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Some properties of bicomplex holomorphic functions

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Abstract

In this paper, we first establish the bicomplex version of Rouche's theorem. Also, a new approach is given to prove the maximum modulus principle for bicomplex holomorphic functions. Our proof is based on the direct method and extends the result proved by Luna-Elizarraras et al. Finally, we generalize the Hurwitz's theorem to bicomplex space.

Keywords: Bicomplex Function, Rouche's Theorem, Maximum Modulus Principle, Hurwitz's theorem
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1 Introduction and Statement of Results

In abstract algebra, a tessarine or bicomplex number is a hypercomplex number in a commutative, associative algebra over real numbers with two imaginary units. In 1892 Corrado Segre introduced [11, 6] bicomplex numbers in Mathematische Annalen, which form an algebra isomorphic to the tessarines. Also, mathematicians proved a fundamental theorem of tessarine algebra: a polynomial of degree n with tessarine coefficients has n^2 roots, counting multiplicity [8]. Rouche's theorem in complex analysis states that if the complex-valued functions f and g are holomorphic inside and on some simple closed contour K, with $|g(z)| < |f(z)|$ on K, then f and $f + g$ have the same number of zeros inside K, where each zero is counted as many times as its multiplicity [7].

In this paper, we prove the bicomplex version of Rouche's theorem; also we prove that a bicomplex function f cannot attain a maximum or minimum of $|f|$ in a bicomplex domain. In this direction, we state some basic definitions and properties of bicomplex number [1, 9]. Let $\mathbb{BC}$ be the bicomplex algebra, i.e.,

$$\mathbb{BC} = \{x_1 + ix_2 + j(x_3 + ix_4) : x_1, x_2, x_3, x_4 \in \mathbb{R}\},$$

with $i^2 = -1, j^2 = -1$ and $ij = ji = 1$. Assuming $Z = x_1 + ix_2 + j(x_3 + ix_4) =: z_1 + jz_2$, we remark that every bicomplex number has the following unique representation:

$$Z = z_1 + jz_2 = (z_1 - iz_2)e_1 + (z_1 + iz_2)e_2,$$

where $e_1 = \dfrac{1 + ij}{2}$ and $e_2 = \dfrac{1 - ij}{2}$. If $Z = z_1 + jz_2 \in \mathbb{BC}$, the norm of Z is defined as follows:

$$\|Z\| = \sqrt{|z_1|^2 + |z_2|^2}.$$

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It can be easily verify that for $Z = \alpha e_1 + \beta e_2$ where $\alpha, \beta \in \mathbb{C}$,

$$\|Z\| = \sqrt{\frac{|\alpha|^2 + |\beta|^2}{2}}.$$

If we define $d : \mathbb{BC} \times \mathbb{BC} \to \mathbb{R}_{\geq 0}, (Z_1, Z_2) \mapsto d(Z_1, Z_2)$, by setting $d(Z_1, Z_2) = \|Z_1 - Z_2\|$ for every Z_1, Z_2 in $\mathbb{BC}$, then $(\mathbb{BC}, d)$ is a metric space.

Suppose that $Z, W \in \mathbb{BC}$ and $ZW = 1$, then each of the elements Z and W is said to be the inverse of each other. An element that has an inverse is said to be invertible (non-singular), and an element that does not have an inverse is said to be non-invertible(singular). For a bicomplex number $Z = \alpha e_1 + \beta e_2 \in \mathbb{BC}$, it is easy to verify that Z is invertible if and only if $\alpha, \beta \neq 0$; in this case, if we denote the inverse of Z by Z^{-1}, then we have

$$Z^{-1} = \alpha^{-1} e_1 + \beta^{-1} e_2.$$

Definition 1.1. Suppose that

$$X_1 = \{z_1 - iz_2 \ : \ z_1, z_2 \in \mathbb{C}\},$$
$$X_2 = \{z_1 + iz_2 \ : \ z_1, z_2 \in \mathbb{C}\},$$

we say that $X = X_1 \times_e X_2 (\subseteq \mathbb{BC})$ is Cartesian set generated by X_1 and X_2, if

$$X = \{z_1 + jz_2 \in \mathbb{BC} \ : \ z_1 - iz_2 \in X_1, z_1 + iz_2 \in X_2\}.$$

