [0, \infty)$ be a function. We say that $\mathcal{Q}$ is triangular $\alpha-$orbital admissible if

(i) $\alpha(\Omega, \mathcal{Q}\Omega) \geq 1 \Rightarrow \alpha(\mathcal{Q}\Omega, \mathcal{Q}^2\Omega) \geq 1$;

(ii) $\alpha(\Omega, \mho) \geq 1$ and $\alpha(\mho, \mathcal{Q}\mho) \geq 1$, then $\alpha(\Omega, \mathcal{Q}\mho) \geq 1$.

Definition 2.5. [14] The mapping $\xi : [0, \infty) \times [0, \infty) \to \mathcal{R}$ is known as simulation function, if the following properties satisfy:

(ξ_1) $\xi(0, 0) = 0$;

(ξ_2) $\xi(e, f) < e - f$, for all $e, f > 0$;

(ξ_3) If $\{e_n\}, \{f_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty}\{e_n\} = \lim_{n\to\infty}\{f_n\} = \ell \in (0, \infty)$, then

$$\lim_{n\to\infty} \sup \xi(e_n, f_n) < 0.$$

In 2015, Argoubi et al. [1] observed that condition (ξ_1) can be relaxed and results can be proved without taking (ξ_1) into consideration.

Definition 2.6. [1] The mapping $\xi : [0, \infty) \times [0, \infty) \to \mathcal{R}$ is known as simulation function if it fulfils (ξ_2) and (ξ_3).

The family of all simulation functions $\xi : [0, \infty) \times [0, \infty) \to \mathcal{R}$ is denoted by $\mathcal{Z}$ in [1]. In 2015, Roldán et al. [16] observed that the third condition (namely: ξ_3) is symmetric in both arguments of ξ but, in proofs, this property is not necessary. In fact, in practice, the arguments of Λ have different meanings and they play different roles. Then, they slightly modify the condition ξ_3 as follows:

(ξ_3') If $\{q_n\}$ and $\{f_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty}\{q_n\} = \lim_{n\to\infty}\{f_n\} = \ell$ and $q_n < f_n$, for each $n \in \mathbb{Z}_+$, then

$$\lim_{n\to\infty} \sup \xi(q_n, f_n) < 0.$$

In 2016, Karapinar [13] utilized $\alpha-$admissible $\mathcal{Z}$-contraction to establish some fixed point results as follows:

Theorem 2.7. [13] Let S be α-admissible $\mathcal{Z}$-contraction with regard to Λ in a complete $(\mathcal{H}, d)$ and the accompanying conditions fulfil:

(i) there exists $x_0 \in \mathcal{H}$ so that $\alpha(x_0, Sx_0) \geq 1$;

(ii) S is triangular $\alpha-$orbital admissible;

(iv) S is continuous.

Then, there exists $\Omega \in \mathcal{H}$, so that $S\Omega = \Omega$.

In 2018, Aydi et al. [2] established fixed point results for triangular α-admissible contraction mapping via simulations functions in the class of quasi metric spaces as follows:

Theorem 2.8. [2] Let S be α-admissible $\mathcal{Z}$-contraction with regard to Λ in a complete quasi metric space and the accompanying conditions fulfil:

(i) there exists $x_0 \in \mathcal{H}$ so that $\alpha(x_0, Sx_0) \geq 1$ and $\alpha(Sx_0, x_0) \geq 1$;

(ii) S is triangular $\alpha-$admissible function;

(iv) S is continuous.

Then, there exists $\Omega \in \mathcal{H}$, so that $S\Omega = \Omega$.

In 2021, Gubran et al. [10] initiated the idea of tri-simulation function involving three variables as follows:

Definition 2.9. [10] The mapping $\Lambda : [0,\infty) \times [0,\infty) \times [0,\infty) \to \mathcal{R}$ is known as tri-simulation function, if the following properties hold:

(Λ_1) $\Lambda(z, y, x) < x - yz$, for all $x, y > 0, z \geq 0$;

(Λ_2) If $\{e_n\}, \{f_n\}, \{g_n\}$ are sequences in $(0,\infty)$ such that $f_n < g_n$, for all $n \in N$, $\lim_{n\to\infty} e_n \geq 1$ and $\lim_{n\to\infty}\{f_n\} = \lim_{n\to\infty}\{g_n\} = \ell \in (0,\infty)$, then

$$\lim_{n\to\infty} \sup \Lambda(e_n, f_n, g_n) < 0.$$

Let us denote the set of all tri-simulation functions by $\mathcal{T}$.

The authors in [10] utilized the class of auxiliary functions to define $\alpha\mathcal{T}$-contraction as follows.

Definition 2.10. [10] Let $\mathcal{Q}$ be a self-mapping in $(\mathcal{H}, d)$ and $\Lambda \in \mathcal{T}$. Then, $\mathcal{Q}$ is said to be $\alpha\mathcal{T}$-contraction with regard to Λ, if $\Lambda(\alpha(\mathcal{Q}\Omega, \mathcal{Q}\mho), d(\mathcal{Q}\Omega, \mathcal{Q}\mho), d(\Omega, \mho)) \geq 0$, for every $\Omega, \mho \in \mathcal{H}$, where $\alpha : \mathcal{H} \times \mathcal{H} \to \mathbb{R}_+$.

