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# Boundary curvature of the numerical range of self-inverse operators

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**Abstract**

In this article, firstly we investigate some properties of the boundary curvature of the numerical range. In the next, we define $M(T)$ as the smallest constant such that $dist(\lambda, \sigma(T)) \leq M(T)R_\lambda(T)$, for all $\lambda \in \partial W(T)$, where $R_\lambda(T)$, the radius curvature at the point $\lambda$, is defined. Also, we investigate for non-convexoid $T$, $M(T) = \sup \frac{dist(\lambda,\sigma(T))}{R_\lambda(T)}$, where the supremum on the right-hand side is taken along all points $\lambda \in \partial W(T)$ with finite non-zero curvature. Finally, the value of $M(T)$ will be calculated for the self-inverse operators.

Keywords: Boundary Curvature, Numerical range, Self-Inverse Operators
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## 1 Introduction and Preliminaries

The numerical range was initially proposed by Toeplitz [29] in 1918 for matrices. This idea was subsequently expanded upon by Lumer [16] in 1961 and Bauer [4] in 1962 for linear operators on Banach spaces. In 1975, Lightbourne and Martin [15] delivered a thorough exposition of it for semi-norms. For further insights into the numerical range of bounded linear operators on Hilbert spaces, the numerical range of bounded linear operators on Banach spaces, and the numerical range of an element of $C^*-$algebras, readers are directed to references [6, 7, 30, 31].

The determination of the norm of self-commutators of the operator $T$ has been a subject of debate in numerous academic works. In the year 2009 , Abdollahi established the correlations among these norm magnitudes, particularly for self-inverse operators. Moreover, in specific scenarios involving automorphic composition operators, it was illustrated that these discoveries facilitated the replacement of laborious calculations with more efficient options, as indicated in [1].

Boundary curvature holds significant importance in the fields of engineering, physics, and chemistry. The concept of curvature, representing the bending or deviation at a specific point, is symbolized by $k(t)$. In the realm of curvature calculations, a fundamental grounding in mathematics suffices. In the year 2009, Pengzi and Luen-Fai [19] delved into a comprehensive exploration of the volume function concerning constant scalar curvature metrics with a designated boundary metric. Their research not only outlined the essential criteria for a metric to be deemed a critical point but also showcased that geodesic balls exclusively function as critical points in space forms. The classical existence and

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uniqueness theory of Jenkins-Serrin was extended for the constant mean curvature equation by Hauswirth et al. in 2009. This study permits prescribed boundary data, including plus or minus infinity, on entire arcs of boundary (see [13, 14]).

In 2013, Pokorn*ý* and Rataj [20] demonstrated that for any compact domain in a Euclidean space with a delta-convex boundary, there exists a unique Legendrian cycle such that the associated curvature measures satisfy a local version of the Gauss–Bonnet formula. In 2016, *Š*prl*á*k and Nov*á*k [27] conducted a study on modeling the Earth's gravitational field, where gravitational curvatures introduce new types of visibility. Initially, the gravitational curvature tensor was decomposed into 6 parts, then expanded using third-order tensor spherical harmonics. Subsequently, boundary-value problems for gravitational curvature were formulated and solved in spectral and spatial domains for four combinations of gravitational curvatures.

In 2017, one of the applications of boundary curvatures was explained by Zhong et al. [32], the distribution of grain boundary curvatures is analyzed as a function of 5 independent crystallographic parameters in austenitic and ferritic steel. Local curvatures and integral mean curvatures are obtained from three-dimensional electron backscattered diffraction data [24, 25, 26].