Definition 1.2. Let $a = \alpha + j\beta$ be a fixed point in $\mathbb{BC}$ and r, r_1 and r_2 denote numbers in $\mathbb{R}$ such that $r > 0$, $r_1 > 0$ and $r_2 > 0$. The open ball $B(a, r)$ and closed ball $\overline{B}(a, r)$ with center a and radius r are defined as follows:

$$B(a, r) = \{z_1 + jz_2 \in \mathbb{BC} \ : \ \|(z_1 + jz_2) - (\alpha + j\beta)\| < r\},$$
$$\overline{B}(a, r) = \{z_1 + jz_2 \in \mathbb{BC} \ : \ \|(z_1 + jz_2) - (\alpha + j\beta)\| \leq r\}.$$

The open discus $D(a; r_1, r_2)$ and closed discus $\overline{D}(a; r_1, r_2)$ with center a and radii r_1 and r_2 are defined as follows:

$$D(a; r_1, r_2) = \big\{z_1 + jz_2 \in \mathbb{BC} \ : \ z_1 + jz_2 = w_1 e_1 + w_2 e_2, |w_1 - (\alpha - i\beta)| < r_1, |w_2 - (\alpha + i\beta)| < r_2\big\},$$

$$\overline{D}(a; r_1, r_2) = \{z_1 + jz_2 \in \mathbb{BC} \ : \ z_1 + jz_2 = w_1 e_1 + w_2 e_2, |w_1 - (\alpha - i\beta)| \leq r_1, |w_2 - (\alpha + i\beta)| \leq r_2\}.$$

It is easy to verify that if $0 < r_1 \leq r_2$, then for every bicomplex number a,

$$D(a; r_1, r_2) \subsetneqq B\Big(a, \sqrt{\frac{r_1^2 + r_2^2}{2}}\Big).$$

Now, to prove the bicomplex version of Rouche's theorem, we first prove the following two theorems.

Theorem 1.3. Let X_1 and X_2 be bounded domains in $\mathbb{C}$, and let X be the Cartesian domain generated by X_1 and X_2. Also,

$$F(z_1 + jz_2) = f_{e_1}(z_1 - iz_2) e_1 + f_{e_2}(z_1 + iz_2) e_2,$$

and

$$G(z_1 + jz_2) = g_{e_1}(z_1 - iz_2) e_1 + g_{e_2}(z_1 + iz_2) e_2,$$

are bicomplex holomorphic functions in X. If

(i) $|F(z)| < |G(Z)|$ for every $Z \in \partial X$,
(ii) $G(Z)$ having at least one zero in X,

then $H(Z) = F(Z) + G(Z)$ has the same number of zeros in X as $G(Z)$ does.

Theorem 1.4. Let X_1 and X_2 be bounded domains in $\mathbb{C}$, and let X be the Cartesian domain generated by X_1 and X_2. Also,

$$F(z_1 + jz_2) = f_{e_1}(z_1 - iz_2)e_1 + f_{e_2}(z_1 + iz_2)e_2,$$

and

$$G(z_1 + jz_2) = g_{e_1}(z_1 - iz_2)e_1 + g_{e_2}(z_1 + iz_2)e_2,$$

are bicomplex holomorphic functions in X. If

(i) $|F(z)| < |G(Z)|$ for every $Z \in \partial X$,
(ii) $G(Z)$ having no zero in X,

then $H(Z) = F(Z) + G(Z)$ has no zero in X as $G(Z)$ does.

By combining Theorems 1.3 and 1.4, we have the following Theorem 1.5.