Let $\mathcal{Q}, \mathcal{W} : \mathcal{H} \to \mathcal{H}$ be two maps. We identify the set of coincidence and common fixed points of $\mathcal{Q}$ and $\mathcal{W}$ by $C(\mathcal{Q}, \mathcal{W})$ and $\mathcal{CF}(\mathcal{Q}, \mathcal{W})$, where $C(\mathcal{Q}, \mathcal{W}) = \{z \in \mathcal{H} : \mathcal{Q}z = \mathcal{W}z\}$ and $\mathcal{CF}(\mathcal{Q}, \mathcal{W}) = \{z \in \mathcal{H} : \mathcal{Q}z = \mathcal{W}z = z\}$.

Throughout the paper, d will stand for metric and $(\mathcal{H}, d)$ will denote metric space.

3 Main Results

In this section, we present the idea of generalized $\alpha\mathcal{T}$-contractive pair of mappings with the assistance of tri-simulation function and utilize this idea to set up outcomes of $C(\mathcal{Q}, \mathcal{W})$ and $\mathcal{CF}(\mathcal{Q}, \mathcal{W})$ in $(\mathcal{H}, d)$. We likewise give an example which yields the principle result. In the displayed work, we broaden the consequences of Gubran et al. [10]. Additionally, many existing outcomes in the casing of metric spaces are built up. We likewise apply our fundamental Theorem to determine coincidence and common fixed point results for $\alpha\mathcal{T}$-contractive pair of mappings. We prove our results by defining generalized $\alpha\mathcal{T}$-contractive pair of mappings with respect to Λ, which is a generalization of the approach of $\alpha\mathcal{T}$-contraction.

Definition 3.1. Let $(\mathcal{H}, d)$ be a metric space and $S, T : \mathcal{H} \to \mathcal{H}$ be given mappings. If there exist $\Lambda \in \mathcal{T}$, $\psi \in \Psi$ and $\alpha : \mathcal{H} \times \mathcal{H} \to [0, \infty)$ such that for each $x, y \in \mathcal{H}$, we have

$$\Lambda(\alpha(Tx, Ty), d(Sx, Sy), \psi(M(Tx, Ty)) \geq 0, \tag{3.1}$$

where

$$M(Tx, Ty) = \max\left\{d(Tx, Ty), \frac{d(Tx, Sx) + d(Ty, Sy)}{2}, \frac{d(Tx, Sy) + d(Ty, Sx)}{2}\right\}.$$

Then, (S, T) is called a generalized $\alpha\mathcal{T}$-contractive pair of mappings. If $\psi(M(Tx, Ty)) = d(x, y)$, then S converts into $\alpha\mathcal{T}$-contractive mapping with respect to Λ.

Theorem 3.2. Let $(\mathcal{H}, d)$ be a complete metric space and $S, T : \mathcal{H} \to \mathcal{H}$ be given mappings such that $S(\mathcal{H}) \subseteq T(\mathcal{H})$, $T(\mathcal{H})$ is closed. Suppose that (S, T) is generalized $\alpha\mathcal{T}$-contractive pair of mappings and the following conditions fulfils:

(i) S is $\alpha-$permissible with respect to T;

(ii) There exists $x_0 \in \mathcal{H}$ so that $\alpha(Tx_0, Sx_0) \geq 1$;

(iii) If $\{Tx_n\}$ is a sequence in $\mathcal{H}$ such that $\alpha(Tx_n, Tx_{n+1}) \geq 1$ and $Tx_n \to Tu \in T(\mathcal{H})$ as n tends to ∞, then there exists a subsequence $\{Tx_{n(k)}\}$ of $\{Tx_n\}$ such that $\alpha(Tx_{n(k)}, Tu) \geq 1$, for all k.

Then, S and T possess a coincidence point.

Proof . In view of $S(\mathcal{H}) \subseteq T(\mathcal{H})$, we can select a point $x_1 \in \mathcal{H}$, so that $Sx_0 = Tx_1$. In a similar way, we can choose x_{n+1} in $\mathcal{H}$ so that

$$Sx_n = Tx_{n+1}. \tag{3.2}$$

Using (ii), we get $\alpha(Tx_0, Sx_0) = \alpha(Tx_0, Tx_1) \geq 1$, Since, S is $\alpha-$permissible w.r.t T, we get

$$\alpha(Tx_n, Tx_m) \geq 1, \tag{3.3}$$

for all $m > n \geq 1$. If $Sx_{n+1} = Sx_n$ for some n, then by (3.2), we obtain $Sx_{n+1} = Tx_{n+1}$. So, S and T have a coincidence point at $x = x_{n+1}$ and so we have completed the proof. Further, we assume that $d(Sx_n, Sx_{n+1}) > 0$. Putting $x = x_n$ and $y = x_{n+1}$ in (3.1), we get

$$0 \leq \Lambda(\alpha(Tx_n, Tx_{n+1}), d(Sx_n, Sx_{n+1}), \psi(M(Tx_n, Tx_{n+1}))),$$

where

$$M(Tx_n, Tx_{n+1}) = \max\left\{d(Tx_n, Tx_{n+1}), \frac{d(Tx_n, Sx_n) + d(Tx_{n+1}, Sx_{n+1})}{2}, \frac{d(Tx_n, Sx_{n+1}) + d(Tx_{n+1}, Sx_n)}{2}\right\}.$$

Using (3.2), we get

$$M(Tx_n, Tx_{n+1}) = \max\left\{d(Sx_{n-1}, Sx_n), \frac{d(Sx_{n-1}, Sx_n) + d(Sx_n, Sx_{n+1})}{2}, \frac{d(Sx_{n-1}, Sx_{n+1}) + d(Sx_n, Sx_n)}{2}\right\}.$$

But,

$$\frac{d(Sx_{n-1}, Sx_{n+1})}{2} \leq max\{d(Sx_{n-1}, Sx_n), d(Sx_n, Sx_{n+1})\}.$$