In the following, we discuss some applications of boundary curvature, Marcel Campen et al. [8] presented an effective algorithm for calculating a discrete metric with specified Gaussian curvature at all interior vertices and geodesic curvature along a mesh boundary in 2021. After that, Niewczas et al. [18] conducted a study to examine the impact of the grain boundary curvature model type on the predictions of static recrystallization (SRX) simulations using cellular automata (CA) in 2022. The issue of calculating $M(A)$ for the matrix $A$ was initially discussed in the works of Toeplitz and Hausdorff. The concept of a constant $M$ and $M = \sup_n M_n$ was introduced by Mantis in 1997, with a negative outcome. The determination of $M(A)$ for $2 \times 2$ matrices, as well as certain special $3 \times 3$ matrices, has been addressed in various articles. In this paper the definitions and important theorems concerning the boundary curvature of the numerical range are presented. Finally, The value of $M(T)$ will be calculated for the numerical range of self-inverse operators $T$.

## 2 Main Results

In this section, consider the numerical range of the operator $T$ from the bounded operators on the Hilbert space $H$ ($T \in B(H)$) defined as $W(T) := \{\langle Tx, x\rangle : x \in H, \|x\| = 1\}$ where $\|.\|$ denotes the norm and $\langle ., .\rangle$ is the inner product on the Hilbert space. $W(T)$ forms a subset of the complex plane. The key property of the numerical range is its convexity, as per the Töplitz-Hausdorff theorem. Another notable aspect is its spectrum operator, which has been studied by some researchers (for example, [3, 23]). $W(T)$ is a connected set with a piecewise analytic boundary $\partial W(T)$. For further details, see [12]. Therefore, for all except finitely many points $\lambda \in \partial W(T)$, the radius of curvature $R_\lambda(T)$ of $\partial W(T)$ at $\lambda$ is well defined. According to the convention, $R_\lambda(T) = 0$ if $\lambda$ is a corner point of $W(T)$, and $R_\lambda(T) = \infty$ if $\lambda$ lies inside a flat portion of $\partial W(T)$. Suppose $dist(\lambda, \sigma(T))$ denote the distance from $\lambda$ to $\sigma(T)$, we define $M(T)$ the smallest constant so that

$$dist(\lambda, \sigma(T)) \leq M(T)R_\lambda(T), \qquad \forall\ \lambda\ \in\ \partial W(T), \tag{2.1}$$

where $R_\lambda(T)$ is defined. By attention to Donoghe's theorem $dist(\lambda, \sigma(T)) = 0$ whenever $R_\lambda(T) = 0$. Therefor, $M(T) = 0$ for all convexoid element $T$. As a reminder, a convexoid element is an element in which the numerical range is equal to the convex hull of its spectrum. For non-convexoid $T$, For non-convexoid $T$,

$$M(T) = sup\frac{dist(\lambda, \sigma(T)}{R_\lambda(T)}. \tag{2.2}$$

The supremum on the right-hand side is taken over all points $\lambda \in \partial W(T)$ with finite non-zero curvature. Computing $M(T)$ for any given $T$ is an intriguing challenge. Similarly, computing $M(A)$ for any arbitrary $n \times n$ matrix $A$ remains an interesting open problem.

**Definition 2.1.** For each infinite, convex and compact subset $C$ of the complex plane, $\lambda(\theta)$ we define it as follows.

$$\lambda(\theta) = \max\{Re(e^{-i\theta}z) : \quad z \in C\}, \quad 0 \leq \theta \leq 2\pi.$$

For each infinite, convex and compact subset $\lambda(\theta)$ of the complex plane X, we define it as follows.

**Lemma 2.2.** If $A \in \mathbb{C}^{n\times n}$, then the following relation holds for $\lambda(\theta)$.

$$\lambda(\theta) = \max \sigma(A_R \cos\theta + A_J \sin\theta),\ 0 \le \theta \le 2\pi,$$

where $A_R$ is the real part and $A_J$ is the imaginary part of the matrix $A$ in the following form.

$$A_R = \frac{1}{2}(A + A^*), \quad A_J = \frac{1}{2i}(A - A^*).$$

**Proof .** If we put $C = W(A)$ in definition 2.1, where $A$ is matrix $n \times n$. Then we have:

$$\begin{aligned}\lambda(\theta) &= \max W(Re(e^{-i\theta}A)) \\ &= \max \sigma(Re(e^{-i\theta}A)) \\ &= \max \sigma(\cos\theta.Re(A) + \sin\theta.Im(A)), \quad 0 \le \theta \le 2\pi \\ &= \max \sigma(A_R \cos\theta + A_J \sin\theta), \quad 0 \le \theta \le 2\pi.\end{aligned}$$