Theorem 1.5. (*Analogue of the Rouche's Theorem*) Let X_1 and X_2 be bounded domains in $\mathbb{C}$, and let X be the Cartesian domain generated by X_1 and X_2. Also,

$$F(z_1 + jz_2) = f_{e_1}(z_1 - iz_2)e_1 + f_{e_2}(z_1 + iz_2)e_2,$$

and

$$G(z_1 + jz_2) = g_{e_1}(z_1 - iz_2)e_1 + g_{e_2}(z_1 + iz_2)e_2,$$

are bicomplex holomorphic functions in X. If $|F(z)| < |G(Z)|$ for every $Z \in \partial X$, then $H(Z) = F(Z) + G(Z)$ has the same number of zeros in X as $G(Z)$ does.

Example 1.6. Assume that

$$F(Z) = e_2 Z^{10} - 6e_1 Z^9 + e_2 Z^5 + 6e_2 Z^3 - e_2 Z + 101e_1 = P(Z) + Q(Z),$$

where

$$P(Z) = -6e_1 Z^9 + e_2 Z^5 + 6e_2 Z^3 + 2e_2 Z + e_1 = (-6\alpha^9 + 1)e_1 + (\beta^5 + 6\beta^3 + 2\beta)e_2, = \phi(\alpha)e_1 + \psi(\beta)e_2,$$

and

$$Q(Z) = e_2 Z^{10} - 3e_2 Z + 100e_1 = 100e_1 + (\beta^{10} - 3\beta)e_2 = \eta(\alpha)e_1 + \zeta(\beta)e_2,$$

then $Q(Z)$ has no zeros, also

$$|\zeta(\beta)| < |\phi(\alpha)|, \qquad \text{and} \qquad |\psi(\beta)| < |\eta(\alpha)|,$$

for $|\alpha| = |\beta| = 1$ ([7], p. 342, 343). For every bicomplex number $Z = \alpha e_1 + \beta e_2$, and every complex number a, we have

$$|ae_1 Z^n| \leq \frac{\sqrt{2}}{2}|a||\alpha|^n, \qquad \text{and} \qquad |ae_2 Z^n| \leq \frac{\sqrt{2}}{2}|a||\beta|^n.$$

Hence, if $Z \in \partial D(0; 1, 1)$, since $|\alpha| \leq 1$, $|\beta| \leq 1$ based on triangle inequality, we have

$$|P(Z)| \leq \frac{16\sqrt{2}}{2}, \qquad \text{and} \qquad |Q(Z)| \geq \frac{96\sqrt{2}}{2}.$$

By Theorem 1.5, $Q(Z)$ and $F(Z)$ have the same number of zeros inside $D(0; 1, 1)$. Hence, $F(Z)$ having no zeros inside $D(0; 1, 1)$.

Example 1.7. Assume that

$$P(Z) = 9Z^5 + Z^3 - \Big(\frac{3+4i}{8} + \frac{11}{24}ij\Big)Z^2 + \frac{1}{48}((2-i) + (6i-1)j)Z - \frac{7}{48}(i-j),$$

and put

$$\begin{aligned} F(Z) &= Z^3 - \Big(\frac{3+4i}{8} + \frac{11}{24}ij\Big)Z^2 + \frac{1}{48}\Big((2-i) + (6i-1)j\Big)Z - \frac{7}{48}(i-j) \\ &= \Big(\alpha^3 - \frac{5+3i}{6}\alpha^2 + \frac{1}{6}\alpha - \frac{i}{3}\Big)e_1 + \Big(\beta^3 + \frac{1-6i}{12}\beta^2 - \frac{2+i}{24}\beta + \frac{i}{24}\Big)e_2 \\ &=: \phi(\alpha)e_1 + \psi(\beta)e_2, \end{aligned}$$

and $G(Z) = 9Z^5$, then for every $Z \in \partial D(0;1,1)$, we have

$$|G(Z)| \geq \frac{9\sqrt{2}}{2}, \qquad \text{and} \qquad |F(Z)| \leq 2\sqrt{2} < |G(Z)|.$$

By Theorem 1.5, $P(Z)$ and $G(Z)$ have the same number of zeros in $D(0;1,1)$, but $G(Z)$ has all its zeros in $D(0;1,1)$. Therefore, $P(Z)$ has all its zeros in $D(0;1,1)$.