Therefore,

$$M(Tx_n, Tx_{n+1}) \leq max\{d(Sx_{n-1}, Sx_n), d(Sx_n, Sx_{n+1})\}. \tag{3.4}$$

Using (3.1), we get

$$
\begin{aligned}
0 &\le \Lambda(\alpha(Tx_n, Tx_{n+1}), d(Sx_n, Sx_{n+1}), \psi(M(Tx_n, Tx_{n+1}))) \\
&< \psi(M(Tx_n, Tx_{n+1})) - \alpha(Tx_n, Tx_{n+1}) d(Sx_n, Sx_{n+1}),
\end{aligned}
$$

which indicate that

$$\alpha(Tx_n, Tx_{n+1}) d(Sx_n, Sx_{n+1}) < \psi(M(Tx_n, Tx_{n+1})).$$

Therefore,

$$d(Sx_n, Sx_{n+1}) \le \alpha(Tx_n, Tx_{n+1}) d(Sx_n, Sx_{n+1}). \tag{3.5}$$

By combining above two inequalities, we get

$$d(Sx_n, Sx_{n+1}) < \psi(M(Tx_n, Tx_{n+1})).$$

Using (3.4), we obtain

$$d(Sx_n, Sx_{n+1}) \le \psi(\max\{d(Sx_{n-1}, Sx_n), d(Sx_n, Sx_{n+1})\}). \tag{3.6}$$

If $\max\{d(Sx_{n-1}, Sx_n), d(Sx_n, Sx_{n+1})\} = d(Sx_n, Sx_{n+1})$, therefore, $d(Sx_{n-1}, Sx_n) \le d(Sx_n, Sx_{n+1})$. By (3.6), we have $d(Sx_n, Sx_{n+1}) < \psi(d(Sx_n, Sx_{n+1}))$. Using property of comparison function, $\psi(d(Sx_n, Sx_{n+1})) < d(Sx_n, Sx_{n+1})$. So, $d(Sx_n, Sx_{n+1}) < d(Sx_n, Sx_{n+1})$, which is a contradiction. So, $\max\{d(Sx_{n-1}, Sx_n), d(Sx_n, Sx_{n+1})\} = d(Sx_{n-1}, Sx_n)$. Thus, $d(Sx_n, Sx_{n+1}) < \psi(d(Sx_{n-1}, Sx_n))$. Using property of comparison function, $\psi(d(Sx_{n-1}, Sx_n)) < d(Sx_{n-1}, Sx_n)$.] Therefore, $d(Sx_n, Sx_{n+1})) < d(Sx_{n-1}, Sx_n)$. Hence, we conclude that the sequence $\{d(Sx_{n-1}, Sx_n)\}$ is decreasing sequence of non-negative real numbers. Accordingly, there is some $r \ge 0$, such that $\lim_{n\to\infty} d(Sx_{n-1}, Sx_n) = r \ge 0$. We assert that $r = 0$. Let us assume that, $r > 0$. Therefore,

$$\lim_{n\to\infty} \alpha(Tx_n, Tx_{n+1}) d(Sx_n, Sx_{n+1}) = r.$$

Letting $n \to \infty$ on both sides of (3.5), we get $\lim_{n\to\infty} \alpha(Tx_n, Tx_{n+1}) = 1$. Moreover by (Λ_2), we have

$$0 \le \lim_{n\to\infty} \sup \Lambda(\alpha(Tx_n, Tx_{n+1}), d(Sx_n, Sx_{n+1}), d(Sx_{n-1}, Sx_n)) < 0,$$

which is a contradiction. Thus, we have

$$\lim_{n\to\infty} d(Sx_{n-1}, Sx_n) = 0. \tag{3.7}$$

Now, we assert that $\{Sx_n\}$ is a Cauchy sequence. Let us imagine, there exists $\varepsilon > 0$, for each $n \in \mathbb{Z}_+$ and $n, m \in \mathbb{Z}_+$ with $n > m > \mathbb{Z}_+$ such that $d(x_m, x_n) > \varepsilon$. From (3.7), there exists $n_0 \in \mathbb{Z}_+$ such that

$$d(Sx_n, Sx_{n+1}) < \varepsilon, \ \ \forall\, n > n_0. \tag{3.8}$$

Consider $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$, such that

$$n_0 \le n_k < m_k < m_{k+1} \textit{ and } d(Sx_{m_k}, Sx_{n_k}) > \varepsilon, \tag{3.9}$$

for all k. Also,

$$d(Sx_{m_{k-1}}, Sx_{n_k}) \le \varepsilon, \tag{3.10}$$

for every k, where m_k is picked as the smallest number $m \in \{n_k, n_{k+1}, n_{k+2}, \ldots\}$ so that (3.9) is satisfied. Also, $n_k + 1 \le m_k$ for every k. But, $n_k + 1 \le m_k$ is infeasible due to (3.8) and (3.9). Therefore, $n_k + 2 \le m_k$, for each k. It yields that $n_k + 1 < m_k < m_k + 1$ for all k. According to the triangle inequality, (3.9) and (3.10), we obtain

$$\varepsilon < d(Sx_{m_k}, x_{n_k}) \le d(Sx_{m_k}, Sx_{m_{k-1}}) + d(Sx_{m_{k-1}}, x_{n_k}) \le d(x_{Sm_k}, Sx_{m_{k-1}}) + \varepsilon.$$