□

**Proposition 2.3.** The parametric representation of $\partial W(T)$ in the form of $A = (x(\theta), y(\theta))$ which is $0 \le \theta \le 2\pi$ such that:

$$\begin{cases} x(\theta) = \lambda(\theta)\cos\theta - \lambda'(\theta)\sin\theta \\ y(\theta) = \lambda(\theta)\sin\theta + \lambda'(\theta)\cos\theta. \end{cases}$$

**Proof .** The equation of the tangent line at point $A \in \partial W(T)$ is:

$$x(\theta)\cos\theta + y(\theta)\sin\theta - \lambda(\theta) = 0. \tag{2.3}$$

On the other hand, the slope of the tangent line is as follows.

$$\frac{y'(\theta)}{x'(\theta)} = -\cot\theta.$$

As a result, we have

$$x'(\theta)\cos\theta + y'(\theta)\sin\theta = 0. \tag{2.4}$$

Now, if we take the derivative of the equation of the tangent line of relation (2.3) with respect to $\theta$, we have:

$$x'(\theta)\cos\theta - x(\theta)\sin\theta + y'(\theta)\sin\theta + y(\theta)\cos\theta - \lambda'(\theta) = 0,$$

as a result

$$-x(\theta)sin\theta + y(\theta)cos\theta = \lambda'(\theta). \tag{2.5}$$

It can be concluded from relations (2.3) and (2.5),

$$\begin{cases} x(\theta)\cos\theta + y(\theta)\sin\theta = \lambda(\theta), \\ -x(\theta)\sin\theta + y(\theta)\cos\theta = \lambda'(\theta). \end{cases} \tag{2.6}$$

Now we have (2.6) equations from solving the device:

$$\begin{cases} x(\theta)\cos\theta\sin\theta + y(\theta)\sin^2\theta = \lambda(\theta)\sin\theta, \\ -x(\theta)\cos\theta\sin\theta + y(\theta)\cos^2\theta = \lambda'(\theta)\cos\theta. \end{cases}$$

Hence, $y(\theta)(\sin^2\theta + \cos^2\theta) = \lambda(\theta)\sin\theta + \lambda'(\theta)\cos\theta$. Then, $y(\theta) = \lambda(\theta)\sin\theta + \lambda'(\theta)\cos\theta$ . Therefore,

$$\begin{cases} x(\theta)\cos^2\theta + y(\theta)\sin\theta\cos\theta = \lambda(\theta)\cos\theta, \\ x(\theta)\sin^2\theta - y(\theta)\cos\theta\sin\theta = -\lambda'(\theta)\sin\theta. \end{cases}$$

Then $x(\theta)(\cos^2\theta+\sin^2\theta)=\lambda(\theta)\cos\theta-\lambda^{'}(\theta)\sin\theta$ and so $x(\theta)=\lambda(\theta)\cos\theta-\lambda^{'}(\theta)\sin\theta$. Therefore, the machine's calculations lead to the following answer and the end of the proof.

$$\begin{cases} x(\theta)=\lambda(\theta)\cos\theta-\lambda^{'}(\theta)\sin\theta \\ y(\theta)=\lambda(\theta)\sin\theta+\lambda^{'}(\theta)\cos\theta \end{cases}$$

□

**Lemma 2.4.** (Elliptic Lemma): Let $A\in\mathbb{C}^{2\times 2}$, then $W(A)$ is an ellipse, and the eigenvalues of the matrix $A$ are the foci of the ellipse.

In the continuation of this article, while examining the numerical range of self-inverse operators of $T$, we will calculate the value of $M(T)$ exactly for them.

**Definition 2.5.** The operator $T$ is said to be self inverse if we have $T^2=I$. The self-inverse operator $T$ is called non-trivial if $T\neq\pm I$.