Proposition 1.8. Assume $F(Z) = f_{e_1}(\alpha)e_1 + f_{e_2}(\beta)e_2$ be non-constant and holomorphic in an open set containing $\overline{D}(0;1,1)$. If $|F(Z)| = 1$ for every $Z \in \partial D(0;1,1)$, then the image of $F(Z)$ contains the $D(0;1,1)$.

In the following two theorems, we prove that for a bicomplex holomorphic function $F(Z)$ in a bicomplex domain X, $|F(Z)|$ cannot attain a maximum (minimum) in D, unless $F(Z)$ is constant.

Theorem 1.9. (*Analogue of the Maximum Modulus Theorem*) If

$$F(z_1 + jz_2) = f_{e_1}(z_1 - iz_2)e_1 + f_{e_2}(z_1 + iz_2)e_2,$$

is a holomorphic function in a domain X, then $|F(Z)|$ cannot attain a maximum in X unless $F(Z)$ is constant.

Theorem 1.10. (*Analogue of the Minimum Modulus Theorem*) If

$$F(z_1 + jz_2) = f_{e_1}(z_1 - iz_2)e_1 + f_{e_2}(z_1 + iz_2)e_2,$$

is a holomorphic function in a domain X, and $F(Z)$ is invertible in X. Then $|F(Z)|$ cannot attain a minimum in X unless $F(Z)$ is constant.

Remark 1.11. If X is a bounded domain in $\mathbb{BC}$, then $\overline{X} = X \bigcup \partial X$ is a compact set in $\mathbb{BC}$ and hence $|F(Z)|$ attains a maximum and minimum on $\overline{X}$, therefore based on Theorems 1.9 and 1.10, maximum and minimum of $|F(Z)|$ occur on ∂X.

Finally, we prove the bicomplex version of Hurwitz's theorem.

Theorem 1.12. Let X_1 and X_2 be bounded domains in $\mathbb{C}$ such that $\overline{X_1}$ and $\overline{X_2}$ compacts, also X be cartesian set determined by X_1 and X_2. If a sequence $\{F_n\}$ of holomorphic functions in X and continues on ∂X, is uniformly convergent on every compact subset of X to a function F and F does not vanish in ∂X, then there exists some positive integer N, such that for every $n \geq N$, $F_n(Z)$ in X has the same number of zeros as the function $F(Z)$ does counting every root as many times as its multiplicity indicates.

2 Lemma

We will need the following lemmas to prove our results.

Lemma 2.1. Let X_1 and X_2 be domains in $\mathbb{C}$, and let X be the Cartesian domain generated by X_1 and X_2, then boundary X is the union of the following three disjoint sets:

$$T_1 = \{\alpha e_1 + \beta e_2 : \alpha \in X_1, \beta \in \partial X_2\}, \quad T_2 = \{\alpha e_1 + \beta e_2 : \alpha \in \partial X_1, \beta \in X_2\}, \quad T_3 = \{\alpha e_1 + \beta e_2 : \alpha \in \partial X_1, \beta \in \partial X_2\}.$$