Due to (3.7), we deduce that

$$\lim_{k\to\infty} d(Sx_{m_k}, Sx_{n_k}) = \varepsilon. \tag{3.11}$$

By using the triangle inequality, we obtain

$$d(Sx_{m_k}, Sx_{n_k}) \le d(Sx_{m_k}, Sx_{m_{k+1}}) + d(Sx_{m_{k+1}}, Sx_{n_{k+1}}) + d(Sx_{n_{k+1}}, Sx_{n_k}).$$

Also, we have

$$d(Sx_{m_{k+1}}, Sx_{n_{k+1}}) \le d(Sx_{m_{k+1}}, Sx_{m_k}) + d(Sx_{m_k}, Sx_{n_k}) + d(Sx_{n_k}, Sx_{n_{k+1}}).$$

With the aid of (3.7), we find that

$$\lim_{k\to\infty} d(Sx_{m_k+1}, Sx_{n_k+1}) = \varepsilon. \tag{3.12}$$

Specifically, there occur $n_1 \in \mathbb{Z}_+$ in order that for all $k \ge n_1$, we acquire

$$d(Sx_{m_k}, Sx_{n_k}) > \frac{\varepsilon}{2} > 0,\ d(Sx_{m_k+1}, Sx_{n_k+1}) > \frac{\varepsilon}{2} > 0. \tag{3.13}$$

Moreover, since S fulfils (iii) of Theorem 3.2, we acquire

$$\alpha(Tx_{m_k}, Tx_{n_k}) \ge 1.$$

Since, (S,T) is generalized $\alpha\mathcal{T}-$contractive pair of mappings, we get

$$\begin{aligned}
0 &\le \Lambda(\alpha(Tx_{m_k}, Tx_{n_k}), d(Sx_{m_k}, Sx_{n_k}), \psi(M(Tx_{m_k}, Tx_{n_k}))\\
&= \Lambda(\alpha(Tx_{m_k}, Tx_{n_k}), d(Tx_{m_k+1}, Tx_{n_k+1}), \psi(M(Tx_{m_k}, Tx_{n_k}))\\
&< \psi(M(Tx_{m_k}, Tx_{n_k}) - \alpha(Tx_{m_k}, Tx_{n_k})d(Tx_{m_k+1}, Tx_{n_k+1})\\
&< \psi(M(Tx_{m_k}, Tx_{n_k})).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
0 &< d(Tx_{m_k+1}, Tx_{n_k+1})\\
&< \alpha(Tx_{m_k}, Tx_{n_k})d(Tx_{m_k+1}, Tx_{n_k+1})\\
&< \psi(M(Tx_{m_k}, Tx_{n_k}))\\
&< d(Tx_{m_k}, Tx_{n_k}).
\end{aligned}$$

From above inequality, together with (3.11) and (3.12), we conclude that

$$s_n = \alpha(Tx_{m_k}, Tx_{n_k})d(Tx_{m_k+1}, Tx_{n_k+1}) \to \varepsilon \text{ and } t_n = d(Tx_{m_k}, Tx_{n_k}) \to \varepsilon.$$

With the aid of (Λ_2), we acquire

$$0 \le \lim_{k\to\infty} sup\, \Lambda(\alpha(Tx_{m_k}, Tx_{n_k}), d(Tx_{m_k+1}, Tx_{n_k+1}), d(Tx_{m_k}, Tx_{n_k})) < 0,$$

which is a contradiction. Hence, $\{Tx_n\}$ is a Cauchy sequence. Since $T(\mathcal{H})$ is closed, so there exists $u \in \mathcal{H}$ such that

$$\lim_{n\to\infty} Tx_n = Tu.$$

Now, we show that S and T possess a coincidence point $u \in \mathcal{H}$. On the contrary, assume that $d(Su, Tu) > 0$.

$$0 \le \Lambda(\alpha(Tx_{n_k}, Tu), d(Sx_{n_k}, Su), \psi(M(Tx_{n_k}, Tu)))$$

$$< \psi(M(Tx_{n_k}, Tu)) - \alpha(Tx_{n_k}, Tu)d(Sx_{n_k}, Su).$$

Thus,

$$\alpha(Tx_{n_k}, Tu)d(Sx_{n_k}, Su) < \psi(M(Tx_{n_k}, Tu)). \tag{3.14}$$

Since by condition (iii) of Theorem 3.2, we have $\alpha(Tx_{n(k)}, Tu) \geq 1$. By the use of triangle inequality, we obtain

$$\begin{aligned} d(Tu, Su) &\leq d(Tu, Sx_{n_k}) + d(Sx_{n_k}, Su) \\ &\leq d(Tu, Sx_{n_k}) + \alpha(Tx_{n_k}, Tu)d(Sx_{n_k}, Su). \end{aligned}$$

Using (3.14), we get $d(Tu, Su) \leq d(Tu, Sx_{n_k}) + \psi(M(Tx_{n_k}, Tu))$, where

$$M(Tx_{n_k}, Tu) = max\left\{d(Tx_{n_k}, Tu), \frac{d(Tx_{n_k}, Sx_{n_k}) + d(Tu, Su)}{2}, \frac{d(Tx_{n_k}, Su) + d(Tu, Sx_{n_k})}{2}\right\}.$$

According to the above equality, we get

$$\begin{aligned} d(Tu, Su) &\leq d(Tu, Sx_{n_k}) + \psi(M(Tx_{n_k}, Tu)), \\ &\leq d(Tu, Sx_{n_k}) + \psi\left(\max\left\{d(Tx_{n_k}, Tu), \frac{d(Tx_{n_k}, Sx_{n_k}) + d(Tu, Su)}{2}, \frac{d(Tx_{n_k}, Su) + d(Tu, Sx_{n_k})}{2}\right\}\right). \end{aligned}$$

Letting $k \to \infty$ in the above inequality yields

$$d(Tu, Su) \leq \psi(\frac{d(Tu, Su)}{2}) < (\frac{d(Tu, Su)}{2}),$$

which is a contradiction. Hence, our contemplation is faulty and $d(Su, Tu) = 0$, which indicates that $u \in C(T, S)$. □

Theorem 3.3. In conjunction with the assumptions of above Theorem, assume that for all $z_1, z_2 \in C(T, S)$, there exists $z_3 \in \mathcal{H}$ such that $\alpha(Tz_1, Tz_3) \geq 1$, $\alpha(Tz_2, Tz_3) \geq 1$ and S, T commute at $u \in C(T, S)$. Then, there exists a unique $u \in \mathcal{H}$ such that $u \in \mathcal{CF}(S, T)$.