Now we calculate the non-trivial inverse of the $M(T)$ value for the self-operators. Let the operator $T$ be self-inverse, i.e., $T^2=I$ but $T\neq\pm I$, so $\sigma(T)=\pm 1$. Also $\partial W(T)$ is an ellipse with foci at $\pm 1$ and major/minor axis $\|T\|\pm\frac{1}{\|T\|}$. The equations of $\partial W(T)$ are as follows:

$$\partial W(T)=a\cos\theta+ib\sin\theta \quad \text{with } a^2=b^2+1.$$

Therefore, we perform preliminary calculations as follows:

$$\begin{cases} x=a\cos\theta \Longrightarrow x^{'}=-a\sin\theta \Longrightarrow x^{''}=-a\cos\theta, \\ y=b\sin\theta \Longrightarrow y^{'}=b\cos\theta \Longrightarrow y^{''}=-b\sin\theta. \end{cases}$$

So

$$\begin{aligned} K_\lambda(T) &= \frac{\|x^{'}y^{''}-y^{'}x^{''}\|}{(x^{'2}+y^{'2})}^{\frac{3}{2}} \\ &= \frac{|ab\sin^2\theta+ab\cos^2\theta|}{(a^2\sin^2\theta+b^2\cos^2\theta)^{\frac{3}{2}}} \\ &= \frac{|ab|}{(a^2\sin^2\theta+b^2\cos^2\theta)^{\frac{3}{2}}}. \end{aligned}$$

In the end it is the result:

$$R_\lambda(T)=\frac{(a^2\sin^2\theta+b^2\cos^2\theta)^{\frac{3}{2}}}{|ab|}. \tag{2.7}$$

According to the relation (7), the radius of curvature is:

$$\begin{aligned} R_\lambda(T) &= \frac{(a^2(1-\cos^2\theta)+b^2\cos^2\theta)^{\frac{3}{2}}}{|ab|} \\ &= \frac{(a^2-a^2\cos^2\theta+b^2\cos^2\theta)^{\frac{3}{2}}}{|ab|} \\ &= \frac{(a^2-\cos^2\theta(a^2-b^2))^{\frac{3}{2}}}{|ab|} \\ &= \frac{(a^2-\cos^2\theta)^{\frac{3}{2}}}{|ab|}. \end{aligned}$$

So we have

$$R_\lambda(T)=\frac{(a^2-\cos^2\theta)^{\frac{3}{2}}}{|ab|}. \tag{2.8}$$

On the other hand, the distance of point $\lambda = (x, y)$ on the border of the ellipse from the nearest center of the ellipse, i.e. $F = (1, 0)$, is calculated as follows:

$$
\begin{aligned}
dist(\lambda, \sigma(T)) &= \sqrt{(x-1)^2 + y^2} \\
&= \sqrt{(a\cos\theta - 1)^2 + (b\sin\theta)^2} \\
&= \sqrt{a^2\cos^2\theta - 2a\cos\theta + 1 + b^2\sin^2\theta} \\
&= \sqrt{a^2\cos^2\theta - 2a\cos\theta + 1 + b^2(1-\cos^2\theta)} \\
&= \sqrt{\cos^2\theta(a^2 - b^2) - 2a\cos\theta + 1 + b^2} \\
&= \sqrt{\cos^2\theta - 2a\cos\theta + a^2} \\
&= \sqrt{(a-\cos\theta)^2} = |a - \cos\theta|.
\end{aligned}
$$

As a result, we have

$$dist(\lambda, \sigma(T)) = |a - \cos\theta|. \tag{2.9}$$

We have relations (2.8) and (2.9),

$$
\begin{aligned}
\frac{dist(\lambda, \sigma(T))}{R_\lambda(T)} &= \frac{ab(a-\cos\theta)}{(a^2 - \cos^2\theta)}^{\frac{3}{2}} \\
\Longrightarrow M(T) = \sup &\Big(\frac{ab(a-\cos\theta)}{(a^2-\cos^2\theta)}^{\frac{3}{2}}\Big).
\end{aligned}
$$