Lemma 2.2. Let X_1 and X_2 be open sets in $\mathbb{C}$. If $f_{e_1} : X_1 \longrightarrow \mathbb{C}$ and $f_{e_2} : X_2 \longrightarrow \mathbb{C}$ are holomorphic functions on X_1 and X_2 respectively, then the function $f : X_1 \times_e X_2 \longrightarrow \mathbb{BC}$ defined as

$$f(z_1 + jz_2) = f_{e_1}(z_1 - jz_2)e_1 + f_{e_2}(z_1 + jz_2)e_2, \quad \forall\ z_1 + jz_2 \in X_1 \times_e X_2,$$

is $\mathbb{BC}-$holomorphic on the open set $X_1 \times_e X_2$ and

$$f'(z_1 + jz_2) = f'_{e_1}(z_1 - iz_2)e_1 + f'_{e_2}(z_1 + iz_2)e_2, \quad \forall\ z_1 + jz_2 \in X_1 \times_e X_2.$$

These lemmas and the next lemma are proved by Charak et al. [2, 3, 10].

Lemma 2.3. If X is an open set in $\mathbb{BC}$, and let $f : X \longrightarrow \mathbb{BC}$ be a $\mathbb{BC}-$holomorphic function on X, then there exist holomorphic functions $f_{e_1} : X_1 \longrightarrow \mathbb{C}$ and $f_{e_2} : X_2 \longrightarrow \mathbb{C}$ with

$$X_1 = \{z_1 - iz_2 : z_1 + jz_2 \in X\}, \quad X_2 = \{z_1 + iz_2 : z_1 + jz_2 \in X\},$$

such that

$$f(z_1 + jz_2) = f_{e_1}(z_1 - iz_2)e_1 + f_{e_2}(z_1 + iz_2)e_2, \qquad \forall\ z_1 + jz_2 \in \mathbb{BC}.$$

Lemma 2.4. (Weierstrass) Let $\{F_n\}$ be a sequence of bicomplex holomorphic functions on a domain D, which converges uniformly on compact subsets of D to a function F. Then F is bicomplex holomorphic in D [3, 4].

Lemma 2.5. Let $f(z)$ be an analytic complex function in a domain D and continue on its closure $\overline{D}$. If $|f(z)|$ is constant on the boundary of D, then $f(z)$ has a zero in D [7, 5].

3 Proofs of the Theorems

3.1 Proof of Theorem 1.3

Let $Z_1 = \alpha_1 e_1 + \beta_1 e_2 \in X$ such that $G(Z_1) = 0$ and $\alpha \in \partial X_1$, then by lemma 2.1, $Z = \alpha e_1 + \beta_1 e_2 \in \partial X$ and based on (i), $|F(Z)| < |G(Z)|$, i.e.

$$\sqrt{\frac{|f_{e_1}(\alpha)|^2 + |f_{e_2}(\beta_1)|^2}{2}} < \sqrt{\frac{|g_{e_1}(\alpha)|^2 + |g_{e_2}(\beta_1)|^2}{2}} = \sqrt{\frac{|g_{e_1}(\alpha)|^2}{2}}.$$

Hence, $|f_{e_1}(\alpha)| < |g_{e_1}(\alpha)|$, for every complex number α with $\alpha \in \partial X_1$. Since, X_1 is a bounded domain, hence by Lemma 2.3, f_{e_1} and g_{e_1} are holomorphic functions on X_1 and by applying Rouches's theorem, we find that $f_{e_1}(\alpha) + g_{e_1}(\alpha)$ has the same number of zeros in X_1 as $g_{e_1}(\alpha)$ does.

Let β be a complex number with $\beta \in \partial X_2$, then $Z = \alpha_1 e_1 + \beta e_2 \in \partial X$ and based on (i), $|F(Z)| < |G(Z)|$, i.e.

$$\sqrt{\frac{|f_{e_1}(\alpha_1)|^2 + |f_{e_2}(\beta)|^2}{2}} < \sqrt{\frac{|g_{e_1}(\alpha_1)|^2 + |g_{e_2}(\beta)|^2}{2}} = \sqrt{\frac{|g_{e_2}(\beta)|^2}{2}}.$$

Hence, $|f_{e_2}(\beta)| < |g_{e_2}(\beta)|$, for every complex number β with $\beta \in \partial X_2$. Since, X_2 is a bounded domain, then by Lemma 2.3, f_{e_2} and g_{e_2} are holomorphic functions on X_2 and by applying Rouches's theorem, we find that $f_{e_2}(\beta) + g_{e_2}(\beta)$ has the same number of zeros in X_2 as $g_{e_2}(\beta)$ does.