Proof . We wish to take three steps:

Step 1. We want to prove that if $z_1, z_2 \in C(T, S)$, then $gz_1 = gz_2$. By given assumption, there exists $z \in \mathcal{H}$ such that

$$\alpha(Tz_1, Tz) \geq 1,\ \alpha(Tz_2, Tz) \geq 1. \tag{3.15}$$

Also, $S(\mathcal{H}) \subseteq T(\mathcal{H})$. Now, we define the sequence $\{z_n\}$ in $\mathcal{H}$ by $Tz_{n+1} = Sz_n$, for all $n \geq 0$ and $z_0 = z$. Since, every $\alpha-$permissible pair of mappings are $\alpha-$admissible, we have from (3.15) that $\alpha(Tz_1, Tz_n) \geq 1$ and $\alpha(Tz_2, Tz_n) \geq 1$. Applying inequality (3.1), we obtain

$$0 \leq \Lambda(\alpha(Tz_1, Tz_n), d(Sz_1, Sz_n), \psi(M(Tz_1, Tz_n))) < \psi(M(Tz_1, Tz_n)) - \alpha(Tz_1, Tz_n)d(Sz_1, Sz_n).$$

Then $\alpha(Tz_1, Tz_n)d(Sz_1, Sz_n) < \psi(M(Tz_1, Tz_n))$. Using (3.5), we obtain

$$d(Sz_1, Sz_n) \leq \alpha(Tz_1, Tz_n)d(Sz_1, Sz_n). \tag{3.16}$$

Also, $d(Sz_1, Sz_n) = d(Tz_1, Tz_{n+1})$. Therefore,

$$d(Tz_1, Tz_{n+1}) \leq \psi(M(Tz_1, Tz_n)), \tag{3.17}$$

where

$$M(Tz_1, Tz_n) = \max\left\{d(Tz_1, Tz_n), \frac{d(Tz_1, Sz_1) + d(Tz_n, Sz_n)}{2}, \frac{d(Tz_1, Sz_n) + d(Tz_n, Sz_1)}{2}\right\}$$

$$\leq \max\{d(Tz_1, Tz_n), d(Tz_1, Tz_{n+1}).$$

Using (3.17), we get $d(Tz_1, Tz_{n+1}) \leq \psi(\max\{d(Tz_1, Tz_n), d(Tz_1, Tz_{n+1})\})$. Let us imagine that $d(Tz_1, Tz_n) > 0$, for each n. If $\max\{d(Tz_1, Tz_n), d(Tz_1, Tz_{n+1})\} = d(Tz_1, Tz_{n+1})$, then,

$$d(Tz_1, Tz_{n+1}) \leq \psi(d(Tz_1, Tz_{n+1})) < d(Tz_1, Tz_{n+1}),$$

which is a contradiction. Thus, we have $\max\{d(Tz_1, Tz_n), d(Tz_1, Tz_{n+1})\} = d(Tz_1, Tz_n)$. So, $d(Tz_1, Tz_{n+1}) \leq \psi(d(Tz_1, Tz_n))$, that is, $\{d(Tz_1, Tz_n)\}$ is a monotonically decreasing sequence in $\mathcal{R}_+$. Thus, we can find $\ell \geq 0$ such that $\lim_{n\to\infty} d(Tz_1, Tz_n) = \ell$. We claim that $\ell = 0$. Let us imagine that $0 < \ell$. With the assistance of (3.15), we get

$$\alpha(Tz_1, Tz_n)d(Tz_1, Tz_{n+1}) = \ell.$$

Letting $s_n = \alpha(Tz_1, Tz_n)d(Tz_1, Tz_{n+1}), t_n = d(Tz_1, Tz_n)$ and taking (Λ_2) into account, we get that

$$0 \leq \lim_{n\to\infty} \sup \Lambda(\alpha(Tz_1, Tz_n), d(Tz_1, Tz_{n+1}), d(Tz_1, Tz_n)) < 0,$$

which is a contradiction. Thus, we have $\lim_{n\to\infty} d(Tz_1, Tz_n) = \ell = 0$. With the assistance of same approach, we can prove that $\lim_{n\to\infty} d(Tz_2, Tz_n) = 0$. Therefore, $Tz_1 = Tz_2$. Now, we exhibit the presence of a common fixed point. Let $z_1 \in C(T, S)$, that is, $Tz_1 = Sz_1$. Due to the commutativity of S and T at their coincidence points, we get

$$T^2 z_1 = TTz_1 = TSz_1 = STz_1. \tag{3.18}$$

Let us suppose that, $Tz_1 = u$. From (3.18), we get $Tu = Su$. Thus, $u \in C(T, S)$. From the given assumption, $Tz_1 = Tu = u = Su$. Then, $u \in \mathcal{CF}(T, S)$. Now, we exhibit that fixed point is unique. Imagine that S and T possess another common fixed point z_3. Then, $z_3 \in \mathcal{CF}(T, S)$. From the given assumption, we have $z_3 = Tz_3 = Tu = u$. Consequently, the common fixed point of S and T is unique. □