Now, with a simple general mathematical calculation, the value of $M(T)$ is obtained as follows:

$$M(T) = \max\left\{\frac{\sqrt{a^2-1}}{a}, \frac{a}{a+1}\right\}. \tag{2.10}$$

We know

$$2b = ||T|| - \frac{1}{||T||}, \quad 2a = ||T|| + \frac{1}{||T||}.$$

Therefore, by placing in relation (2.10), the value of $M(T)$ is:

$$
\begin{aligned}
M(T) &= \max\left\{\frac{\sqrt{\frac{1}{4}(||T||^2 + 2 + \frac{1}{||T||^2}) - 1}}{\frac{1}{2}(||T|| + \frac{1}{||T||})}, \frac{\frac{1}{2}(||T|| + \frac{1}{||T||})}{\frac{1}{2}(||T|| + \frac{1}{||T||}) + 1}\right\} \\
&= \max\left\{\frac{\sqrt{\frac{||T||^4+1-2||T||^2}{4||T||^2}}}{\frac{||T||^2+1}{2||T||}}, \frac{\frac{||T||^2+1}{2||T||}}{\frac{||T||^2+2||T||+1}{2||T||}}\right\} \\
&= \max\left\{\frac{\sqrt{\frac{(||T||^2-1)^2}{4||T||^2}}}{\frac{||T||^2+1}{2||T||}}, \frac{\frac{||T||^2+1}{2||T||}}{\frac{(||T||+1)^2}{2||T||}}\right\} \\
&= \max\left\{\frac{||T||^2-1}{||T||^2+1}, \frac{||T||^2+1}{(||T||+1)^2}\right\}
\end{aligned}
$$

Therefore, the important result of the article in summary from the above calculation process leads to the proof of the following basic theorem.

**Theorem 2.6.** Suppose the $T$ operator is inverse and non-trivial, then we have

$$M(T) = \max\left\{\frac{\|T\|^2-1}{\|T\|^2+1}, \frac{\|T\|^2+1}{(\|T\|+1)^2}\right\}. \tag{2.11}$$

**Example 2.7.** Let the operator $T:\mathbb{R}^2\to\mathbb{R}^2$ be defined by $T(x,y)=(-y,-x)$. This operator is self-inverse since $T^2=I$, where $I$ is the identity operator. Moreover, $T$ is non-trivial since $T\neq\pm I$.

To calculate the maximum value of the norm of the operator $T$ using the above theorem, we first need to compute the norm of the operator $T$ as follows:

$$\|T\|=\sup_{\|x\|=1}\|T(x)\|.$$

Now, applying the above definition, we can calculate the norm of $T$ as follows:

$$\|T(x,y)\|=\|(-y,-x)\|=\sqrt{(-y)^2+(-x)^2}=\sqrt{x^2+y^2}.$$

By substituting $x=\cos(\theta)$ and $y=\sin(\theta)$, we have:

$$\|T(x,y)\|=\sqrt{x^2+y^2}=1.$$

Now, by substituting $\|T\|=1$ into the formula, we can find the maximum value of the norm of the operator $T$ as follows:

$$M(T)=\max\left\{\frac{1^2-1}{1^2+1},\frac{1^2+1}{(1+1)^2}\right\}=\max\left\{0,\frac{2}{4}\right\}=\frac{1}{2}.$$

## 3 Conclusions

In consideration of the applications of boundary curvature in engineering, physics, and chemistry, we initially explored certain properties of the boundary curvature of the numerical range. Secondly, we introduced $M(T)$ as the smallest constant such that $dist(\lambda,\sigma(T))\leq M(T)R_\lambda(T)$, for all $\lambda\in\partial W(T)$. Subsequently, for non-convexoid $T$, we have $M(T)=\sup\frac{dist(\lambda,\sigma(T))}{R_\lambda(T)}$, where the supremum on the right-hand side is taken over all points $\lambda\in\partial W(T)$ with finite non-zero curvature. Lastly, we determined the value of $M(T)$ for the self-inverse operators.

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