Therefore, $H(Z) = F(Z) + G(Z) = \big(f_{e_1}(\alpha) + g_{e_1}(\alpha)\big)e_1 + \big(f_{e_2}(\beta) + g_{e_2}(\beta)\big)e_2$, has the same number of zeros in X as does $G(Z)$ and this completes the proof of Theorem 1.3.

3.2 Proof of Theorem 1.4

Assume that $Z^* = \alpha^* e_1 + \beta^* e_2 \in X$ such that $H(Z^*) = 0$. Hence, $(f_{e_1} + g_{e_1})(\alpha^*) = 0$ and $(f_{e_2} + g_{e_2})(\beta^*) = 0$, and therefore,

$$|f_{e_1}(\alpha^*)| = |g_{e_1}(\alpha^*)| \qquad \text{and} \qquad |f_{e_2}(\beta^*)| = |g_{e_2}(\beta^*)|. \tag{3.1}$$

For every $\alpha \in \partial X_1$, since, $Z = \alpha e_1 + \beta^* e_2 \in \partial X$, then $|P(Z)| < |Q(Z)|$. It follows that

$$|f_{e_1}(\alpha)|^2 + |f_{e_2}(\beta^*)|^2 < |g_{e_1}(\alpha)|^2 + |g_{e_2}(\beta^*)|^2,$$

and by (3.1), we obtain $|f_{e_1}(\alpha)| < |g_{e_1}(\alpha)|$, for every complex number α with $\alpha \in \partial X_1$. By Lemma 2.3 and applying Rouche's theorem for $f_{e_1}(\alpha)$ and $g_{e_1}(\alpha)$, we find that $g_{e_1}(\alpha)$ has at least one zero in X_1. Using a similar argument used above, we find that $g_{e_2}(\beta)$ has at least one zero in X_2. Hence, $G(Z)$ has at least one zero in X, which contradicts with this fact that $G(Z)$ does not vanish in X. Hence, $(F+G)(z)$ has no zeros in X and this proves the desired result.

3.3 Proof of Proposition 1.8

For every complex number α with $|\alpha| = 1$, since, $\alpha e_1 \in \partial D(0; 1, 1)$, hence, $|F(\alpha e_1)| = 1$, and

$$|f_{e_1}(\alpha)|^2 + |f_{e_2}(0)|^2 = 1.$$

Therefore, $|f_{e_1}(\alpha)|$ is constant for every complex number α with $|\alpha| = 1$ and by Lemma 2.5, $\phi(\alpha)$ has a zero in $\{\alpha \in \mathbb{C} : |\alpha| < 1\}$. Similarly, we can prove that $|f_{e_1}(\beta)|$ also has a zero in $|\beta| < 1$ and hence, $F(Z)$ has a zero in $D(0; 1, 1)$. For every bicomplex number W in $D(0; 1, 1)$, we have

$$|W| < 1 < |F(Z)|,$$

for every $Z \in \partial D(0; 1, 1)$. So, Theorem 1.5 implies that $F(Z) - W$ and $F(Z)$ have the same number of zeros in D, therefore $F(Z) - W$ has at least one zero in D. This proves the desired result.