Corollary 3.4. [10] Let $S : \mathcal{H} \to \mathcal{H}$ be given map in complete $(\mathcal{H}, d)$. If there exists $\Lambda \in \mathcal{T}$ and $\alpha : \mathcal{H} \times \mathcal{H} \to [0, \infty]$ such that

$$\Lambda(\alpha(\Omega, \mho), d(S\Omega, S\mho), d(\Omega, \mho)) \geq 0,$$

for every $\Omega, \mho \in \mathcal{H}$ and fulfilling the following situations:

(i) There exists $\Omega_0 \in \mathcal{H}$ such that $\alpha(\Omega_0, S\Omega_0) \geq 1$;

(ii) S is $\alpha-$ permissible;

(iii) If $\{\Omega_n\} \in \mathcal{H}$, so that $\alpha(\Omega_n, \Omega_{n+1}) \geq 1$, for all n and $\Omega_n \to \Theta_1 \in \mathcal{H}$ as $n \to \infty$, then there exists a subsequence $\{\Omega_{n(k)}\}$ of $\{\Omega_n\}$, such that $\alpha(\Omega_{n(k)}, \Theta_1) \geq 1$, for every k. Then, S possess a fixed point.

Proof . The result proceeds from main Theorem 3.2. □

Corollary 3.5. Let $S : \mathcal{H} \to \mathcal{H}$, be a given function in complete metric space. If there exist $\Lambda \in \mathcal{T}$ and $\alpha : \mathcal{H} \times \mathcal{H} \to [0, \infty]$ such that

$$\Lambda(\alpha(\Omega, \mho), d(S\Omega, S\mho), M(S\Omega, S\mho)) \geq 0,$$

where

$$M(S\Omega, S\mho) = max\left\{d(\Omega, \mho), \frac{d(\Omega, S\Omega) + d(\mho, S\mho)}{2}, \frac{d(\Omega, S\mho) + d(\mho, S\Omega)}{2}\right\},$$

for every $\Omega, \mho \in \mathcal{H}$ and fulfilling the following conditions

(i) there exists $\Omega_0 \in \mathcal{H}$ such that $\alpha(\Omega_0, S\Omega_0) \geq 1$;

(ii) S is α-permissible;

(iii) If $\{\Omega_n\} \in \mathcal{H}$, such that $\alpha(\Omega_n, \Omega_{n+1}) \geq 1$ and $\Omega_n \to \Theta_1 \in \mathcal{H}$ as $n \to \infty$, then there exists a subsequence $\{\Omega_{n(k)}\}$ of $\{\Omega_n\}$, such that $\alpha(\Omega_{n(k)}, \Theta_1) \geq 1$, for every k. Then, S possess a fixed point.

Proof . The result proceeds from main Theorem 3.2 by placing T as Identity function. $\square$

Example 3.6. Consider $\mathcal{H} = [0, +\infty)$ associated with the metric

$$d(\Omega, \mho) = \begin{cases} 0, & \text{if } \Omega = \mho, \\ \max\{\Omega, \mho\}, & \text{otherwise,} \end{cases}$$

for all $\Omega, \mho \in \mathcal{H}$. Define the self mappings S and T by $S(\ell) = \ell$ and $T(\ell) = 2\ell$ for $\ell \in \mathcal{H}$, with $\psi(t) = \frac{t}{2}$. Let $\Lambda : \mathcal{H} \times \mathcal{H} \times \mathcal{H} \to \mathcal{R}$ be defined as

$$\Lambda(\rho, \sigma, \wp) = \rho\wp - \frac{\sigma + 2}{\sigma + 1}\sigma.$$

Now, we formalize the mapping α as

$$\alpha(\Theta_1, \Theta_2) = \begin{cases} 1, & \text{if } (\Theta_1, \Theta_2) \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

If $M(T\Omega, T\mho) = d(T\Omega, T\mho)$. Thus,

$$\begin{aligned} \Lambda(\alpha(T\Omega, T\mho), d(S\Omega, S\mho), \psi(M(T\Omega, T\mho)) &= \Lambda(\alpha(T\Omega, T\mho), d(S\Omega, S\mho), \psi(d(T\Omega, T\mho)) \\ &= \Lambda(1, \mho, 2\mho) \\ &= 2\mho - \frac{\mho + 2}{(\mho + 1)} \frac{\mho}{2} \\ &= \frac{4\mho(\mho + 1) - \mho(\mho + 2)}{2(\mho + 1)} \\ &= \frac{4\mho^2 + 4\mho - \mho^2 - 2\mho}{2(\mho + 1)} \\ &= \frac{3\mho^2 + 2\mho}{2(\mho + 1)} \geq 0. \end{aligned}$$

If

$$M(T\Omega, T\mho) = \frac{d(T\Omega, S\Omega) + d(T\mho, S\mho)}{2} \text{ or } \frac{d(T\Omega, S\mho) + d(T\mho, S\Omega)}{2}.$$

Thus,

$$\begin{aligned} \Lambda(\alpha(T\Omega, T\mho), d(S\Omega, S\mho), \psi(M(T\Omega, T\mho)) &= \Lambda(1, \mho, \Omega + \mho) \\ &= \Omega + \mho - \frac{\mho + 2}{(\mho + 1)} \frac{\mho}{2} \\ &= \frac{(2\Omega + 2\mho)(\mho + 1) - \mho(\mho + 2)}{2(\mho + 1)} \\ &= \frac{2(\Omega\mho + \Omega) + \mho^2}{2(\mho + 1)} \geq 0. \end{aligned}$$

Clearly, (S, T) is a generalized $\alpha\mathcal{T}$-contractive pair of mappings and $\psi(t) = \frac{t}{2}$. Now, all the assumptions of Theorem 3.2 and Theorem 3.3 are satisfied. Therefore, S and T have a coincidence point. Also, $0 \in C(S, T)$ and $0 \in \mathcal{CF}(S, T)$.