3.4 Proof of Theorem 1.9

Let $X_1 := \{z_1 - iz_2 : z_1 + jz_2 \in X\}$ and $X_2 := \{z_1 + iz_2 : z_1 + jz_2 \in X\}$, then by Lemma 2.3, there exist holomorphic functions $f_{e_1} : X_1 \to X$ and $f_{e_2} : X_2 \to X$ such that

$$F(z_1 + jz_2) = f_{e_1}(z_1 - iz_2)e_1 + f_{e_2}(z_1 + iz_2)e_2, \quad \forall\ z_1 + jz_2 \in \mathbb{BC}.$$

Suppose that $|F(Z)|$ attains maximum at a point $Z_0 = \alpha_0 e_1 + \beta_0 e_2$ in X. If $B(Z_0, r) \subset X$, then $D(Z_0; r, r) \subsetneqq B(Z_0, r)$, and

$$|f_{e_1}(\alpha)|^2 + |f_{e_2}(\beta)|^2 \le |f_{e_1}(\alpha_0)|^2 + |f_{e_2}(\beta_0)|^2, \qquad \forall\ \alpha e_1 + \beta e_2 \in D(Z_0; r, r).$$

For every $Z = \alpha e_1 + \beta_0 e_2$ such that $|\alpha - \alpha_0| < r$, since, $Z \in D(Z_0; r, r)$ and $\alpha \in X_1$, we have $|F(Z)| \le |F(Z_0)|$, i.e.

$$|f_{e_1}(\alpha)|^2 + |f_{e_2}(\beta_0)|^2 \le |f_{e_1}(\alpha_0)|^2 + |f_{e_2}(\beta_0)|^2.$$

Therefore, $|f_{e_1}(\alpha)| \le |f_{e_1}(\alpha_0)|$, for every complex number α with $|\alpha - \alpha_0| < r$ $(\alpha \in X_1)$. Since, f_{e_1} is analytic in X_1, hence, based on the maximum modulus theorem, f_{e_1} is constant. Using a similar argument as used for f_{e_1}, we find that f_{e_2} is also constant. Therefore, $F(z)$ is constant and this proves the desired result.

3.5 Proof of Theorem 1.10

Let $Z_0 = \alpha_0 e_1 + \beta_0 e_2 \in X$ such that for every $Z \in X$, we have $|F(Z_0)| \le |F(Z)|$. Choose a disk $B(Z_0, r)$ contained in X. For every complex number α with $|\alpha - \alpha_0| < r$, since, $Z = \alpha e_1 + \beta_0 e_2 \in X$, hence, $|F(Z_0)| \le |F(Z)|$, i.e.

$$|f_{e_1}(\alpha_0)|^2 + |f_{e_2}(\beta_0)|^2 \le ||f_{e_1}(\alpha)|^2 + |f_{e_2}(\beta_0)|^2,$$

or

$$|f_{e_1}(\alpha_0)| \le |f_{e_1}(\alpha)|,$$

for every α with $|\alpha - \alpha_0| < r$. Also, as $F(Z)$ is invertible in X, then $f_{e_1}(z_1 - iz_2) \ne 0$ in X_1 and $f_{e_2}(z_1 + iz_2) \ne 0$ in X_2 (X_1, X_2 is defined in Theorem 1.5). Now, by applying maximum modulus theorem for f_{e_1}, we find that f_{e_1} is constant in disk $|\alpha - \alpha_0| < r$ and therefore (by uniqueness theorem) f_{e_1} is constant in X_1.

Similarly, we can get f_{e_2} is constant in X_2 and therefore, $F(Z)$ is constant in X. This completes the proof of Theorem 1.10.

3.6 Proof of Theorem 1.12

Assume that $m = \operatorname*{Min}_{Z \in \partial X} |F(Z)|$, based on hypothesis, $m > 0$ and we can find positive integer n_0 such that for every $n \ge n_0$, we have

$$|F_n(Z) - F(Z)| < m \le |F(Z)|,$$

for all $Z \in \partial X$. By Lemma 2.4, F is holomorphic, hence, by Theorem 1.5, $F_n(Z) = (F_n(Z) - F(Z)) + F(Z)$ and $F(Z)$ have the same number of zeros in X. This completes the proof of Theorem 1.12.

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