4 Application to the integral equation

In this section, we give an application of the integral equation.

Theorem 4.1. Let us consider the non-linear Fredholm integral equation

$$S_1x(t) = S_2(t) + \int_a^b F(t, s, x(s))ds, \tag{4.1}$$

for some $a, b \in R$ with $a < b, S_2 : [a, b] \to R$ and $F : [a, b]^2 \times R \to R$ be two continuous functions. Also, imagine that the subsequent properties hold:

(i) $S_1 : C[a, b] \to C[a, b]$ is a continuous mapping;

(ii) F satisfying

$$\begin{aligned} \mid F(t, s, x(s)) \mid + \mid F(t, s, y(s)) \mid \leq \frac{1}{b-a}\psi(\max \Bigg\{ & \mid S_2x(t) \mid + \mid S_2y(t) \mid, \\ & \frac{(\mid S_2x(t) \mid + \mid S_1x(t) \mid) + (\mid S_2y(t) \mid + \mid S_1y(t) \mid)}{2}, \\ & \frac{(\mid S_2x(t) \mid + \mid S_1y(t) \mid) + (\mid S_2y(t) \mid + \mid S_1x(t) \mid)}{2} \Bigg\} - 2 \mid S_2(t) \mid, \end{aligned}$$

for all $t, s \in [a, b]$. Then, the non-linear Fredholm integral equation (4.1) owns a unique solution in $C[a, b]$.

Proof . We know that $C[a, b]$ is complete with respect to the metric $\sigma : C[a, b] \times C[a, b] \longrightarrow R^+$ defined as

$$\sigma(x, y) = \sup_{t \in [a,b]} \mid x(t) - y(t) \mid,$$

where $x, y \in C[a, b]$. Let $d : C[a, b] \times C[a, b] \longrightarrow R^+$ having

$$d(x, y) = \sup_{t \in [a,b]} \mid x(t) \mid + sup_{t \in [a,b]} \mid y(t) \mid,$$

where $x, y \in C[a, b]$. Now,

$$\begin{aligned} \mid S_1x(t) \mid + \mid S_1y(t) \mid &= \mid S_2(t) + \int_a^b F(t, s, x(s))ds \mid + \mid S_2(t) + \int_a^b F(t, s, y(s))ds \mid \\ &\leq \mid S_2(t) \mid + \mid \int_a^b F(t, s, x(s))ds \mid + \mid S_2(t) \mid + \mid \int_a^b F(t, s, y(s))ds \mid \\ &\leq 2 \mid S_2(t) \mid + \mid \int_a^b F(t, s, x(s))ds \mid + \mid \int_a^b F(t, s, y(s))ds \mid \\ &\leq 2 \mid S_2(t) \mid + \int_a^b \mid F(t, s, x(s)) \mid ds + \int_a^b \mid F(t, s, y(s)) \mid ds \\ &\leq 2 \mid S_2(t) \mid + \int_a^b (\mid F(t, s, x(s)) \mid + \mid F(t, s, y(s)) \mid)ds \\ &\leq 2 \mid S_2(t) \mid + \int_a^b \Bigg(\frac{1}{b-a}\psi(\max \Bigg\{ \mid S_2x(t) \mid + \mid S_2y(t) \mid, \\ &\quad \frac{(\mid S_2x(t) \mid + \mid S_1x(t) \mid) + (\mid S_2y(t) \mid + \mid S_1y(t) \mid)}{2}, \\ &\quad \frac{(\mid S_2x(t) \mid + \mid S_1y(t) \mid) + (\mid S_2y(t) \mid + \mid S_1x(t) \mid)}{2} \Bigg\} - 2 \mid S_2(t) \mid \Bigg) ds \\ &= 2 \mid S_2(t) \mid + \Bigg(\frac{1}{b-a}\psi(\max \Bigg\{ \mid S_2x(t) \mid + \mid S_2y(t) \mid, \\ &\quad \frac{(\mid S_2x(t) \mid + \mid S_1x(t) \mid) + (\mid S_2y(t) \mid + \mid S_1y(t) \mid)}{2}, \\ &\quad \frac{(\mid S_2x(t) \mid + \mid S_1y(t) \mid) + (\mid S_2y(t) \mid + \mid S_1x(t) \mid)}{2} \Bigg\} - 2 \mid S_2(t) \mid \Bigg) \int_a^b ds \\ &= \psi \Bigg(\max \Bigg\{ \mid S_2x(t) \mid + \mid S_2y(t) \mid, \frac{(\mid S_2x(t) \mid + \mid S_1x(t) \mid) + (\mid S_2y(t) \mid + \mid S_1y(t) \mid)}{2}, \end{aligned}$$

$$\left.\left.\frac{(\mid S_2x(t)\mid + \mid S_1y(t)\mid) + (\mid S_2y(t)\mid + \mid S_1x(t)\mid)}{2}\right\}\right)$$

$$\leq \psi\left(\max\left\{p(S_2x,S_2y), \frac{p(S_2x,S_1x)+p(S_2y,S_1y)}{2}, \frac{p(S_2x,S_1y)+p(S_2y,S_1x)}{2}\right\}\right)$$

$$= \psi(M(S_2x,S_2y)),$$

for all $x, y \in C[a,b]$ and $t \in [0,\infty]$. Consequently,

$$\sup_{t\in[a,b]} \mid S_1x(t)\mid + \sup_{t\in[a,b]} \mid S_1y(t)\mid \leq \psi(M(S_2x,S_2y)),$$

which indicates that

$$d(S_1x,S_1y) \leq \psi(M(S_2x,S_2y)).$$

Now, we formalize the mapping $\eta : \mathcal{H}\times\mathcal{H} \to [0,\infty)$ as

$$\alpha(\Theta_1,\Theta_2) = \begin{cases} 1, & \text{if } (\Theta_1,\Theta_2) \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $\Lambda(\alpha(\Theta_1,\Theta_2), d(S_1x,S_1y), \psi(M(S_2x,S_2y))) \geq 0$. Therefore, by Theorem 3.2, the non-linear Fredholm integral equation (4.1) owns a solution. □

5 Conclusion

In this work, we investigate the existence of a coincident point of generalized α-permissible $\mathcal{T}$-contraction. The proposed work contributes to the formulation of a unique common fixed point with the help of commutative property of two maps. The presented theorems enhance various results present in the literature. Specifically speaking, we established the results for new contraction via new kind of simulation function in three variables. Additionally, an illustrative example and corollaries are provided to demonstrate the main results. Moreover, as an application, we employ the achieved result to earn the existence criteria of the solution of a kind of non-linear Fredholm integral equation.

References

[1] H. Argoubi, B. Samet, and C. Vetro, *Nonlinear contractions involving simulation functions in a metric space with a partial order*, J. Nonlinear Sci. Appl. **8** (2015), 1082–1094.

[2] H. Aydi, A. Felhi, E. Karapinar, and F.A. Alojail, *Fixed points on quasi-metric spaces via simulation functions and consequences*, J. Math. Anal. **9** (2018), no. 2, 10–24.

[3] S. Banach, *Sur les operations dans les ensembles abstraits et leur application aux equations itegrales*, Fund. Math. **3** (1922), 133–181.

[4] S.H. Bonab, R. Abazari, A.B. Vakilabad, and H. Hosseinzadeh, *Generalized metric spaces endowed with vector-valued metrics and matrix equations by tripled fixed point theorems*, J. Inequal. Appl. **204** (2020), 1–16.

[5] S.H. Bonab, V. Parvaneh, H. Hosseinzadeh, A. Dinmohammadi, and B. Mohammadi, *Some common fixed point results via α-series for a family of js-contraction type mappings*, Fixed Point Theory and Fractional Calculus, Forum for Interdisciplinary Mathematics, Springer, Singapore, 2022.

[6] S.H. Bonab, H. Hosseinzadeh V. Parvaneh, N.A. Shotorbani, H. Aydi, and S.J. Ghoncheh, *n-tuple fixed point theorems via α-series in C^*-algebra-valued metric spaces with an application in integral equations*, Int. J. Ind. Math. **15** (2023), no. 2, 95–105.

[7] S. Chandok, A. Chanda, L.K. Dey, M. Pavlovič, and S. Radenovič, *Simulation functions and Geraghty type results*, Bol. Soc. Paranaense Mat. **39** (2021), no. 1, 35–50.

[8] P. Debnath, H.M. Srivastava, K. Chakraborty, and P. Kumam, *(η,ψ)-rational f-contractions and weak-Wardowski contractions in a triple-controlled modular-type metric space*, Adv. Number Theory Appl. Anal., 2023, pp. 279-308.

[9] R. Gubran, W.M. Alfaqieh, and M. Imdad, *Common fixed point results for α-admissible mappings via simulation function*, J. Anal. **25** (2017), 281–290.

[10] S. Gubran, W.M. Alfaqih, and M. Imdad, *Fixed point results via tri-simulation function*, Ital. J. Pure Appl. Math. **45** (2021), 419–430.

[11] H. Hosseinzadeh, S.H. Bonab, and K.A. Sefidab, *Some common fixed point theorems for four mapping in generalized metric spaces*, Thai J. Math. **20** (2022), no. 1, 425–437.

[12] H. Hosseinzadeha, H. Isik, and S.H. Bonaba, *n-tuple fixed point theorems via α-series on partially ordered cone metric spaces*, Int. J. Nonlinear Anal. Appl. **13** (2022), no. 2, 3115–3126.

[13] E. Karapinar, *Fixed points results via simulation functions*, Filomat **30** (2016), no. 8, 2343–2350.

[14] F. Khojasteh, S. Shukla, and S. Radenovic, *A new approach to the study of fixed point theory for simulation functions*, Filomat **29** (2015), no. 6, 1189–1194.

[15] O. Popescu, *Some new fixed point theorems for α-Geraghty contractive type maps in metric spaces*, Fixed Point Theory Appl. **190** (2014), 1–12.

[16] A. Roldán, E. Karapinar, C. Roldán, and J. Martínez-Moreno, *Coincidence point theorems on metric spaces via simulation functions*, J. Comput. Appl. Math. **275** (2015), 345–355.

[17] B. Samet, C. Vetro, and P. Vetro, *Fixed point theorem for $(\alpha-\psi)$-contractive type mappings*, Nonlinear Anal. **75** (2012), 2154–2165.

[18] Y. Talaei, S. Micula, H. Hosseinzadeh, and S. Noeiaghdam, *A novel algorithm to solve nonlinear fractional quadratic integral equations*, AIMS Math. **7** (2022), no. 7, 13237–13257.